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by

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INITIALLY DEFORMED FLOWS

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Abstract. In this paper we analyze systems with a continuous initial value deformation. These systems, defined using generalizations of the classical flow axioms, have a deformed flow-like structure. Trivial examples include the classical dynamical systems and phase shifted classical flows. Moreover, whenever the initial deformation is a homeomorphism on the phase space, these systems are induced by classical dynamical systems. For an initial phase space deformation function f, we denote the generalized flow by π_f in keeping with the spirit of the π notation for a classical system or flow, and we refer to the generalized system as an f-flow.

1. Introduction

In this section we introduce the axioms for a generalized dynamical system. The author first encountered such discrete notions in [5, p. 239] where Gordon Thomas Whyburn considered dynamics for iterates of continuous functions f with $f(X) \subset X$ rather than $f(X) = X$. Although we shall use various notations that are not standardized, the definitions and notations used herein are consistent with those used by many researchers (including the author who used them in [1], [2], [3], [4], and several other papers.)

For a flow (X, π), we have initial value $\pi(x, 0) = x$ and we have $\pi(\pi(x, 0), t + s) = \pi(\pi(x, t), s)$. Analogously, in the following generalization of (X, π), we introduce an initial value deformation

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$\pi_f(x,0) = f(x)$, and again, $\pi_f(\pi_f(x,0), t+s) = \pi_f(\pi_f(x,t), s)$. As we shall see, the generalized flows are not always induced by classical flows, and hence, a given generalized flow may have no direct connection to the class of flows except for a parallel initial value format.

Definition 1.1. Let f be a function defined on a space X. A continuous (discrete) f-flow or simply an f-flow (X, π_f) consists of a topological space X and a function π_f from the product space $X \times \mathbb{R}$ $(X \times \mathbb{Z})$ into the space X satisfying the following axioms.

(1) Initial deformation: $\pi_f(x,0) = f(x)$ for each $x \in X$.
(2) Group deformation action: $\pi_f\big(\pi_f(x,t), s\big) = \pi_f(f(x), t+s)$ for each $x \in X$ and each $t, s \in \mathbb{R}$ $(t, s \in \mathbb{Z})$.
(3) Continuity: $\pi_f : X \times \mathbb{R} \longrightarrow X$ $(\pi_f : X \times \mathbb{Z} \longrightarrow X)$ is continuous.

For brevity, whenever X is understood, we shall say that π_f is an f-flow when referring to an f-flow (X, π_f). If $\mathbb{R}$ $(\mathbb{Z})$ is replaced by $\mathbb{R}^+ = [0, \infty)$ or $\mathbb{R}^- = (-\infty, 0]$ $(\mathbb{Z}^+$ or $\mathbb{Z}^-)$, then (X, π_f) is called a positive or negative (discrete) f-semiflow, respectively. Each type of f-flow π_f is the corresponding type of classical flow if $f = i_X$, the identity on X, in which case we use the usual notation $\pi = \pi_{i_X}$ and refer to it as a flow.

A given f-flow (X, π_f) on a Hausdorff phase space X will be assumed throughout this paper. Despite the fact that corresponding results hold for discrete f-flows and f-semiflows, we state and prove results throughout this treatise primarily for continuous f-flows.

2. Preliminaries

Our first task is to develop a few basic properties for f-flows and to determine some connections between f-flows and flows.

Since π_f is continuous, $\pi_f | X \times \{0\} = f$ is continuous on X, and hence, we have the following proposition.

Proposition 2.1. *The function f is continuous on X.*

The result of Proposition 2.2, although simple, will be used throughout this paper.

Proposition 2.2. *For each integer $n \geq 0$,*

$$\pi_f \circ (f^n \times i_{\mathbb{R}}) = f^n \circ \pi_f \,. \tag{2.1}$$

Moreover, (2.1) *is valid for all integers n when f is bijective (injective but f^n is defined on $f^{-n}(X)$ for $n<0$).*

Proof: Equation (2.1) trivially holds for $n=0$ where $f^0 = i_X$. For $x \in X$ and $t \in \mathbb{R}$, $\pi_f\big(f(x),t\big) = \pi_f\big(f(x),t+0\big) = \pi_f\big(\pi_f(x,t),0\big) = f\big(\pi_f(x,t)\big)$ yields $\pi_f \circ (f \times i_{\mathbb{R}}) = f \circ \pi_f$. By assuming the result for $n=k$, we have $\pi_f \circ (f^{k+1} \times i_{\mathbb{R}})(x,t) = \pi_f\big(f^{k+1}(x),t\big) = f\Big(\pi_f(f^k(x),t)\Big) = f\Big(f^k\big(\pi_f(x,t)\big)\Big) = f^{k+1}\big(\pi_f(x,t)\big) = f^{k+1} \circ \pi_f(x,t)$, and hence, we have (2.1) by induction for all non-negative integers n.

Finally, for $n=-1$ and f bijective (injective) we have

$$\begin{aligned}\big(\pi_f \circ (f \times i_{\mathbb{R}})\big) \circ (f^{-1} \times i_{\mathbb{R}}) &= (f \circ \pi_f) \circ (f^{-1} \times i_{\mathbb{R}}),\\ \pi_f &= f \circ \big(\pi_f \circ (f^{-1} \times i_{\mathbb{R}})\big),\\ f^{-1} \circ \pi_f &= \pi_f \circ (f^{-1} \times i_{\mathbb{R}}),\end{aligned}$$

and hence, using induction, we again have (2.1) valid for each negative integer n. □

Corollary 2.3. *If f is an embedding of X, then the mapping π_f induces a flow $(f(X),\pi)$ on $f(X)$ where $\pi = \pi_f \circ (f^{-1} \times i_{\mathbb{R}})$ and $f \circ \pi = \pi \circ (f \times i_{\mathbb{R}})$.*

Proof: The continuity of π is obvious, $\pi(x,0) = \pi_f\big(f^{-1}(x),0\big) = f\big(f^{-1}(x)\big) = x$ for each $x \in f(X)$, and finally $\pi\big(\pi(x,t),s\big) = \pi_f\Big(f^{-1}\Big(\pi_f\big(f^{-1}(x),t\big)\Big),s\Big) = \pi_f\Big(\pi_f\Big(f^{-1}\big(f^{-1}(x)\big),t\Big),s\Big) = \pi_f\Big(\pi_f\big(f^{-2}(x),t\big),s\Big) = \pi_f\Big(f\big(f^{-2}(x)\big),t+s\Big) = \pi_f\big(f^{-1}(x),t+s\big) = \pi(x,t+s)$ for each $x \in f(X)$ and $t,s \in \mathbb{R}$. Hence, π is a flow. On the other hand, according to Proposition 2.2, $f\circ\pi = f\circ\pi_f\circ(f^{-1}\times i_{\mathbb{R}}) = \pi_f\circ(f\times i_{\mathbb{R}})\circ(f^{-1}\times i_{\mathbb{R}}) = \pi_f = \pi\circ(f\times i_{\mathbb{R}})$, completing the proof. □

Corollary 2.4. *Let f be continuous on X and (X,π) be a flow. Then the mapping π induces an f-flow (X,π_f) on X where $\pi_f = \pi \circ (f \times i_{\mathbb{R}})$ provided $f \circ \pi = \pi \circ (f \times i_{\mathbb{R}})$.*

Proof: The continuity of π_f is trivial and $\pi_f(x,0) = \pi\big(f(x),0\big) = f(x)$ for each $x \in X$. Moreover, for each $x \in X$ and $t,s \in \mathbb{R}$, we have $\pi_f\big(\pi_f(x,t),s\big) = \pi\Big(f\Big(\pi\big(f(x),t\big)\Big),s\Big) = \pi\Big(\pi\big(f^2(x),t\big),s\Big) = \pi\big(f^2(x),t+s\big) = \pi_f\big(f(x),t+s\big)$ and π_f is an f-flow. □

Henceforth, given the conditions of corollaries 2.3 and 2.4, we shall refer to π and π_f as the respective flow and f-flow *induced* by π_f and π.

Corollary 2.5. *Let f be an embedding of X. Then $\pi_f = \pi \circ (f \times i_{\mathbb{R}})$ is an f-flow on X if and only if $\pi = \pi_f \circ (f^{-1} \times i_{\mathbb{R}})$ is a flow on $f(X)$ and $f \circ \pi = \pi \circ (f \times i_{\mathbb{R}})$.*

Corollary 2.6. *Let (X, π_f) be an f-flow. Then $(X, \pi'_{f^{n+1}})$ defined by $\pi'_{f^{n+1}} = f^n \circ \pi_f$ is an f^{n+1}-flow for each $n \geq 0$. Also, for $n < 0$, $(f^n(X), \pi'_{f^{n+1}})$ is an f^{n+1}-flow whenever f is an embedding.*

Proof: Let x be a point of X and let t and s be elements of $\mathbb{R}$. For $n \geq 0$ (n in $\mathbb{Z}^-$ if f is an embedding), we have $\pi'_{f^{n+1}} = f^n \circ \pi_f$ a composition of continuous functions as well as $\pi'_{f^{n+1}}(x, 0) = f^n\big(\pi_f(x,0)\big) = f^n\big(f(x)\big) = f^{n+1}(x)$ and $\pi'_{f^{n+1}}\big(\pi'_{f^{n+1}}(x,t), s\big) = \pi'_{f^{n+1}}\Big(f^n\big(\pi_f(x,t)\big), s\Big) = \pi'_{f^{n+1}}\Big(\pi_f\big(f^n(x), t\big), s\Big) = f^n\bigg(\pi_f\Big(\pi_f\big(f^n(x), t\big), s\Big)\bigg) = f^n\Big(\pi_f\big(f^{n+1}(x), t+s\big)\Big) = \pi'_{f^{n+1}}\big(f^{n+1}(x), t+s\big)$. □

Note that axioms (1) and (2) of Definition 1.1 hold for $\pi'_{f^{n+1}}$ in Corollary 2.6 for each $n \in \mathbb{Z}$ whenever f is just a bijection. Continuity is required only for axiom (3).

We shall refer to the f-flow π_f defined in Proposition 2.7 as a *generalized product* f-flow. It is used in Example 3.6 of the next section to define an f-flow which neither induces a flow nor is induced by a flow.

Proposition 2.7. *Let π_{g_i} be a g_i-flow determined by $g_i : X_i \longrightarrow X_i$ for each i in a nonempty index set J. The mapping $\pi_f : X \times \mathbb{R} \longrightarrow X$ defined by $\pi_f\big((x_i), t\big) = \big(\pi_{g_i}(x_i, t)\big)$ for $(x_i) \in X$ and $t \in \mathbb{R}$ where $X = \times_{i \in J} X_i$ and $f = \times_{i \in J} g_i$ is an f-flow.*

Proof: For $(x_i) \in X$ and $t, s \in \mathbb{R}$, we have $\pi_f\big((x_i), 0\big) = \big(\pi_{g_i}(x_i, 0)\big) = \big(g_i(x_i)\big) = f\big((x_i)\big)$ and $\pi_f\Big(\pi_f\big((x_i), t\big), s\Big) = \pi_f\Big(\big(\pi_{g_i}(x_i, t)\big), s\Big) = \Big(\pi_{g_i}\big(\pi_{g_i}(x_i, t), s\big)\Big) = \Big(\pi_{g_i}\big(g_i(x_i), t+s\big)\Big) = \pi_f\Big(\big(g_i(x_i)\big), t+s\Big) = \pi_f\Big(f\big((x_i)\big), t+s\Big)$. Next, if f_i and p_i are the i^{th} coordinate map of f and the i^{th} projection of X, respectively, then $f_i = g_i \circ p_i$ so that f is continuous. Likewise, the i^{th} coordinate map of π_f is $\pi_{g_i} \circ (p_i \times i_{\mathbb{R}})$, and hence, π_f is continuous. □

It is often advantageous to consider an f-flow on a compact phase space. For locally compact spaces this may be possible depending, of course, on the function f. Theorem 2.8 and its corollaries address this by extension of the functions to the one point compactification of the phase space. Whenever $(X^*, \pi^*_{f^*})$ as defined in Theorem 2.8 is an f^*-flow, we shall refer to it as the *extended* f-flow.

In order to state Theorem 2.8 succinctly, we shall define now a point y in X to be an f-*critical* point of π_f whenever $\pi_f(y,t) = f(y)$ for each t in $\mathbb{R}$. In §4 and §5, we shall consider such points more extensively.

Theorem 2.8. *Let $X^* = X \cup \{\infty\}$ be the one point compactification of a noncompact locally compact space X. Let $f^* : X^* \to X^*$ be a continuous extension of f where (X, π_f) is an f-flow such that $f^*(\infty)$ is either ∞ or an f-critical point of π_f. Let $\pi^*_{f^*}$ be the extension of π_f where $\pi^*_{f^*}(\infty, t) = f^*(\infty)$ for each $t \in \mathbb{R}$. Then, $(X^*, \pi^*_{f^*})$ is an f^*-flow whenever $f^*(\infty) = \infty$ or $f^*(\infty) \neq \infty$ and $f(y) = f\,(f^*(\infty))$ for a unique point y in X.*

Proof: The initial deformation axiom of Definition 1.1 is evident for the hypothesis in view of the definition of $\pi^*_{f^*}$. To see that the group deformation axiom of Definition 1.1 holds, we first observe that when $f^*(\infty) = \infty$, $\pi^*_{f^*}\big(\pi^*_{f^*}(\infty, t), s\big) = \pi^*_{f^*}\big(f^*(\infty), s\big) = \pi^*_{f^*}(\infty, s) = f^*(\infty) = \pi^*_{f^*}(\infty, t+s) = \pi^*_{f^*}\big(f^*(\infty), t+s\big)$ for $t, s \in \mathbb{R}$. On the other hand, whenever $f^*(\infty)$ is an f-critical point of π_f, $\pi^*_{f^*}\big(\pi^*_{f^*}(\infty, t), s\big) = \pi^*_{f^*}\big(f^*(\infty), s\big) = \pi_f\big(f^*(\infty), s\big) = f\big(f^*(\infty)\big) = \pi_f\big(f^*(\infty), t+s\big) = \pi_{f^*}\big(f^*(\infty), t+s\big)$ for $t, s \in \mathbb{R}$. Thus, since $\pi^*_{f^*}|X \times \mathbb{R} = \pi_f$, the group deformation action axiom follows.

To see that the continuity axiom of Definition 1.1 holds when $f^*(\infty) = \infty$, let $x_i \to \infty$ and $t_i \to t$ where $x_i \neq \infty$ for each i. Assume that a subnet $\big(\pi_f(x_j, t_j)\big)$ of $\big(\pi_f(x_i, t_i)\big)$ converges to some $z \neq \infty$. Then $f^2(x_j) \to f^*(\infty)$ since $f(x_j) = f^*(x_j) \to f^*(\infty) = \infty$. Also, $f^2(x_j) = \pi_f(f(x_j), 0) = \pi_f\big(f(x_j), t_j - t_j\big) = \pi_f\big(\pi_f(x_j, t_j), -t_j\big) \to \pi_f(z, -t)$. But this means $\infty = f^*(\infty) = \pi_f(z, -t) \in X$, which is absurd. Thus, $\pi_f(x_j, t_j) \to \infty$, and hence, $\pi^*_{f^*}$ is continuous at (∞, t) for each t in $\mathbb{R}$ whenever $f^*(\infty) = \infty$.

Finally, in order to see that the continuity axiom of Definition 1.1 holds when $f^*(\infty) = y \neq \infty$, let $x_i \to \infty$ and $t_i \to t$ where $x_i \neq \infty$ for each i. Assume that a subnet $\big(\pi_f(x_j, t_j)\big)$ of $\big(\pi_f(x_i, t_i)\big)$ converges to some $z \neq y$. We have $f(x_j) = f^*(x_j) \to f^*(\infty) = y$ and $\pi_f\big(f(x_j), t_j\big) \longrightarrow \pi_f(y,t) = f(y)$. Also, $\pi_f\big(f(x_j), t_j\big) = f\big(\pi_f(x_j, t_j)\big) \longrightarrow f(z)$. Thus, $f(y) = f(z)$ or $y = z$, and hence, we again have $\pi^*_{f^*}$ continuous at (∞, t) for each t in $\mathbb{R}$, completing the proof. □

The only part of the proof of Theorem 2.8 that can fail is the continuity of π^*_f at (∞, t) for some t in $\mathbb{R}$ for $f^*(\infty) \neq \infty$. Consequently, we have the following corollary.

Corollary 2.9. *Let $(X^*, \pi^*_{f^*})$ be as defined in Theorem 2.8 and $f^*(\infty) \neq \infty$. Then, $(X^*, \pi^*_{f^*})$ is an f^*-flow if and only if $\pi^*_{f^*}$ is continuous at (∞, t) for each t in $\mathbb{R}$.*

Corollary 2.10. *Let $(X^*, \pi^*_{f^*})$ be as defined in Theorem 2.8. Then, $(X^*, \pi^*_{f^*})$ is an f^*-flow whenever $f^*(\infty) = \infty$, f is injective, or f is surjective.*

Proof: If f is injective, then $f(y) = f(f^*(\infty))$ for some y in X implies that y is unique. Note that the set $f^*(X^*)$ is compact in X^*, and hence, since either $f^*(X^*) = X$ or else $f^*(X^*) = X^*$. Whenever f is surjective, we must have $f^*(X^*) = X^*$ with $f^*(\infty) = \infty$. □

3. Examples

Before proceeding further, we give a few examples of f-flows. Although Example 3.6 gives a heuristic class of f-flows, it demonstrates that f-flows are indeed distinct from the classical continuous flows. Obviously, each f-flow (continuous or discrete) defines an f-semiflow by restricting π_f to $X \times \mathbb{R}^+$, $X \times \mathbb{R}^-$, $X \times \mathbb{Z}^+$, or $X \times \mathbb{Z}^-$.

Example 3.1. Let $f : X \longrightarrow X$ be any continuous function on X. Define $\pi_f(x,t) = f(x)$ for each $x \in X$ and $t \in \mathbb{R}$. Then, (X, π_f) is an f-flow. The mapping π_f is induced by the fixed flow (X, π) since $\pi_f(x,t) = \pi\big(f(x), t\big)$. Whenever f is a homeomorphism, π_f induces the flow (X, π) since $\pi(x,t) = \pi_f\big(f^{-1}(x), t\big)$.

Example 3.2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x + b$ for each $x \in \mathbb{R}$ where $b \in \mathbb{R}$ is fixed. Then, $\pi_f(x,t) = f(x) + t$ defines an f-flow on $\mathbb{R}$. When $b = 0$ this is the parallel flow. Note also that $f(x) = mx + b$ with $m \neq 1$ does not define an f-flow in this manner. The parallel flow π and π_f induce each other by $\pi_f(x,t) = \pi\big(f(x),t\big)$ and $\pi(x,t) = \pi_f\big(f^{-1}(x),t\big)$.

Example 3.3. In the following parts (a), (b), and (c), the f-flows $\pi_f : X \times \mathbb{R} \longrightarrow X$ are defined by $\pi_f(x,t) = f(x)\exp(i\alpha t)$ where α is fixed in $\mathbb{R}$ and f maps X to X.

(a) Let $X = S_1$ and let $f(x) = x\exp(i\beta)$ where β is fixed in $\mathbb{R}$. As in Example 3.2, π and π_f induce each other.

(b) Let X be the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$ and let $f(x) = \frac{x}{\|x\|}\exp(i\beta)$ where β is fixed in $\mathbb{R}$. Both f and π_f map onto S_1. This is not induced by a flow. However, $\pi = \pi_f|S_1 \times \mathbb{R}$ is the induced f-flow in part (a).

(c) Let $X = \mathbb{R}^2$ and let $f(x) = rx\exp(i\beta)$ where both r and β are fixed in $\mathbb{R}$. When $r = 1$, we have the f-flow in part (a). If we let X be the unit disc D and let $0 < r < 1$, then π_f and f map onto the central r-radius disc in D. These f-flows are not induced by a flow.

(d) In part (c) with either $X = \mathbb{R}^2$ or D, define π_f by $\pi_f(x,t) = f(x)\exp\big(i\alpha t\ \|x\|\big)$, or for the punctured plane or punctured unit disc, define π_f by $\pi_f(x,t) = f(x)\exp\Big(\frac{i\alpha t}{\|x\|}\Big)$. These are f-flows not induced by flows.

Example 3.4. Let (X,π) be a flow and let $t_o \in \mathbb{R}$ be fixed. Define $\pi_f(x,t) = \pi(x,t+t_o)$ for each $x \in X$ and $t \in \mathbb{R}$. For $f(x) = \pi(x,t_o)$, the mapping π_f defines and f-flow. Note that examples 3.2 and 3.3 for $\alpha \neq 0$ are all of this type since $\pi_f(x,t) = \pi(x,t+b)$ and $\pi_f(x,t) = \pi(x,t+\beta\alpha^{-1})$, respectively. This is not the case for Example 3.1 which includes Example 3.3 with $\alpha = 0$.

Example 3.5. It is tempting to try to construct an example in the following manner, but it does not in general lead to an f-flow. Let (X,π) be a flow and let $f : X \longrightarrow X$ be continuous. Define $\pi_f = f \circ \pi$. Then, the initial deformation and continuity axioms of Definition 1.1 hold but the group deformation axiom does not. However, some of these are f-flows, for instance, examples 3.1 and 3.4. In both cases, we have $\pi_f = f \circ \pi = \pi \circ (f \times i_{\mathbb{R}})$ of Corollary 2.4.

Example 3.6. The family of f-flows $\pi_f : (\mathbb{R} \times \mathbb{C}) \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{C}$ defined by $\pi_f(x,t) = \big(g(x), p_2(x)\exp(it)\big)$ for $x \in \mathbb{R} \times \mathbb{C}$ and $t \in \mathbb{R}$ are products of generalized flows where $f : \mathbb{R} \times \mathbb{C} \longrightarrow \mathbb{R} \times \mathbb{C}$ is defined by $f(x) = \big(g(x), p_2(x)\big)$ with $g : \mathbb{R} \times \mathbb{C} \longrightarrow \mathbb{R}$ continuous and $p_2 : \mathbb{R} \times \mathbb{C} \longrightarrow \mathbb{C}$ the second coordinate projection map. This type of f-flow is not induced by a flow and does not induce a flow for such maps as $g(x,y) = x^3 - x$. This family of f-flows arises from solutions to systems of partial differential equations. The system $u_t = 0$, $v_t = iv$ with initial conditions $u(x_o, 0) = g(x_o)$ and $v(x_o, 0) = p_2(x_o)$ for $x_o \in \mathbb{R} \times \mathbb{C}$ has solutions $u(x,t) = g(x)$ and $v(x,t) = p_2(x)\exp(it)$ for each $t \in \mathbb{R}$ and $x \in \mathbb{R} \times \mathbb{C}$ determining these f-flows for any continuous g.

Example 3.7. For a continuous function $f : X \longrightarrow X$, let (X, π_f) be defined by $\pi_f(x,n) = f^{n+1}(x)$ on $X \times \mathbb{Z}^+$ or defined on $X \times \mathbb{Z}$ whenever f is an embedding. These discrete f-semiflows and f-flows are just translations of discrete semiflows and flows as long as $f(X) = X$. With $f(X) \subset X$, we have $\pi_f(X \times \mathbb{Z}^+) \subset f(X)$. (Whyburn considered some dynamical system properties for continuous f with $f(X) \subset X$ in [5, p. 239].) When a discrete f-flow or f-semiflow is defined, then it always contains flows like those of this example. Let $g(x) = \pi_f(x, 1)$. Then $g^n(x) = \pi_f(f^{n-1}(x), n)$ and $\pi_g(x,n) = g^{n+1}(x)$ is contained in the f-flow or f-semiflow. Also, one might let $g_k(x) = \pi_f(x,k)$ for fixed k in $\mathbb{Z}$ or in $\mathbb{Z}^+$. Then $g_k^n(x) = \pi_f\big(f^{n-1}(x), nk\big)$ for each $n \in \mathbb{Z}$ or each $n \in \mathbb{Z}^+$ and $\pi_{g_k}(x,n) = g_k^{n+1}(x)$ is contained in the f-flow or f-semiflow.

4. f-SOLUTIONS AND f-TRANSITIONS

Theorem 4.10, our major theorem in this section, classifies f-flows in terms of the periods of their points. In order to classify f-flows, we obtain two classes of mappings by fixing the first and second coordinates of the mapping π_f. Some basic properties of these classes will be obtained prior to proving Theorem 4.10.

First, we shall consider the f-transition mappings defined by fixing the second coordinate of π_f.

Definition 4.1. Let $t \in \mathbb{R}$ be fixed and let $\pi_f(x,t)$ be denoted by $\pi_f^t(x)$ for each $x \in X$. Then the map $\pi_f^t : X \longrightarrow X$ is called the f-transition relative to t.

Proposition 4.2. *The set $\mathcal{T}_f = \{\pi_f^t : t \in \mathbb{R}\}$ of f-transitions forms an abelian group under the operation $\odot$ defined by $\pi_f^t \odot \pi_f^s = \pi_f^{t+s}$ for each $t, s \in \mathbb{R}$. If f is bijective (injective), then the collection $\mathcal{T}_f$ consists of bijective mappings on X (bijective mappings of X on $f^2(X)$.)*

Proof: First, since f is injective, if $\pi_f^t(x) = \pi_f^t(y)$ for some $x, y \in X$ and $t \in \mathbb{R}$, then

$$\begin{array}{rcl} \pi_f(x,t) & = & \pi_f(y,t) \\ \pi_f\big(\pi_f(x,t),-t\big) & = & \pi_f\big(\pi_f(y,t),-t\big) \\ \pi_f\big(f(x),0\big) & = & \pi_f\big(f(y),0\big) \\ f^2(x) & = & f^2(y) \\ x & = & y \end{array}$$

yields π_f^t injective. Moreover, the f-transition π_f^t is surjective when f is surjective, because $\pi_f^t\Big(\pi_f^{-t}\big(f^{-2}(x)\big)\Big) = \pi_f^t\Big(\pi_f\big(f^{-2}(x),-t\big)\Big) = \pi_f\Big(\pi_f\big(f^{-2}(x),-t\big),t\Big) = \pi_f\Big(f\big(f^{-2}(x)\big),t-t\Big) = \pi_f\big(f^{-1}(x),0\big) = f\big(f^{-1}(x)\big) = x$ for each $x \in X$ means $\pi_f^t(X) = X$. If f is injective, then the equation array above holds for $x \in f^2(X)$ so that π_f^t maps X onto $f^2(X)$.

Finally, $\pi_f^t \odot \pi_f^s = \pi_f^s \odot \pi_f^t$ and $\pi_f^t \odot (\pi_f^s \odot \pi_f^r) = (\pi_f^t \odot \pi_f^s) \odot \pi_f^r$ for each $t, s, r \in \mathbb{R}$ are evident from the definition of $\odot$. The identity for $\mathcal{T}_f$ is obviously $\pi_f^0 = f$ and the inverse of π_f^t for each $t \in \mathbb{R}$ is π_f^{-t}. □

Corollary 4.3. *If f is an embedding of X, then π_f^t is an embedding of X onto $f^2(X)$ for each t in $\mathbb{R}$ with inverse $f^{-2} \circ \pi_f^{-t}$.*

Next, we shall consider the f-solution mappings of π_f and some of their properties.

Definition 4.4. Let $x \in X$ be fixed and let $\pi_f(x,t)$ be denoted by ${}_x\pi_f(t)$ for each $t \in \mathbb{R}$. Then the map ${}_x\pi_f : \mathbb{R} \longrightarrow X$ is called the f-solution relative to x.

Definition 4.5. A point $x \in X$ (the f-solution ${}_x\pi_f$) is called f-periodic with f-period t if and only if ${}_x\pi_f(t) = f(x)$. The set of f-periods of x (of ${}_x\pi_f$) is denoted by $\mathcal{P}_f^x$. Whenever $\mathcal{P}_f^x = \mathbb{R}$, we shall refer to the point x (the f-solution ${}_x\pi_f$) as f-critical.

An f-periodic f-solution ${}_x\pi_f$ need not be periodic since the former condition is defined only for $0 \in \mathbb{R}$, while the latter is defined for all elements of $\mathbb{R}$.

Proposition 4.6. *An f-solution ${}_x\pi_f$ for $x \in X$ is f-periodic if it is periodic. The converse holds whenever f is injective.*

Proof: If ${}_x\pi_f(r+t) = {}_x\pi_f(r)$ for each $r \in \mathbb{R}$ and some $x \in X$, then it is certainly true for $r = 0$. Conversely, let t be an f-period of an f-solution ${}_x\pi_f$ for $x \in X$. Then

$$\begin{aligned}
{}_x\pi_f(t) &= f(x) \\
\pi_f(x,t) &= \pi_f(x,0) \\
\pi_f\big(\pi_f(x,t),r\big) &= \pi_f\big(\pi_f(x,0),r\big) \\
\pi_f\big(f(x),r+t\big) &= \pi_f\big(f(x),r\big) \\
f\big(\pi_f(x,r+t)\big) &= f\big(\pi_f(x,r)\big) \\
\pi_f(x,r+t) &= \pi_f(x,r) \\
{}_x\pi_f(r+t) &= {}_x\pi_f(r)
\end{aligned}$$

for each $r \in \mathbb{R}$, and hence, ${}_x\pi_f$ is periodic. □

The following proposition classifies the f-solutions into three distinct categories which will ultimately classify orbits for such f in the next section.

Proposition 4.7. *If f is injective, then each f-solution ${}_x\pi_f$ is either an injection, constant, or periodic.*

Proof: Suppose $({}_x\pi_f)(t) = ({}_x\pi_f)(s)$ for $t \neq s$. Then

$$\begin{aligned}
\pi_f(x,t) &= \pi_f(x,s) \\
\pi_f\big(\pi_f(x,t),-t\big) &= \pi_f\big(\pi_f(x,s),-t\big) \\
\pi_f\big(f(x),0\big) &= \pi_f\big(f(x),s-t\big) \\
f^2(x) &= f\big(\pi_f(x,s-t)\big) \\
f(x) &= \pi_f(x,s-t)
\end{aligned}$$

and $s-t$ is an f-period of ${}_x\pi_f$. Thus, ${}_x\pi_f$ is either periodic or else it is constant. □

Proposition 4.8. *For each $x \in X$, $\mathcal{P}_f^{f^n(x)}$ is a topologically closed subset of $\mathcal{P}_f^{f^{n+1}(x)}$ for each integer $n \geq 0$ and $\mathbb{P}_f^x = \cup_{n=0}^{\infty}\mathcal{P}_f^{f^n(x)}$ is an additive group. If f is bijective (injective), then $\mathcal{P}_f^x = \mathcal{P}_f^{f^n(x)}$ for each $n \in \mathbb{Z}$ ($n \in \mathbb{Z}^+$) and $\mathcal{P}_f^x = \mathbb{P}_f^x$ is a topologically closed additive group.*

Proof: To see that $\mathcal{P}_f^{f^n(x)}$ is topologically closed for each integer $n \geq 0$, let (t_j) be a net in $\mathcal{P}_f^{f^n(x)}$ such that $t_j \to t$. Then we have $f^{n+1}(x) = \pi_f\big(f^n(x), t_j\big) \to \pi_f\big(f^n(x), t\big)$ so that $f^{n+1}(x) = \pi_f(f^n(x), t)$, and hence, $t \in \mathcal{P}_f^{f^n(x)}$.

Next, for $t \in \mathcal{P}_f^{f^n(x)}$, we have $t \in \mathcal{P}_f^{f^{n+1}(x)}$ since $\pi_f\big(f^{n+1}(x), t\big) = f\big(\pi_f(f^n(x), t)\big) = f\big(f^{n+1}(x)\big) = f^{n+2}(x)$, and hence, $\mathcal{P}_f^{f^n(x)} \subset \mathcal{P}_f^{f^{n+1}(x)}$. Whenever f is bijective (injective), the preceding argument holds for each $n \in \mathbb{Z}$ $(n \geq 0)$; moreover, for $t \in \mathcal{P}_f^{f^{n+1}(x)}$, we have $\pi_f\big(f^n(x), t\big) = f^{-1}\Big(\pi_f\big(f^{n+1}(x), t\big)\Big) = f^{-1}\big(f^{n+2}(x)\big) = f^{n+1}(x)$, and thus, $t \in \mathcal{P}_f^{f^n(x)}$ for each $n \in \mathbb{Z}$ $(n \geq 0)$, i.e., $\mathcal{P}_f^{f^n(x)} = \mathcal{P}_f^{f^{n+1}(x)}$ for each $n \in \mathbb{Z}$ $(n \geq 0)$.

Finally, in order to show that $\mathbb{P}_f^x$ is a group, at least $0 \in \mathcal{P}_f^x$ so that $\mathbb{P}_f^x \neq \emptyset$. For $t, s \in \mathbb{P}_f^x$ there exist integers $n \geq 0$ and $m \geq 0$ such that $t \in \mathcal{P}_f^{f^n(x)}$ and $s \in \mathcal{P}_f^{f^m(x)}$, and hence, $t, s \in \mathcal{P}_f^{f^k(x)}$ where $k = \max\{n, m\}$. Thus, we have $\pi_f\big(f^{k+1}(x), t - s\big) = \pi_f\Big(\pi_f\big(f^k(x), t\big), -s\Big) = \pi_f\Big(\pi_f\big(f^k(x), s\big), -s\Big) = \pi_f\big(f^{k+1}(x), s - s\big) = \pi_f\big(f^{k+1}(x), 0\big) = f^{k+2}(x)$. Therefore, $t - s \in \mathbb{P}_f^x$, and consequently, $\mathbb{P}_f^x$ is a group. □

Corollary 4.9. *If $t \in \mathcal{P}_f^x$, then $nt \in \mathcal{P}_f^{f^n(x)}$ for each integer $n \geq 0$. For f injective, $nt \in \mathcal{P}_f^x$ whenever $t \in \mathcal{P}_f^x$ and $n \in \mathbb{Z}$.*

The following theorem completely classifies (X, π_f) in terms of the periods of its points when f is injective.

Theorem 4.10. *If f is injective, then $\mathcal{P}_f^x$ satisfies one of the following for each $x \in X$.*

(1) $\mathcal{P}_f^x = \{0\}$,
(2) $\mathcal{P}_f^x$ *is infinite cyclic, or*
(3) $\mathcal{P}_f^x = \mathbb{R}$.

Proof: The topologically closed subgroups of $\mathbb{R}$ are of these three types. □

5. f-ORBITS

In this section, we shall consider three subsets of the ranges of f-solutions (or f-orbits and semi-f-orbits) and some of their associated properties. Theorem 5.12, the major theorem of this section, classifies these f-orbits and semi-f-orbits into three distinct categories.

Definition 5.1. For each $x \in X$, we define and denote the f-orbit, positive f-orbit, and negative f-orbit of x by

(1) $C_f(x) = \{\pi_f(x,t) : t \in \mathbb{R}\}$,
(2) $C_f^+(x) = \{\pi_f(x,t) : t \in \mathbb{R}^+\}$, and
(3) $C_f^-(x) = \{\pi_f(x,t) : t \in \mathbb{R}^-\}$, respectively.

Also, $C_f(x)$ and x are called f-regular, f-critical (f-fixed), and f-periodic whenever ${}_x\pi_f$ is injective, constant, and nonconstant f-periodic, respectively. We denote the sets of all f-regular, f-critical, and f-periodic points of X by $\mathcal{R}_f$, $\mathcal{S}_f$, and $\mathcal{P}_f$, respectively.

We shall adopt the following notations. If π is induced by π_f, we shall use $C(x)$, $C^+(x)$, and $C^-(x)$ to denote the orbit and semi-orbits of x. Also, for each subset M of X, we shall denote $\cup_{x\in M} C_f(x)$, $\cup_{x\in M} C_f^+(x)$, and $\cup_{x\in M} C_f^-(x)$ by $C_f(M)$, $C_f^+(M)$, and $C_f^-(M)$, respectively.

If X is noncompact locally compact, then the positive f^*-orbit $C^{*+}(x)$ of x in X is

$$C_{f^*}^{*\,+}(x) = \begin{cases} C_f^+(x) & \text{if } x \in X \\ f^*(\infty) & \text{if } x = \infty. \end{cases}$$

The dual and bilateral extended orbits are similarly defined.

The first proposition examines invariance of $\mathcal{S}_f$, $\mathcal{P}_f$, and $\mathcal{R}_f$ with respect to f.

Proposition 5.2. *$f(\mathcal{S}_f) \subset \mathcal{S}_f$ and $f(\mathcal{P}_f) \subset \mathcal{P}_f \cup \mathcal{S}_f$. If f is bijective (injective), then $f(\mathcal{S}_f) = \mathcal{S}_f$, $f(\mathcal{P}_f) = \mathcal{P}_f$ $(f(\mathcal{P}_f) \subset \mathcal{P}_f)$, and $f(\mathcal{R}_f) = \mathcal{R}_f$.*

Proof: Let $x \in \mathcal{S}_f$. Then by Proposition 4.8, we have $\mathbb{R} = \mathcal{P}_f^x \subset \mathcal{P}_f^{f(x)}$, and hence, $f(x) \in \mathcal{S}_f$, i.e., $f(\mathcal{S}_f) \subset \mathcal{S}_f$. Moreover, if f is an injection, $\mathbb{R} = \mathcal{P}_f^x = \mathcal{P}_f^{f(x)}$ and $x \in \mathcal{S}_f$ if and only if $f(x) \in \mathcal{S}_f$, and consequently, $f(\mathcal{S}_f) = \mathcal{S}_f$ for f a bijection because $x = f\big(f^{-1}(x)\big) \in f(\mathcal{S}_f)$ and $\mathcal{P}_f^x \subset \mathcal{P}_f^{f^{-1}(x)}$.

Similarly, let $x \in \mathcal{P}_f$. We have $\mathcal{P}_f^x \subset \mathcal{P}_f^{f(x)} \subset \mathbb{R}$, so that $f(x) \in \mathcal{P}_f \cup \mathcal{S}_f$, i.e., $f(\mathcal{P}_f) \subset \mathcal{P}_f \cup \mathcal{S}_f$. Again, if f is an injection, $\mathcal{P}_f^x = \mathcal{P}_f^{f(x)}$ so that $x \in \mathcal{P}_f$ if and only if $f(x) \in \mathcal{P}_f$. Hence, $f(\mathcal{P}_f) \subset \mathcal{P}_f$, and moreover, again $f(\mathcal{P}_f) = \mathcal{P}_f$ for f a bijection.

Finally, let $x \in \mathcal{R}_f$. We have $\{0\} = \mathcal{P}_f^x \subset \mathcal{P}_f^{f(x)}$ so that $f(x)$ might be in $\mathcal{S}_f$, $\mathcal{P}_f$, or $\mathcal{R}_f$. However, when f is an injection, $\{0\} = \mathcal{P}_f^x = \mathcal{P}_f^{f(x)}$, implying that $x \in \mathcal{R}_f$ if and only if $f(x) \in \mathcal{R}_f$, and therefore, once more, $f(\mathcal{R}_f) = \mathcal{R}_f$ for f, a bijection. □

The next theorem and its corollaries examine the closedness of the f-critical set $\mathcal{S}_f$ and the subsets of the f-periodic set $\mathcal{P}_f$.

Theorem 5.3. *Let f be injective and let $\big((x_j, t_j)\big)$ be a net in $X \times \mathbb{R}$ with $t_j \in \mathcal{P}_f^{x_j}$ for each j. If $x_j \to x$ and $t_j \to 0$, then $x \in \mathcal{S}_f$.*

Proof: Let nets $x_j \to x$ and $t_j \to 0$ where each t_j is a period for x_j. We wish to show that $x \in \mathcal{S}_f$. Assume the contrary and let $\pi_f(x, \tau) \neq \pi_f(x, 0)$ for some $\tau \neq 0$. Choose disjoint $U \in \eta\big(\pi_f(x,\tau)\big)$ and $V \in \eta\big(\pi_f(x, 0)\big)$. Then $\pi_f^{-1}(V) \in \eta\big((x,0)\big)$ in $X \times \mathbb{R}$. There exists $W \in \eta(x)$ and an $\epsilon > 0$ such that $\pi_f\big(W \times (-\epsilon, \epsilon)\big) \subset V$. Thus, $\pi_f(x_j, t) \in V$ for $j \geq j_o$ for some j_o and each $t \in (-\epsilon, \epsilon)$. Note that, according to Corollary 4.9, $\pi_f(x_j, t) = \pi_f(x_j, t - nt_j)$ for all $z \in \mathbb{Z}$ so that $\pi_f(x_j, t) \in V$ whenever $x_j \in W$ and $|t - nt_j| < \epsilon$. For any $t \in \mathbb{R}$, there is a j_o' such that $|t_j| < \epsilon$ for $j \geq j_o'$ and we can choose an integer $n_j > 0$ where $|t - n_j t_j| < \epsilon$ for $j \geq j_o'$. Hence, $C_f(x_j)$ is in V. On the other hand, $\pi_f(x_j, \tau) \to \pi_f(x, \tau)$ in U which is disjoint from V. But we have $\pi_f(x, \tau) \in U \cap V$. Hence, $\pi_f(x, 0) = \pi_f(x, \tau)$ for each $\tau \in \mathbb{R}$. □

Corollary 5.4. *If f is injective, then $\mathcal{S}_f$ is topologically closed.*

Corollary 5.5. *If f is injective, then the set $\mathcal{P}_\tau$ of f-periodic or f-critical points of (X, π_f) with periods less than or equal to a fixed non-negative real number τ is topologically closed.*

Proof: Let (x_j) be a net in $\mathcal{P}_\tau$ such that $x_j \to x$. Let τ_j be a period of x_j such that $0 < \tau_j \leq \tau$ for each j. Some subnet of (τ_j) converges to a $\tau_o \leq \tau$ so we may assume (x_j) and (τ_j) are chosen such that $x_j \to x$ and $\tau_j \to \tau_o$. Either $\tau_o = 0$ or $\tau_o > 0$. If $\tau_o = 0$, then $x \in \mathcal{S}_f$. If $\tau_o > 0$, then $\pi_f(x_j, \tau_j) = f(x_j) \to f(x)$ and $\pi_f(x_j, \tau_i) \to \pi_f(x, \tau_o)$, and hence, $f(x) = \pi_f(x, \tau_o)$. Therefore, $x \in \mathcal{P}_\tau$. □

The following proposition, which is a result of Proposition 2.2 and the definitions, sheds light on f-orbits in general and under special conditions. It is useful in the proof to note that $f^2(y) = \pi_f\big(f(y), 0\big) = \pi_f\big(f(y), t-t\big) = \pi_f\big(\pi_f(y,t), -t\big) = \pi_f(x, -t)$ if and only if $\pi_f(y,t) = x$.

Proposition 5.6. *For each $x, y \in X$ and $t, s \in \mathbb{R}$, we have the following and their duals.*

(1) $f\big(C_f(x)\big) = C_f\big(C_f(x)\big) = C_f\big(f(x)\big) = C_f^+\big(C_f^-(x)\big) = C_f^+\big(C_f(x)\big) = C_f\big(C_f^+(x)\big)$.
(2) $f\big(C_f^+(x)\big) = C_f^+\big(C_f^+(x)\big) = C_f^+\big(f(x)\big)$.
(3) $\pi_f\big(C_f(x), t\big) = C_f\big(\pi_f(x,t)\big)$.
(4) $\pi_f\big(C_f^+(x), t\big) = C_f^+\big(\pi_f(x,t)\big)$.
(5) $C_f\big(\pi_f(x,t)\big) = C_f\big(\pi_f(x,s)\big) = C_f\big(f(x)\big)$.
(6) $C_f^+\big(\pi_f(x,t)\big) \subset C_f^+\big(\pi_f(x,s)\big)$ *for* $s \le t$.
(7) $C_f(x) = C_f^+(x) \cup C_f^-(x)$.
(8) $x \in C_f(y)$ *implies* $f^2(y) \in C_f(x)$, *or* $y \in C_f\big(f^{-2}(x)\big)$ *when* f *is injective.*
(9) $x \in C_f^+(y)$ *implies* $f^2(y) \in C_f^-(x)$, *or* $y \in C_f^-\big(f^{-2}(x)\big)$ *when* f *is injective.*
(10) $x \in C_f^+(y)$ *if and only if* $y \in C_f^-\big(f^{-2}(x)\big)$ *when* f *is injective.*
(11) $C_f(x)$ *(*$C_f^+(x)$*) is connected.*
(12) $f^n\big(C_f(x)\big) = C_f\big(f^n(x)\big)$ *for each integer* $n \in \mathbb{Z}^+$ *(*$n \in \mathbb{Z}$ *if* f *is bijective or else injective and* $x \in f^2(X)$*).*
(13) $f^n\big(C_f^+(x)\big) = C_f^+\big(f^n(x)\big)$ *for each integer* $n \in \mathbb{Z}^+$ *(*$n \in \mathbb{Z}$ *if* f *is bijective or else injective and* $x \in f^2(X)$*).*
(14) $C(x) = C_f\big(f^{-1}(x)\big)$ *and* $C_f(x) = C\big(f(x)\big)$ *whenever one of* π *or* π_f *induces the other.*
(15) $C^+(x) = C_f^+\big(f^{-1}(x)\big)$ *and* $C_f^+(x) = C^+\big(f(x)\big)$ *whenever one of* π *or* π_f *induces the other.*

Proof: For (1), we have $f\big(C_f(x)\big) \subset C_f\big(C_f(x)\big)$ because $f(y) \in C_f(y)$ for each $y \in C_f(x)$. For $y \in C_f\big(C_f(x)\big)$, we have, for some $t, s \in \mathbb{R}$, $y = \pi_f\big(\pi_f(x,t), s\big) = \pi_f\big(f(x), t+s\big) \in C_f\big(f(x)\big)$, and hence, $C_f\big(C_f(x)\big) \subset C_f\big(f(x)\big)$. Moreover, if $t+s<0$, then $y = \pi_f\big(f(x), t+s\big) = \pi_f\big(\pi_f(x,0), t+s\big) \in C_f^+\big(C_f^-(x)\big)$, and if $t+s \geq 0$, then $y = \pi_f\big(f(x), t+s\big) = \pi_f\big(\pi_f(x, t+s), 0\big) \in C_f^+\big(C_f^-(x)\big)$. Thus, we have $C_f\big(f(x)\big) \subset C_f^+\big(C_f^-(x)\big)$. Evidently, we have $C_f^+\big(C_f^-(x)\big) \subset C_f^+\big(C_f(x)\big)$. Next, let $y \in C_f^+\big(C_f(x)\big)$ and choose $T > 0$. Then, for some $t \in \mathbb{R}^+$ and $s \in \mathbb{R}$, $y = \pi_f\big(\pi_f(x,t), s\big) = \pi_f\big(\pi_f(x, t + s - T), T\big) \in C_f\big(C_f^+(x)\big)$. Finally, let $y \in C_f\big(C_f^+(x)\big)$ and $y = \pi_f\big(\pi_f(x,t), s\big)$ where $t \in \mathbb{R}^+$ and $s \in \mathbb{R}$. Then $y = \pi_f\big(f(x), t+s\big) = f\big(\pi_f(x, t+s)\big) \in f\big(C_f(x)\big)$, and hence, $C_f\big(C_f^+(x)\big) \subset f\big(C_f(x)\big)$ completing the proof of (1).

From $\pi_f\big(\pi_f(x,s), t\big) = \pi_f\big(\pi_f(x,t), s\big) = \pi_f\big(f(x), t+s\big)$, (3), (4), and (5) follow.

To see that (6) holds for $s \leq t$, we need only choose $T \geq t - s$ in $\pi_f\big(\pi_f(x,t), s-t+T\big) = \pi_f\big(f(x), s+T\big) = \pi_f\big(\pi_f(x,s), T\big)$.

Statement (7) is evident.

To verify (8), let $x = \pi_f(y,t) \in C_f(y)$. Then $f^2(y) = \pi_f\big(f(y), 0\big) = \pi_f\big(f(y), t-t\big) = \pi_f\big(\pi_f(y,t), -t\big) = \pi_f(x, -t) \in C_f(x)$. In addition, this means that $x \in C_f^+(y)$ implies $f^2(y) \in C_f^-(x)$. Also, whenever f is injective, we have $y = f^{-2}\big(f^2(y)\big) = f^{-2}\big(\pi_f(x,-t)\big) = \pi_f\big(f^{-2}(x), -t\big) \in C_f\big(f^{-2}(x)\big)$. If $t \geq 0$, (9) also follows.

Statement (10) follows from (9) and its dual since $y \in C_f^-\big(f^{-2}(x)\big)$ implies $x = f^2\big(f^{-2}(y)\big) \in C_f^+(y)$.

Statement (11) is a result of the continuity of π_f and the connectedness of $\mathbb{R}$ ($\mathbb{R}^+$).

Using induction on $f\big(C_f(x)\big) = C_f\big(f(x)\big)$ of (1) yields (12), and similarly, (2) yields (13). Whenever f is bijective, (12) and (13) follow for $n \in \mathbb{Z}$ from Proposition 2.2.

Statements (14) and (15) follow from the equations $\pi = \pi_f \circ (f^{-1} \times i_{\mathbb{R}})$ and $\pi_f = \pi \circ (f \times i_{\mathbb{R}})$ of corollaries 2.3 and 2.4.

The duals follow in a similar manner completing the proof. $\square$

When two f-orbits intersect, they need not be the same f-orbit but the f-orbits of their images are equal, or equivalently, the images of the f-orbits under f are the same set. However, if f is injective, the f-orbits partition the phase space.

Proposition 5.7. *If $C_f(x) \cap C_f(y) \neq \emptyset$ for some $x, y \in X$, then $C_f\big(f(x)\big) = C_f\big(f(y)\big)$. Whenever f is injective, $C_f(x) = C_f(y)$.*

Proof: Let $C_f(x) \cap C_f(y) \neq \emptyset$ for x and y in X. Then $\pi_f(x,t) = \pi_f(y,s)$ for some $t, s \in \mathbb{R}$, and hence, $C_f\big(f(x)\big) = C_f\big(\pi_f(x,t)\big) = C_f\big(\pi_f(y,s)\big) = C_f\big(f(y)\big)$ by Proposition 5.6. Also, $f\big(C_f(x)\big) = f\big(C_f(y)\big)$, so that, if f is an injection, $C_f(x) = C_f(y)$. □

The next proposition characterizes the f-critical points for f injective.

Proposition 5.8. *If f is injective, the following and their duals are equivalent for each $x \in X$.*

(1) $x \in \mathcal{S}_f$.
(2) $C_f(x) = \{f(x)\}$.
(3) $C_f^+(x) = \{f(x)\}$.
(4) $\pi_f(x, [a,b]) = \{f(x)\}$ *for some* $[a,b] \subset \mathbb{R}$, $a < b$.
(5) *Every neighborhood of $f(x)$ contains an f-orbit.*
(6) *Every neighborhood of $f(x)$ contains a positive f-orbit.*
(7) *There is an $\epsilon > 0$ such that, for every neighborhood U of x, there is a y in U such that $\pi_f(y, [0,\epsilon]) \subset f(U)$.*

Proof: Obviously, (1) and (2) are equivalent. The equivalence of (2), (3), and (4) follows from Proposition 4.8 since $\mathcal{P}_f^x$ is an additive group, and we have $\mathbb{R} = \mathcal{P}_f^x$, $\mathbb{R}^+ \subset \mathcal{P}_f^x$, $[a,b] \subset \mathcal{P}_f^x$, respectively. Obviously, (2) implies (5); (5) implies (6); and (6) implies (3) and (7).

To complete the proof, we show that (7) implies (4). For $x \in X$, choose $V \in \eta\big(f(x)\big)$ and $U \in \eta(x)$ such that $f(U) \subset V$. There exists an $\epsilon > 0$ for which $\pi_f(y,[0,\epsilon]) \subset f(U) \subset V$ for some $y \in U$. We can choose a net (x_i) converging to the point x such that $\pi_f(x_i,[0,\epsilon]) \subset f(U) \subset V$ for each i. Thus, $\pi_f(x_i,[0,\epsilon]) \to \pi_f(x,[0,\epsilon])$ and $\pi_f(x,[0,\epsilon]) \subset V$. Since ϵ depends on x but not on U, (4) follows from $\pi_f(x,[0,\epsilon]) \subset \cap\{V | V \in \eta\big(f(x)\big)\} = \{f(x)\}$. The duals follow in a similar manner, completing the proof. □

We now characterize the non-f-regular points whenever f is injective.

Proposition 5.9. *If f is injective, the following and their duals are equivalent for each $x \in X$.*

(1) $x \in \mathcal{P}_f \cup \mathcal{S}_f$.
(2) $C_f(x) = \pi_f(x, [0, t_o])$ *for some* $t_o > 0$.
(3) $C_f^+(x) = \pi_f(x, [0, t_o])$ *for some* $t_o > 0$.
(4) $C_f(x) = C_f^+(x)$.

Proof: Let (1) hold. If $x \in \mathcal{S}_f$, then (2), (3), and (4) follow from Proposition 5.8. Let $x \in \mathcal{P}_f$ and let $t_o \in \mathcal{P}_f^x$. Since $\pi_f(x, [0, t_o]) \subset C_f(x)$, let $y \in C_f(x)$. Then, $y \in \pi_f(x, s)$ for some $s \in \mathbb{R}$. There exists an $n \in \mathbb{Z}$ such that $0 \leq s + nt_o \leq t_o$. Thus, $y = \pi_f(x, s) = \pi_f(x, s + nt_o) \in \pi_f(x, [0, t_o])$, and hence, (2), (3), and (4) follow. Conversely, (2), (3), and (4) imply ${}_x\pi_f$ is not an injection, and hence, (1) follows by Proposition 4.7 by selecting a positive period t_o of ${}_x\pi_f$. Again, the duals follow in a similar manner, completing the proof. □

The final two theorems and their corollaries characterize all f-orbits and f-semiorbits of X when f is an embedding in terms of the compactness and homeomorphisms of the f-orbits and f-semiorbits.

Theorem 5.10. *Let f be an embedding of X. Then $x \in \mathcal{P}_f \cup \mathcal{S}_f$ if and only if $C_f^+(x)$, $C_f^-(x)$, or $C_f(x)$ is compact.*

Proof: Certainly, $x \in \mathcal{P}_f \cup \mathcal{S}_f$ implies $C_f^+(x)$, $C_f^-(x)$, and $C_f(x)$ are compact in view of Proposition 5.9. Conversely, let $C_f(x)$ be compact. Then we claim that $C_f^+(x)$ is also compact. Suppose not and let $M = C_f(x) \setminus \overline{C_f^+(x)}$ which is open in $C_f(x)$. For some point y_o in M, choose $t_o \in \mathbb{R}^-$ such that $y_o = \pi_f(x, t_o)$. There is a net (t_i) in $\mathbb{R}^+$ such that $\big(\pi_f(x, t_i)\big)$ converges to a point $y_o' \notin C_f^+(x)$. It must be the case that t_i can be chosen so that $t_i \to +\infty$ for, if (t_i) is bounded, there is a subnet (t_j) converging to a $t \geq 0$, and hence, $\pi_f(x, t_j) \to \pi_f(x, t)$ in $C_f^+(x)$. Thus, there is a $t_o' < 0$ for which $y_o' = \pi_f(x, t_o')$. Now, $\pi_f(x, t_i - t_o' + t_o) = f^{-1}\Big(\pi_f\big(f(x), t_i - t_o' + t_o\big)\Big) = f^{-1}\Big(\pi_f\big(\pi_f(x, t_i), -t_o' + t_o\big)\Big) \to$

$f^{-1}\Big(\pi_f\big(\pi_f(x,t_o'),-t_o'+t_o\big)\Big) = f^{-1}\big(\pi_f(f(x),t_o)\big) = \pi_f(x,t_o)$ and, because the net $\big(\pi_f(x,t_i-t_o'+t_o)\big)$ is ultimately in $C_f^+(x)$, we have $\pi_f(x,t_o)$ in $\overline{C_f^+(x)}$ as well as in the set M, which is absurd. Hence, $C_f^+(x)$ is compact.

Now, consider the compact set $C_f^+(x)$. Let (t_i) be a net in $\mathbb{R}$ such that $t_i \to +\infty$. The net $\big(\pi_f(x,t_i)\big)$ is ultimately in $C_f^+(x)$; therefore, a subnet $\big(\pi_f(x,t_j)\big)$ of $\big(\pi_f(x,t_i)\big)$ converges to some point $y \in C_f^+(x)$. Let $y = \pi_f(x,t_o)$ for $t_o \in \mathbb{R}^+$. Now, $\pi_f(x,t_j - 2t_o) = f^{-1}\Big(\pi_f\big(f(x),t_j - 2t_o\big)\Big) = f^{-1}\Big(\pi_f\big(\pi_f(x,t_j),-2t_o\big)\Big) \to f^{-1}\Big(\pi_f\big(\pi_f(x,t_o),-2t_o\big)\Big) = f^{-1}\big(\pi_f(f(x),-t_o)\big) = \pi_f(x,-t_o)$. The net $\big(\pi_f(x,t_j-2t_o)\big)$ is ultimately in $C_f^+(x)$ since the net (t_j-2t_o) is ultimately in $\mathbb{R}^+$, and hence, $\pi_f(x,-t_o) \in C_f^+(x)$. There is a $t_o' \in \mathbb{R}^+$ such that $\pi_f(x,t_o') = \pi_f(x,-t_o)$. Thus,

$$\begin{aligned}
\pi_f\big(\pi_f(x,t_o'),t_o\big) &= \pi_f\big(\pi_f(x,-t_o),t_o\big)\\
\pi_f\big(f(x),t_o+t_o'\big) &= \pi_f\big(f(x),0\big)\\
f\big(\pi_f(x,t_o+t_o')\big) &= f^2(x)\\
\pi_f(x,t_o+t_o') &= f(x)
\end{aligned}$$

implies x is in $\mathcal{P}_f \cup \mathcal{S}_f$. The dual follows in a similar manner, completing the proof. □

In view of Theorem 5.10 and Proposition 5.9, the f-orbit and f-semiorbits are all compact and equal whenever one of them is compact, and hence, we have the following corollary.

Corollary 5.11. *Let f be an embedding of X. Then $x \in \mathcal{R}_f$ if and only if $C_f^+(x)$, $C_f^-(x)$, or $C_f(x)$ is noncompact.*

Theorem 5.12. *Let f be an embedding of X. For each $x \in X$, $C_f^+(x)$, $C_f^-(x)$, and $C_f(x)$ are each*

(1) *the singleton $\{f(x)\}$ if and only if $x \in \mathcal{S}_f$,*
(2) *homeomorphic to S^1 if and only if $x \in \mathcal{P}_f$, and*
(3) *noncompact if and only if $x \notin \mathcal{S}_f \cup \mathcal{P}_f$.*

Proof: We have left to prove only that if $x \in \mathcal{P}_f$, then $C_f^+(x)$, $C_f^-(x)$, and $C_f(x)$ are each homeomorphic to S^1. There is a least positive real number τ such that $C_f^+(x) = \pi_f\big(x,[0,\tau)\big)$ and $\pi_f(x,t) \neq$

$\pi_f(x,s)$ for distinct $t,s \in [0,\tau)$. The mapping $h : S^1 \longrightarrow C_f^+(x)$ of the complex plane representation $\{\exp(\frac{2\pi it}{\tau}) : t \in [0,\tau)\}$ of S^1 onto $C_f^+(x)$ defined by $h\big(\exp(\frac{2\pi it}{\tau})\big) = \pi_f(x,t)$ for each $t \in [0,\tau)$ is a homeomorphism since it is a continuous bijection of a compact space onto a Hausdorff space. □

Corollary 5.13. *Let f be an embedding of X. Then the following are equivalent to $x \in \mathcal{R}_f$.*

(1) *$C_f(x)$ is homeomorphic to $\mathbb{R}$.*
(2) *$C_f^+(x)$ (or $C_f^-(x)$) is homeomorphic to $\mathbb{R}^+$.*

6. Epilogue

Research on classical flows began in the nineteenth century with Henri Poincaré's topological analysis of planer flows defined by autonomous systems of differential equations. Although f-flows have kindled the interest of the author, in the future several significant examples of f-flows should be cataloged in order to completely justify the topological study begun herein where we have also given but one category of nonclassical continuous f-flows including a class determined by systems of partial differential equations.

Properties of f-orbit closures, f-limit sets, f-prolongations, f-prolongational limit sets, generalized attraction, and generalized stability will be addressed in subsequent sequels to this paper.

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