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ON SOME QUESTIONS ABOUT POSETS OF TOPOLOGIES ON A FIXED SET

CAMILLO COSTANTINI

ABSTRACT. Letting $\Sigma_2(X)$, $\Sigma_3(X)$, and $\Sigma_{lc}(X)$ be the poset of all regular and locally compact T_2 topologies on a set X, respectively, we show that under the Continuum Hypothesis there exists a non-minimal topology in $\Sigma_2([0,1])$ which is not upper there. We also show that a pseudoradial T_2 topology and a countably compact regular topology on a set X cannot be lower in $\Sigma_2(X)$ and that there are two locally compact topologies σ, τ on a subset X of the real plane which differ at more than one point and form a gap in $\Sigma_{lc}(X)$. In addition, we show that a Tychonoff non-minimal topology on a set X must be upper in $\Sigma_3(X)$. This gives three ZFC answers, one consistent answer and a partial answer to questions raised by Ofelia T. Alas and Richard G. Wilson, [Appl. Gen. Topol. **5** (2004), no. 2, 231–242]; O. T. Alas, S. Hernández, M. Sanchis, M. G. Tkachenko, and R. G. Wilson, [Acta Math. Hungar. **112** (2006), no. 3, 199–219]; and Nathan Carlson, [Topology Appl. **154** (2007), no. 3, 619–624]. We also introduce the newly defined notions of strongly upper and strongly lower topologies and investigate some of their properties.

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Key words and phrases. compact topology, Continuum Hypothesis, countably compact topology, independent family, lattice of topologies, locally compact topology, maximal point, minimal topology, poset of topology, pseudoradial topology, regular point, regular topology, semiregularization, semiregular space, (strongly) lower topology, (strongly) upper topology, T_2 topology, Tychonoff topology.

1. Introduction

This paper refers rather directly to some recent investigations concerning lattices or posets of topologies which may be defined starting from either a set or a group. Namely, in [3], [1], and [5], the authors consider the lattice $\mathscr{L}_1(X)$ of all T_1 topologies on a set X and the subposets of $\mathscr{L}_1(X)$ consisting of all Hausdorff, regular, Tychonoff, or locally compact topologies on X; they also consider, for a fixed group G, the poset Σ_G of all Hausdorff topologies on G which make it a topological group.

Even if in the literature there is a large number of results concerning the above mentioned structures (see, for example, the survey paper [10] or the studies on mutually complementary and self-complementary topologies in [16], [17], [20], [4], [21], [22], [18], and [15]), the main subject (which traces back to [11] and [19] and has strict connections with the study of intervals in posets of topologies carried out in [14], [8], [7], and [12]) of the three articles noted before seems not to be so extensively investigated. Actually, such articles focus on the problem of the existence of gaps in the poset Σ under consideration, where a *gap* is determined by a pair of topologies $\sigma, \tau \in \Sigma$ with $\sigma \subsetneqq \tau$, such that for every $\gamma \in \Sigma$ with $\sigma \subseteq \gamma \subseteq \tau$, we must have either $\gamma = \sigma$ or $\gamma = \tau$. Also, a topology $\sigma \in \Sigma$ is said to be *lower* if there exists some $\tau \in \Sigma$ for which the above property holds, i.e., such that (σ, τ) forms a gap; and, symmetrically, a topology $\tau \in \Sigma$ will be called *upper* if there exists a topology $\sigma \in \Sigma$ such that (σ, τ) forms a gap.

Of course, such notions make sense in every poset. But their relevance in $\mathscr{L}_1(X)$, or in suitable subposets of it, consists mainly in that one can try to characterize, on the base of inner properties, the topologies which are lower or upper—or, at least, give necessary or sufficient conditions for a topology to have either property. Nearly all the results in [3], [1], and [5] suit this spirit of research, and it is remarkable to observe how various and meaningful the topological notions involved in such investigations can be.

In the present paper, we solve three questions from [3] and [1], give a consistent answer to the main question raised in [5], and also provide a partial answer to another question from [1]. Namely, we prove that

— a pseudoradial Hausdorff topology and a countably compact regular topology on a set X cannot be lower in $\mathscr{L}_1(X)$ (or, equivalently, in the poset $\Sigma_2(X)$ of all Hausdorff topologies on X), solving Question 2.12 and the second part of Question 2.6 [3];
— there exists a locally compact topology τ on the set $X = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$, strictly finer than the Euclidean topology σ on X, such that (σ,τ) forms a gap in the poset $\Sigma_{lc}(X)$ of all locally compact T_2 topologies on X, and such that σ and τ differ at each point of the set $\{(x,0) \mid x \in \mathbb{R}\}$, solving in the negative an implicit question raised in the considerations before Lemma 3.1 [1];
— under the Continuum Hypothesis, there is a Hausdorff topology τ on the unit interval $I = [0,1]$, such that τ is strictly finer than the Euclidean topology on I and τ is not upper in $\Sigma_2(I)$ (see [5, Question 4.5], where the notation TOP_2 is used instead of $\Sigma_2(X)$);
— every Tychonoff topology on a set X, which is not minimal in the poset $\Sigma_3(X)$ of all regular topologies on X, is upper in $\Sigma_3(X)$, giving a partial positive answer to Question 2.15 in [1].

Moreover, we introduce the notions of "strongly upper" and "strongly lower" topology (see Definition 4.5), and we give an example in Σ_2 of an upper topology which is not strongly upper.

In a forthcoming paper, we will show that there is a Hausdorff group topology τ on the additive group $(\mathbb{Q},+)$ of the rational numbers, such that τ is strictly finer than the Euclidean topology on $\mathbb{Q}$ and τ is not upper in $\Sigma_{(\mathbb{Q},+)}$; this will give a negative answer to Problem 4.16 in [1].

This paper was already finished when the author received a copy of [2], where a ZFC example of a non-minimal, non-upper topology in Σ_2 is produced. The construction of this example is certainly simpler than that carried out in §3 of the present paper. However, the author deems that the use of well-orderings, transfinite induction, and (κ)-independent families may have further applications for solving similar or related questions; therefore, his example may be of interest all the same.

2. Preliminary facts

Recall that a subset A of a topological space X is said to be *regular-open* if $A = \operatorname{Int}\overline{A}$. A topological space X is said to be *semiregular* if every $x \in X$ has a fundamental system of neighborhoods consisting of regular-open sets.

Observe that for every open subset Ω of a topological space X, the set $\operatorname{Int}\overline{\Omega}$ is a regular-open set and is actually the smallest regular-open set including Ω. Given an arbitrary topological space (X,τ), the association

$$x \mapsto \left\{ \operatorname{Int}\overline{V} \,\middle|\, V \in \tau,\ x \in V \right\}$$

is an assignment of a fundamental system of (open) neighborhoods for a new topology σ on X. One easily sees that $\sigma \le \tau$ and that σ is the finest semiregular topology coarser than τ.

The following two results can be considered as folklore. The former is easy to prove (see [13, 0.1(a)]); as for the latter, (which holds as well under the weaker assumption that the semiregularization of τ be minimal-Hausdorff instead of compact), we have not been able to find a precise reference in the literature, so we give it a proof for the reader's convenience.

Proposition 2.1. *If (X,τ) is a T_2 topological space and σ is the semiregularization of τ, then σ is T_2, too.*

Proposition 2.2. *If τ is a Hausdorff topology on a set X, whose semiregularization is a compact topology σ, then for every Hausdorff topology γ with $\gamma \subseteq \tau$, we have the inclusion $\sigma \subseteq \gamma$.*

Proof: Towards a contradiction, suppose there exists $A \in \sigma \setminus \gamma$; therefore, there is $\bar{x} \in A$ such that for every γ-neighborhood V of $\bar{x}$, $V \setminus A \neq \emptyset$. Then, letting $\mathscr{V}_{\bar{x}}$ to be the collection of all open γ-neighborhoods of $\bar{x}$, partially ordered by reverse inclusion, and associating to every $V \in \mathscr{V}_{\bar{x}}$ a point $y(V) \in V \setminus A$, we obtain a net $\{y(V)\}_{V\in\mathscr{V}_{\bar{x}}}$. Let $\bar{y}$ be a cluster point of such a net in (X,σ) (see, for example, [6, Theorem 3.1.23]); since $X \setminus A$ is σ-closed and $y(V) \in X \setminus A$ for every $V \in \mathscr{V}_{\bar{x}}$, $\bar{y}$ must belong in its turn to $X \setminus A$, which implies that $\bar{y} \neq \bar{x}$.

Fix two disjoint open γ-neighborhoods V_1 and V_2 of $\bar{x}$ and $\bar{y}$, respectively. Then $V_2 \in \tau$, too; hence, $\mathrm{Int}_\tau\mathrm{Cl}_\tau V_2 \in \sigma$—in particular, $\mathrm{Int}_\tau\mathrm{Cl}_\tau V_2$ is a σ-open neighborhood of $\bar{y}$.

Since $\bar{y}$ is a σ-cluster point of the net $\{y(V)\}_{V\in\mathscr{V}_{\bar{x}}}$ and $V_1 \in \mathscr{V}_{\bar{x}}$, there must exist $V_1' \in \mathscr{V}_{\bar{x}}$ with $V_1' \subseteq V_1$, such that $y(V_1') \in \mathrm{Int}_\tau\mathrm{Cl}_\tau V_2 \subseteq \mathrm{Cl}_\tau V_2$, whence $y(V_1') \in V_1' \cap \mathrm{Cl}_\tau V_2 \subseteq V_1 \cap \mathrm{Cl}_\gamma V_2 \neq \emptyset$. This is a contradiction, because $V_1, V_2 \in \gamma$ and $V_1 \cap V_2 = \emptyset$. $\square$

Let us point out another simple result which will be of use to manage topologies obtained by adding a new set to another previously assigned topology.

Proposition 2.3. *Let σ be a topology on a set X and A a subset of X (which is not in σ); set $\tau = \langle \sigma \cup \{A\} \rangle$. Then for every $x \in A$, a fundamental system of (open) τ-neighborhoods for x is given by $\{V \cap A \,|\, V \in \sigma,\, x \in V\}$.*

Proof: By the very definition of topology generated by a collection of sets, the collection $\mathscr{B}$ of all finite intersections of $\sigma \cup \{A\}$ is a base for τ. Since σ is closed under finite intersections, we have the equality $\mathscr{B} = \{V \cap A \,|\, V \in \sigma\}$. As a consequence, for every $x \in A$, the collection $\mathscr{B} = \{V \cap A \,|\, V \in \sigma,\, x \in V\}$ is a fundamental system of open τ-neighborhoods for x. □

Independent families are a basic tool for the construction which we will carry out in the next section. More precisely, we will use them in the following generalized form.

Definition 2.4. Let X be a set, ζ an infinite cardinal, and $\mathscr{S} = \{S_i\}_{i \in I}$ an indexed family of subsets of X. Then we will say that $\mathscr{S}$ is a *ζ-independent family* on X if for every pair I_1, I_2 of disjoint subsets of I, with $|I_1 \cup I_2| < \zeta$, we have the inequality $\left(\bigcap_{i \in I_1} S_i\right) \setminus \left(\bigcup_{i \in I_2} S_i\right) \neq \emptyset$.

For $\zeta = \omega$, the above definition gives rise to the common notion of independent family. The next proposition is a generalization of the well-known Fichtenholtz-Kantorovich-Hausdorff theorem (see, for example, [9, Exercise VIII,(A6)]) and may be considered as folklore. However, since it is not easy to find a precise reference for it in the literature, we provide a proof for the reader's convenience.

We are using customary set-theoretic notation. If A and B are two cardinals, then ${}^{A}B$ is the set of all functions from A to B; for two cardinals κ and ζ, κ^{ζ} is the cardinality of the set ${}^{\zeta}\kappa$ (so that, for any two sets A and B, we have the equality $\left|{}^{B}A\right| = |A|^{|B|}$). For any set A and any cardinal ζ, $[A]^{\zeta}$ and $[A]^{\leq \zeta}$ denote the collection of all subsets of A having cardinality ζ and [not greater than ζ], respectively. Let us also recall that 2, as an ordinal, coincides with the set $\{0, 1\}$.

Proposition 2.5. *If ζ is an infinite cardinal, then there is a ζ^+-independent family of cardinality 2^{2^ζ} on every set of cardinality 2^ζ.*

Proof: Of course, it will suffice to show the statement for a single set of cardinality 2^ζ.

Let $\{T_\alpha\}_{\alpha\in 2^\zeta}$ list $[2^\zeta]^{\leq\zeta}$ and, for every $\alpha \in 2^\zeta$, let $\{Y_{\alpha,\beta}\}_{\beta\in 2^\zeta}$ list $\left[{}^{T_\alpha}2\right]^{\leq\zeta}$ (not necessarily in a one-to-one way). Put

$$M = \big\{(\alpha,\beta,\varphi)\,\big|\,\alpha,\beta\in 2^\zeta,\ \varphi\in {}^{Y_{\alpha,\beta}}2\big\};$$

then, of course, M has cardinality 2^ζ. We want to associate to every $\psi \in {}^{2^\zeta}2$ a subset M_ψ of M, in such a way that $\{M_\psi\}_{\psi\in {}^{2^\zeta}2}$ is a ζ^+-independent family.

Given $\psi \in {}^{2^\zeta}2$, let

$$M_\psi = \big\{(\alpha,\beta,\varphi)\in M\,\big|\,\psi{\restriction}_{T_\alpha}\in Y_{\alpha,\beta}\ \wedge\ \varphi(\psi{\restriction}_{T_\alpha}) = 1\big\}.$$

Suppose now we have a subset L of ${}^{2^\zeta}2$ with $|L|\leq\zeta$ and a function $\vartheta\colon L\to 2$; then, for every $\{\psi_1,\psi_2\}\in[L]^2$, there is a $\beta(\psi_1,\psi_2)\in 2^\zeta$ such that $\psi_1\big(\beta(\{\psi_1,\psi_2\})\big)\neq\psi_2\big(\beta(\{\psi_1,\psi_2\})\big)$. The set $L' = \big\{\beta(\psi_1,\psi_2)\,\big|\,\{\psi_1,\psi_2\}\in[L]^2\big\}$ is a subset of 2^ζ of cardinality $\leq\zeta$; hence, $L' = T_{\hat\alpha}$ for some $\hat\alpha\in 2^\zeta$. Since $\hat\Psi = \big\{\psi{\restriction}_{T_{\hat\alpha}}\,\big|\,\psi\in L\big\}$ is an element of $[{}^{T_{\hat\alpha}}2]^{\leq\zeta}$, there exists $\hat\beta\in 2^\zeta$ such that $\hat\Psi = Y_{\hat\alpha,\hat\beta}$. We define $\hat\varphi\colon Y_{\hat\alpha,\hat\beta}\to 2$: Given any $\eta\in Y_{\hat\alpha,\hat\beta}$, by our definition of $L'\,(=T_{\hat\alpha})$, there exists a unique $\psi\in L$ with $\psi{\restriction}_{T_{\hat\alpha}} = \eta$. Then we put $\hat\varphi(\eta) = \vartheta(\psi)$.

Letting, for every $\psi\in L$,

$$M_\psi^\vartheta = \begin{cases} M_\psi & \text{if } \vartheta(\psi) = 1,\\ M\setminus M_\psi & \text{if } \vartheta(\psi) = 0,\end{cases}$$

we will prove that $(\hat\alpha,\hat\beta,\hat\varphi)\in\bigcap_{\psi\in L} M_\psi^\vartheta$. Actually, given any $\psi\in L$, we certainly have the relation $\psi{\restriction}_{T_{\hat\alpha}}\in Y_{\hat\alpha,\hat\beta}$. Therefore, if $\vartheta(\psi) = 1$, then $\hat\varphi\big(\psi{\restriction}_{T_{\hat\alpha}}\big) = 1$, so that $(\hat\alpha,\hat\beta,\hat\varphi)\in M_\psi = M_\psi^\vartheta$ by the definition of M_ψ. If, on the contrary, $\vartheta(\psi) = 0$, then $\hat\varphi\big(\psi{\restriction}_{T_{\hat\alpha}}\big) = 0$, so that $(\hat\alpha,\hat\beta,\hat\varphi)\notin M_\psi$ and hence, $(\hat\alpha,\hat\beta,\hat\varphi)\in M\setminus M_\psi = M_\psi^\vartheta$. □

Remark 2.6. Notice that if $\mathscr{S} = \{S_j\}_{j\in J}$ is a ζ-independent family on a set X for a certain infinite cardinal ζ and $|J|\geq\zeta$, then for any two disjoint $J_1, J_2 \subseteq J$ with $|J_1| < \zeta$ and $|J_2| < \zeta$, we have the inequality $\big|\big(\bigcap_{j\in J_1} S_j\big) \setminus \big(\bigcup_{j\in J_2} S_j\big)\big| \geq \zeta$.

Indeed, if we had $|S| < \zeta$, where $S = \big(\bigcap_{j\in J_1} S_j\big) \setminus \big(\bigcup_{j\in J_2} S_j\big)$, then there would exist a one-to-one $\vartheta\colon S \to J \setminus (J_1 \cup J_2)$. Thus, letting

$$J_1' = \big\{\vartheta(x) \,\big|\, x \in S,\, x \notin S_{\vartheta(x)}\big\} \text{ and } J_2' = \big\{\vartheta(x) \,\big|\, x \in S,\, x \in S_{\vartheta(x)}\big\},$$

we would easily obtain the equality $S \cap \big(\big(\bigcap_{j\in J_1'} S_j\big) \setminus \big(\bigcup_{j\in J_2'} S_j\big)\big) = \emptyset$, i.e., $\big(\bigcap_{j\in J_1\cup J_1'} S_j\big) \setminus \big(\bigcup_{j\in J_2\cup J_2'} S_j\big) = \emptyset$. This is a contradiction, as $|J_1 \cup J_1'| < \zeta$, $|J_2 \cup J_2'| < \zeta$ and $(J_1 \cup J_1') \cap (J_2 \cup J_2') = \emptyset$.

Our next result shows how it is possible to obtain, on some topological spaces, families of subsets which turn out to be ζ-independent families (for a fixed, suitable infinite cardinal ζ), once restricted to any nonempty open subset of the space.

Theorem 2.7. *Let X be a topological space and suppose that there is an infinite cardinal ζ such that $|A| \geq 2^\zeta$ for every nonempty open subset A of X. Suppose also that $\pi w(X) \leq 2^\zeta$. Then there exists a (faithfully indexed) family $\big\{S_\delta \,\big|\, \delta \in 2^{2^\zeta}\big\}$ of subsets of X such that for every nonempty open subset A of X, the family*

$$\big\{S_\delta \cap A \,\big|\, \delta \in 2^{2^\zeta}\big\}$$

is ζ^+-independent.

Proof: Fix a map j from 2^ζ onto 2^ζ, such that for every $\beta \in 2^\zeta$ the set $j^{-1}(\beta) = \{\alpha \in 2^\zeta \,|\, j(\alpha) = \beta\}$ has cardinality 2^ζ; fix also a π-base $\{B_\beta \,|\, \beta \in 2^\zeta\}$ of X (with $\beta \mapsto B_\beta$ *not necessarily* one-to-one), and for every $\beta \in 2^\zeta$, let $\{x_{\beta,\gamma} \,|\, \gamma \in 2^\zeta\}$ be a subset of B_β (with $\gamma \mapsto x_{\beta,\gamma}$ one-to-one). By transfinite induction, define $\gamma(\alpha)$ for every $\alpha \in 2^\zeta$ by

$$\gamma(\alpha) = \min\big\{\gamma' \in 2^\zeta \,\big|\, x_{j(\alpha),\gamma'} \notin \{x_{j(\alpha'),\gamma(\alpha')} \,|\, \alpha' < \alpha\}\big\}. \tag{2.1}$$

(Of course, the above definition is correct as $\gamma' \mapsto x_{j(\alpha),\gamma'}$ is one-to-one for every fixed $\alpha \in 2^\zeta$.) Thus, letting $y_\alpha = x_{j(\alpha),\gamma(\alpha)}$ for every $\alpha \in 2^\zeta$, we easily see by (2.1) that $\alpha \mapsto y_\alpha$ is one-to-one. For every $\beta \in 2^\zeta$, set

$$M_\beta = \{y_\alpha \,|\, \alpha \in 2^\zeta \;\wedge\; j(\alpha) = \beta\};$$

then, due to the definition and properties of y_α and the way we have chosen j, we see that the following hold.

(1) $\forall\beta \in 2^\zeta\colon M_\beta \subseteq B_\beta$;
(2) $\forall\beta \in 2^\zeta\colon |M_\beta| = 2^\zeta$;
(3) $\forall\beta,\beta' \in 2^\zeta\colon (\beta \neq \beta' \Longrightarrow M_\beta \cap M_{\beta'} = \emptyset)$.

Now, by (2) and Proposition 2.5, for every $\beta \in 2^\zeta$, there exists a ζ^+-independent family $\left\{T_{\beta,\delta} \,\middle|\, \delta \in 2^{2^\zeta}\right\}$ on M_β. Setting

$$S_\delta = \bigcup_{\beta\in 2^\zeta} T_{\beta,\delta}$$

for every $\delta \in 2^{2^\zeta}$, we claim that the family $\left\{S_\delta \,\middle|\, \delta \in 2^{2^\zeta}\right\}$ has the property required by the statement. Indeed, let A be a nonempty open subset of X; then there exists a $\hat\beta \in 2^\zeta$ such that $B_{\hat\beta} \subseteq A$. We want to prove that $\left\{S_\delta \cap A \,\middle|\, \delta \in 2^{2^\zeta}\right\}$ is ζ^+-independent on A, i.e., that for any two disjoint $I_1, I_2 \subseteq 2^{2^\zeta}$ with $|I_1| \leq \zeta$ and $|I_2| \leq \zeta$, the inequality

$$\textstyle\bigcap_{\delta\in I_1}(S_\delta \cap A) \setminus \bigcup_{\delta\in I_2}(S_\delta\cap A) = A \cap \Big(\big(\bigcap_{\delta\in I_1} S_\delta\big) \setminus \big(\bigcup_{\delta\in I_2} S_\delta\big)\Big) \neq \emptyset$$

holds; and, of course, this will follow as a consequence if we can show that

$$M_{\hat\beta} \cap \Big(\big(\textstyle\bigcap_{\delta\in I_1} S_\delta\big) \setminus \big(\bigcup_{\delta\in I_2} S_\delta\big)\Big) \neq \emptyset \tag{2.2}$$

(taking (1) into account). Actually, if I_1 and I_2 are as above, then consider that

$$\bigcap_{\delta\in I_1} S_\delta = \bigcap_{\delta\in I_1}\bigcup_{\beta\in 2^\zeta} T_{\beta,\delta} \supseteq \bigcap_{\delta\in I_1} T_{\hat\beta,\delta}$$

and that

$$\begin{aligned}\bigcup_{\delta\in I_2} S_\delta = \bigcup_{\delta\in I_2}\bigcup_{\beta\in 2^\zeta} T_{\beta,\delta} &= \left(\bigcup_{\delta\in I_2} T_{\hat\beta,\delta}\right) \cup \left(\bigcup_{\delta\in I_2}\bigcup_{\beta\in 2^\zeta\setminus\{\hat\beta\}} T_{\beta,\delta}\right)\\ &= \left(\bigcup_{\delta\in I_2} T_{\hat\beta,\delta}\right) \cup \left(\bigcup_{\beta\in 2^\zeta\setminus\{\hat\beta\}}\bigcup_{\delta\in I_2} T_{\beta,\delta}\right)\\ &\subseteq \left(\bigcup_{\delta\in I_2} T_{\hat\beta,\delta}\right) \cup \left(\bigcup_{\beta\in 2^\zeta\setminus\{\hat\beta\}} M_\beta\right);\end{aligned}$$

thus, to prove (2.2), it will suffice to show that

$$M_{\hat\beta} \cap \Big(\big(\bigcap_{\delta\in I_1} T_{\hat\beta,\delta}\big) \setminus \Big(\big(\bigcup_{\delta\in I_2} T_{\hat\beta,\delta}\big) \cup \big(\bigcup_{\beta\in 2^\zeta\setminus\{\hat\beta\}} M_\beta\big)\Big)\Big) \neq \emptyset,$$

i.e., that

(2.3)
$$M_{\hat\beta} \cap \Big(\Big(\big(\bigcap_{\delta\in I_1} T_{\hat\beta,\delta}\big) \setminus \big(\bigcup_{\delta\in I_2} T_{\hat\beta,\delta}\big)\Big) \setminus \big(\bigcup_{\beta\in 2^\zeta\setminus\{\hat\beta\}} M_\beta\big)\Big) \neq \emptyset.$$

Notice that, by (3) above, $M_{\hat\beta} \cap \big(\bigcup_{\beta\in 2^\zeta\setminus\{\hat\beta\}} M_\beta\big) = \emptyset$, so that (2.3) is equivalent to

$$M_{\hat\beta} \cap \Big(\big(\bigcap_{\delta\in I_1} T_{\hat\beta,\delta}\big) \setminus \big(\bigcup_{\delta\in I_2} T_{\hat\beta,\delta}\big)\Big) \neq \emptyset, \tag{2.4}$$

an immediate consequence of the fact that $\{T_{\hat\beta,\delta} \mid \delta \in 2^\zeta\}$ is a ζ^+-independent family on $M_{\hat\beta}$. □

Corollary 2.8. *There exists a collection $\{S_\delta \mid \delta \in 2^{\mathfrak c}\}$ of subsets of the unit interval $I = [0,1]$, such that for every nonempty open subset A of I (endowed with the Euclidean topology), the family $\{S_\delta \cap A \mid \delta \in 2^{\mathfrak c}\}$ is ω_1-independent.*

Actually, in the next section, we will not need the above corollary in all its strength. It will be sufficient for us to have a family $\{S_\delta \mid \delta \in \mathfrak c\}$ with the property of the statement. Observe, in passing, that for a similar family, we have the inequality $S_{\delta'} \cap A \neq S_{\delta''} \cap A$ for every nonempty open subset A of I and for any two distinct $\delta', \delta'' \in \mathfrak c$ (as, for example, the set $(S_{\delta'} \cap A) \setminus (S_{\delta''} \cap A)$ must be nonempty).

3. A non-minimal topology in Σ_2 under CH, which is not upper

Let $\mathscr B$ be the collection of all nonempty open subsets of I, endowed with the Euclidean topology. Thanks to Corollary 2.8, we may fix a collection $\mathscr M$ of subsets of $I = [0,1]$, with $|\mathscr M| = \mathfrak c$, such that for every $B \in \mathscr B$, the collection $\{M \cap B \mid M \in \mathscr M\}$ turns out to be an ω_1-independent family on B and, moreover, $M_1 \cap B \neq M_2 \cap B$ for any two distinct $M_1, M_2 \in \mathscr M$ (i.e., the restriction to B acts in a one-to-one way on $\mathscr M$). Henceforth, we will assume the Continuum Hypothesis throughout the construction of our space. Thus, index I in a one-to-one way as $\{x_\alpha \mid \alpha \in \omega_1\}$, and then associate to every $\alpha \in \omega_1$ a subcollection $\mathscr M_\alpha$ of $\mathscr M$ in such a way that $|\mathscr M_\alpha| = \mathfrak c\,(= \omega_1)$

and $\mathscr{M}_{\alpha'} \cap \mathscr{M}_{\alpha''} = \emptyset$ for any two distinct $\alpha', \alpha'' \in \omega_1$. Let us also index each $\mathscr{M}_\alpha$, in a one-to-one way, as $\{M_{\alpha,\beta} \mid \beta \in \omega_1,\ \beta \geq \alpha\}$.

Let σ be the Euclidean topology on I (i.e., $\sigma = \mathscr{B} \cup \{\emptyset\}$); thus, letting $\mathscr{B}_\alpha = \{B \in \mathscr{B} \mid x_\alpha \in B\}$ for every $\alpha \in \omega_1$, each $\mathscr{B}_\alpha$ turns out to be a fundamental system of (open) σ-neighborhoods for x_α. Now we define by transfinite induction, for every $\alpha \in \omega_1$, a subset L_α of $\alpha \times \alpha$:

$$\begin{aligned} L_\alpha = \big\{(\alpha', \beta') \,\big|\, & \alpha' \leq \beta' < \alpha \\ & \wedge \ \ \forall (\alpha'', \beta'') \in L_{\alpha'} \colon x_\alpha \in M_{\alpha'', \beta''} \\ & \quad \wedge \ \ \forall \beta'' \in \omega_1 \colon \big(\alpha' \leq \beta'' \leq \beta' \Longrightarrow x_\alpha \in M_{\alpha', \beta''}\big)\big\}. \end{aligned} \tag{3.1}$$

(In particular, of course, $L_0 = \emptyset$.) Let us show two basic properties of the sets L_α.

Fact 1. *If $(\alpha', \beta') \in L_\alpha$ for some $\alpha \in \omega_1$ and β^* is such that $\alpha' \leq \beta^* \leq \beta'$, then $(\alpha', \beta^*) \in L_\alpha$.*

Proof: First of all, $(\alpha', \beta') \in L_\alpha$ implies that $\alpha' \leq \beta^* \leq \beta' < \alpha$ and that $x_\alpha \in M_{\alpha'', \beta''}$ for every $(\alpha'', \beta'') \in L_{\alpha'}$. Moreover, if β'' is such that $\alpha' \leq \beta'' \leq \beta^*$, then we also have the inequalities $\alpha' \leq \beta'' \leq \beta'$; thus, using again the fact that $(\alpha', \beta') \in L_\alpha$, we conclude that $x_\alpha \in M_{\alpha', \beta''}$. Therefore, $(\alpha', \beta^*) \in L_\alpha$. □

Fact 2. *If $(\alpha_1, \beta_1) \in L_{\alpha_2}$ for some $\alpha_2 \in \omega_1$, then $L_{\alpha_1} \subseteq L_{\alpha_2}$.*

Proof: We will prove by transfinite induction on α_2 that

$$\forall \alpha_2 \in \omega_1 \colon \forall \alpha_1 \in \omega_1 \colon \forall \beta_1 \in \omega_1 \colon \big((\alpha_1, \beta_1) \in L_{\alpha_2} \Longrightarrow L_{\alpha_1} \subseteq L_{\alpha_2}\big). \tag{3.2}$$

Indeed, let $\hat{\alpha}_2 \in \omega_1$ be such that for every $\alpha_2 < \hat{\alpha}_2$, (3.2) holds. To prove (3.2) for $\alpha_2 = \hat{\alpha}_2$, we consider an arbitrary $(\hat{\alpha}_1, \hat{\beta}_1) \in L_{\hat{\alpha}_2}$, and we show that $L_{\hat{\alpha}_1} \subseteq L_{\hat{\alpha}_2}$. Indeed, let $(\hat{\alpha}', \hat{\beta}') \in L_{\hat{\alpha}_1}$. First of all, by (3.1), the relation $(\hat{\alpha}_1, \hat{\beta}_1) \in L_{\hat{\alpha}_2}$ implies that $\hat{\alpha}_1 \leq \hat{\beta}_1 < \hat{\alpha}_2$, while $(\hat{\alpha}', \hat{\beta}') \in L_{\hat{\alpha}_1}$ implies that $\hat{\alpha}' \leq \hat{\beta}' < \hat{\alpha}_1$; hence, we conclude that $\hat{\alpha}' \leq \hat{\beta}' < \hat{\alpha}_1 < \hat{\alpha}_2$.

Next, suppose there is a pair $(\alpha'', \beta'') \in L_{\hat{\alpha}'}$. Since $\hat{\alpha}_1 < \hat{\alpha}_2$, by the inductive hypothesis, we may apply (3.2) with $\alpha_2 = \hat{\alpha}_1$, $\alpha_1 = \hat{\alpha}'$, and $\beta_1 = \hat{\beta}'$. Since $(\hat{\alpha}', \hat{\beta}') \in L_{\hat{\alpha}_1}$, it follows that $L_{\hat{\alpha}'} \subseteq L_{\hat{\alpha}_1}$. Thus, from $(\alpha'', \beta'') \in L_{\hat{\alpha}'}$, we infer that $(\alpha'', \beta'') \in L_{\hat{\alpha}_1}$; then, taking (3.1) into account (for $\alpha = \hat{\alpha}_2$), from $(\hat{\alpha}_1, \hat{\beta}_1) \in L_{\hat{\alpha}_2}$ and $(\alpha'', \beta'') \in L_{\hat{\alpha}_1}$, we conclude that $x_{\hat{\alpha}_2} \in M_{\alpha'', \beta''}$.

Finally, it remains to show that, given any $\beta'' \in \omega_1$ with $\hat\alpha' \le \beta'' \le \hat\beta'$, we have the relation $x_{\hat\alpha_2} \in M_{\hat\alpha',\beta''}$. Indeed, from $\hat\alpha' \le \beta'' \le \hat\beta'$ and $(\hat\alpha', \hat\beta') \in L_{\hat\alpha_1}$, we deduce by Fact 1 that $(\hat\alpha', \beta'') \in L_{\hat\alpha_1}$. Then from $(\hat\alpha_1, \hat\beta_1) \in L_{\hat\alpha_2}$ and $(\hat\alpha', \beta'') \in L_{\hat\alpha_1}$, we deduce—still taking into account the definition of $L_{\hat\alpha_2}$—that $x_{\hat\alpha_2} \in M_{\hat\alpha',\beta''}$. $\square$

Now we want to define a new topology τ on I by associating to each $\alpha \in \omega_1$ a fundamental system $\mathscr{V}_\alpha$ of neighborhoods for x_α, and we will do it in such a way that every element of each $\mathscr{V}_\alpha$ will turn out to be τ-open. For $\alpha \in \omega_1$, set

$$\mathscr{V}_\alpha = \left\{ V_{\alpha,\beta,B} \,\middle|\, \alpha < \beta < \omega_1,\ B \in \mathscr{B}_\alpha \right\}, \tag{3.3}$$

where for every $\beta \in \omega_1$ with $\beta > \alpha$ and for every $B \in \mathscr{B}_\alpha$, we let

$$\begin{aligned} V_{\alpha,\beta,B} = \{x_\alpha\} \cup \Bigg(B \cap \Bigg(\bigcap_{(\alpha',\beta') \in L_\alpha} M_{\alpha',\beta'} \Bigg) \\ \cap \Bigg(\bigcap_{\alpha \le \beta' \le \beta} M_{\alpha,\beta'} \Bigg) \setminus \left\{ x_{\alpha'} \,\middle|\, \alpha' \le \beta \right\} \Bigg). \end{aligned} \tag{3.4}$$

(Of course, if one happens to come up with the intersection of an empty family, it will be meant to be equal to I).

To show that such an assignment actually gives rise to a topology, we will prove the following properties (cf. [6, Proposition 1.2.3]).

(BP1) For every $\alpha \in \omega_1$, $\mathscr{V}_\alpha \ne \emptyset$, and for every $U \in \mathscr{V}_\alpha$, $x_\alpha \in U$.

(BP2) If $x_{\alpha^*} \in U \in \mathscr{V}_\alpha$, then there exists $V \in \mathscr{V}_{\alpha^*}$ such that $V \subseteq U$.

(BP3) For any $U_1, U_2 \in \mathscr{V}_\alpha$, there exists $U \in \mathscr{V}_\alpha$ such that $U \subseteq U_1 \cap U_2$.

(BP1) is trivially fulfilled and (BP3) is straightforward: for every $\alpha \in \omega_1$, $\beta', \beta'' > \alpha$ and $B', B'' \in \mathscr{B}_\alpha$, we have the inclusion $V_{\alpha,\max\{\beta',\beta''\},B'\cap B''} \subseteq V_{\alpha,\beta',B'} \cap V_{\alpha,\beta'',B''}$. We are going to prove (BP2). Thus, suppose we have $\hat\alpha, \hat\beta, \alpha^* \in \omega_1$ and $B \in \mathscr{B}_{\hat\alpha}$, with $\hat\alpha < \hat\beta$ and $x_{\alpha^*} \in V_{\hat\alpha,\hat\beta,B}$. Of course, we may assume $\alpha^* \ne \hat\alpha$; hence, from (3.4), it follows that $\alpha^* > \hat\beta > \hat\alpha$, that $B \in \mathscr{B}_{\alpha^*}$, that $x_{\alpha^*} \in M_{\hat\alpha,\beta'}$ for $\hat\alpha \le \beta' \le \hat\beta$, and that $x_{\alpha^*} \in M_{\alpha',\beta'}$ for $(\alpha', \beta') \in L_{\hat\alpha}$. Then, considering (3.1) for $\alpha = \alpha^*$, we easily see that $\big(\hat\alpha, \hat\beta\big) \in L_{\alpha^*}$, which implies by Fact 1 and Fact 2 that

$\{(\hat{\alpha}, \beta') \mid \hat{\alpha} \leq \beta' \leq \hat{\beta}\} \subseteq L_{\alpha^*}$ and that $L_{\hat{\alpha}} \subseteq L_{\alpha^*}$. Therefore,

$$\bigcap_{(\alpha',\beta') \in L_{\alpha^*}} M_{\alpha',\beta'} \subseteq \left(\bigcap_{(\alpha',\beta') \in L_{\hat{\alpha}}} M_{\alpha',\beta'} \right) \cap \left(\bigcap_{\hat{\alpha} \leq \beta' \leq \hat{\beta}} M_{\hat{\alpha},\beta'} \right);$$

thus, taking, for example, the element $V_{\alpha^*,\alpha^*+1,B}$ of $\mathscr{V}_{\alpha^*}$ (remembering that $B \in \mathscr{B}_{\alpha^*}$), and considering that

$$V_{\alpha^*,\alpha^*+1,B} \subseteq \{x_{\alpha^*}\} \cup \left(B \cap \left(\bigcap_{(\alpha',\beta') \in L_{\alpha^*}} M_{\alpha',\beta'} \right) \setminus \{x_{\alpha'} \mid \alpha' \leq \alpha^* + 1\} \right),$$

we have the inclusion $V_{\alpha^*,\alpha^*+1,B} \subseteq V_{\hat{\alpha},\hat{\beta},B}$.

Of course, since each $V_{\alpha,\beta,B}$ is included in B, the topology τ generated by the above assignment is finer than σ; hence, in particular, it is T_2. Notice that "finer," here, holds also in the strict sense, i.e., $\sigma \subsetneqq \tau$. Indeed, for $n \in \omega$, let α_n be such that $x_{\alpha_0} = 0$ and $x_{\alpha_n} = \frac{1}{n}$ for $n > 0$; also, set $\hat{\alpha} = \sup_{n\in\omega} \alpha_n$. Then, for any fixed $B \in \mathscr{B}_{\alpha_0}$, the set $V_{\alpha_0,\hat{\alpha}+1,B}$ is a τ-neighborhood of 0 missing all elements of the sequence $\{\frac{1}{n}\}_{n\in\omega\setminus\{0\}}$, which σ-converges to 0. Thus, $V_{\alpha_0,\hat{\alpha}+1,B}$ is not a σ-neighborhood of 0.

Now we prove that the semiregularization of τ is σ, so that later, we will be able to apply Proposition 2.2. Actually, let τ_s be the semiregularization of τ; since, as we have already pointed out, the semiregularization of a T_2 topology γ is the finest semiregular topology γ' such that $\gamma' \subseteq \gamma$, and since (I, σ) is clearly a semiregular (in fact, a regular) space, we have the inclusion $\sigma \subseteq \tau_s$. Thus, we have to prove that $\tau_s \subseteq \sigma$, i.e., that for every $x_\alpha \in I$ and for every τ-open neighborhood V of x_α, there exists a σ-neighborhood W of x_α with $W \subseteq \mathrm{Int}_\tau \mathrm{Cl}_\tau V$.

Consider an arbitrary basic τ-neighborhood $V_{\alpha,\beta,B}$ of x_α, with $\alpha < \beta$ and $B \in \mathscr{B}_\alpha$. We will prove that $B \subseteq \mathrm{Cl}_\tau V_{\alpha,\beta,B}$, and since B is σ-open—hence, τ-open, too—we will also have the inclusion $B = \mathrm{Int}_\tau B \subseteq \mathrm{Int}_\tau \mathrm{Cl}_\tau V_{\alpha,\beta,B}$. Actually, let $x_{\alpha^*} \in B$ be arbitrary. We want to show that for every $\beta^* > \alpha^*$ and for every $B^* \in \mathscr{B}_{\alpha^*}$, the inequality $V_{\alpha^*,\beta^*,B^*} \cap V_{\alpha,\beta,B} \neq \emptyset$ holds. Indeed, for β^* and B^* as above, set $B^\sharp = B \cap B^*$ and define $\mathscr{M}_{B^\sharp} = \{M \cap B^\sharp \mid M \in \mathscr{M}\}$

and

$$\begin{aligned}\mathscr{M}'_{B^\sharp} = {} & \bigl\{ M_{\alpha',\beta'} \cap B^\sharp \bigm| (\alpha',\beta') \in L_{\alpha^*} \cup L_\alpha \bigr\} \\ & \cup \bigl\{ M_{\alpha^*,\beta'} \cap B^\sharp \bigm| \alpha^* \le \beta' \le \beta^* \bigr\} \cup \bigl\{ M_{\alpha,\beta'} \cap B^\sharp \bigm| \alpha \le \beta' \le \beta \bigr\};\end{aligned}$$

since $\mathscr{M}'_{B^\sharp}$ is a countable subset of $\mathscr{M}_{B^\sharp}$, it follows from Remark 2.6 that $\bigl|\bigcap \mathscr{M}'_{B^\sharp}\bigr| = \mathfrak{c}\,(=\omega_1)$, and hence,

$$\begin{aligned}B^\sharp \cap \Biggl(\bigcap_{(\alpha',\beta') \in L_{\alpha^*} \cup L_\alpha} M_{\alpha',\beta'} \Biggr) \cap \Biggl(\bigcap_{\alpha^* \le \beta' < \beta^*} M_{\alpha^*,\beta'} \Biggr) \\ \cap \Biggl(\bigcap_{\alpha \le \beta' < \beta} M_{\alpha,\beta'} \Biggr) \setminus \bigl\{ x_{\alpha'} \bigm| \alpha' \le \max\{\beta^*,\beta\} \bigr\} \neq \emptyset.\end{aligned}$$

Then, taking (3.4) into account, we conclude that

$$V_{\alpha^*,\beta^*,B^*} \cap V_{\alpha,\beta,B} \neq \emptyset.$$

Now, we prove that for every Hausdorff topology $\tau' \subsetneqq \tau$ and for every $A \in \tau \setminus \tau'$, we have the inequality $\tau \neq \langle \tau' \cup \{A\} \rangle$, and this will show, by Theorem 4.3 [5], that τ is not an upper topology in $\Sigma_2(I)$. Actually, suppose τ' is any T_2 topology on I with $\tau' \subsetneqq \tau$; then, as noted before, by Proposition 2.2, we have the inclusion

$$\sigma \subseteq \tau'. \tag{3.5}$$

If $A \in \tau \setminus \tau'$, then, on the one hand, there must exist a point $x_{\hat\alpha} \in A$ such that

$$\forall\, W' \in \tau' \colon \bigl(x_{\hat\alpha} \in W' \Longrightarrow W' \setminus A \neq \emptyset \bigr), \tag{3.6}$$

and, on the other hand, for such an $\hat\alpha$, there will exist $\hat\beta > \hat\alpha$ and $B \in \mathscr{B}_{\hat\alpha}$ such that

$$V_{\hat\alpha,\hat\beta,B} \subseteq A. \tag{3.7}$$

We claim that for every τ'-neighborhood W' of $x_{\hat\alpha}$, we have the inequality $(W' \cap A) \setminus V_{\hat\alpha,\hat\beta+1,B} \neq \emptyset$, and by Proposition 2.3, this is sufficient to show that $V_{\hat\alpha,\hat\beta+1,B}$ is not a $\langle \tau' \cup \{A\} \rangle$-neighborhood of $x_{\hat\alpha}$ (so that, in particular, $\langle \tau' \cup \{A\} \rangle \neq \tau$).

Actually, let $x_{\hat\alpha} \in W' \in \tau'$. Since $B \in \mathscr{B}_{\hat\alpha} \subseteq \sigma \subseteq \tau'$ (due to (3.5)), and $x_{\hat\alpha} \in V_{\hat\alpha,\hat\beta,B} \subseteq B$, we also have the relations $x_{\hat\alpha} \in W' \cap B \in \tau'$, so that by (3.6), we may fix an element

$$x_{\alpha^*} \in (W' \cap B) \setminus A. \tag{3.8}$$

In particular, the fact that $x_{\alpha^*} \in W' \in \tau'$ implies (as $\tau' \subseteq \tau$) that there are $\beta^* > \alpha^*$ and $B^* \in \mathscr{B}_{\alpha^*}$ such that

$$V_{\alpha^*,\beta^*,B^*} \subseteq W'. \tag{3.9}$$

Notice that $\big(\hat\alpha, \hat\beta + 1\big) \notin L_{\alpha^*}$. Indeed, if $\big(\hat\alpha, \hat\beta + 1\big)$ were in L_{α^*}, then, first of all, it would follow from (3.1) that $\alpha^* > \hat\beta + 1 > \hat\beta$ and that $x_{\alpha^*} \in M_{\hat\alpha,\beta}$ for $\hat\alpha \le \beta \le \hat\beta + 1$ (in particular, for $\hat\alpha \le \beta \le \hat\beta$); moreover, by Fact 2, we would have the inclusion $L_{\hat\alpha} \subseteq L_{\alpha^*}$, so that $x_{\alpha^*} \in M_{\alpha',\beta'}$ for every $(\alpha', \beta') \in L_{\hat\alpha}$ (again using (3.1)). Since, by (3.8), $x_{\alpha^*} \in B$, we would finally conclude that

$$x_{\alpha^*} \in \left(B \cap \left(\bigcap_{(\alpha',\beta') \in L_{\hat\alpha}} M_{\alpha',\beta'} \right) \cap \left(\bigcap_{\hat\alpha \le \beta' \le \hat\beta} M_{\hat\alpha,\beta'} \right) \setminus \big\{ x_{\alpha'} \,\big|\, \alpha' \le \hat\beta \big\} \right) \subseteq V_{\hat\alpha,\hat\beta,B},$$

thus contradicting the combination of (3.7) and (3.8).

Now, consider the collection

$$\mathscr{M}' = \big\{ M_{\hat\alpha,\beta} \,\big|\, \hat\alpha \le \beta \le \hat\beta \big\} \cup \big\{ M_{\alpha^*,\beta} \,\big|\, \alpha^* \le \beta \le \beta^* \big\} \cup \big\{ M_{\alpha,\beta} \,\big|\, (\alpha, \beta) \in L_{\hat\alpha} \cup L_{\alpha^*} \big\}.$$

Since (as we have just proved) $\big(\hat\alpha, \hat\beta + 1\big) \notin L_{\alpha^*}$, since $\big(\hat\alpha, \hat\beta + 1\big) \notin L_{\hat\alpha}$ (because $\big(\hat\alpha, \hat\beta + 1\big) \in L_{\hat\alpha}$ would imply that $\hat\alpha \le \hat\beta + 1 < \hat\alpha$), and since $\alpha^* \ne \hat\alpha$ (as $x_{\hat\alpha} \in A$ by (3.7), while $x_{\alpha^*} \notin A$), it follows that $\mathscr{M}'$ is a countable subset of $\mathscr{M}$ which does not contain $M_{\hat\alpha,\hat\beta+1}$. Let $B^\sharp = B \cap B^*$, so that $B^\sharp \in \mathscr{B}_{\alpha^*}$ (remembering that $x_{\alpha^*} \in B$ by (3.8) and that B^* has been chosen to belong to $\mathscr{B}_{\alpha^*}$); then consider the collection $\mathscr{M}_{B^\sharp} = \{ M \cap B^\sharp \,|\, M \in \mathscr{M} \}$. By the initial properties of the collection $\mathscr{M}$, it follows that $\mathscr{M}_{B^\sharp}$ is an ω_1-independent family on $B^\sharp$. Since $\mathscr{M}'_{B^\sharp} = \big\{ M' \cap B^\sharp \,\big|\, M' \in \mathscr{M}' \big\}$ is a countable subcollection of $\mathscr{M}_{B^\sharp}$, and since $M_{\hat\alpha,\hat\beta+1} \cap B^\sharp \notin \mathscr{M}'_{B^\sharp}$ (as the restriction to $B^\sharp$ acts on $\mathscr{M}$ in a one-to-one way), we conclude by Remark 2.6 that there exists

$$\bar{x} \in \left(\bigcap_{(\alpha,\beta)\in L_{\hat{\alpha}}\cup L_{\alpha^*}} \left(B^\sharp \cap M_{\alpha,\beta}\right) \right) \cap \left(\bigcap_{\hat{\alpha}\le\beta\le\hat{\beta}} \left(B^\sharp \cap M_{\hat{\alpha},\beta}\right) \right) \cap \left(\bigcap_{\alpha^*\le\beta\le\beta^*} \left(B^\sharp \cap M_{\alpha^*,\beta}\right) \right) \cap \left(B^\sharp \setminus M_{\hat{\alpha},\hat{\beta}+1}\right) \setminus \left\{x_\alpha \,\middle|\, \alpha \le \max\{\hat{\beta},\beta^*\}\right\}.$$

Thus, taking (3.4) into account, we deduce that $\bar{x} \in V_{\hat{\alpha},\hat{\beta},B}$, $\bar{x} \in V_{\alpha^*,\beta^*,B^*}$, and $\bar{x} \notin V_{\hat{\alpha},\hat{\beta}+1,B}$; therefore, by (3.7) and (3.9), it follows that $\bar{x} \in (A \cap W') \setminus V_{\hat{\alpha},\hat{\beta}+1,B}$; hence, $(A \cap W') \setminus V_{\hat{\alpha},\hat{\beta}+1,B} \neq \emptyset$.

Remark 3.1. It is worth considering some possible attempts to define the sets $V_{\alpha,\beta,B}$ in a way simpler than (3.4). At first glance, we could be convinced that we do not need to introduce the sets L_α for such a definition to work; however, a more careful analysis of the situation will turn up some serious problems.

On the one hand, if we simply set

$$V_{\alpha,\beta,B} = \{x_\alpha\} \cup \left(B \cap \left(\bigcap_{\alpha'\le\beta'<\alpha} M_{\alpha',\beta'} \right) \cap \left(\bigcap_{\alpha\le\beta'\le\beta} M_{\alpha,\beta'} \right) \setminus \left\{x_{\alpha'} \,\middle|\, \alpha' \le \beta\right\} \right)$$

for $\alpha < \beta < \omega_1$ and $B \in \mathscr{B}_\alpha$, then (BP1)–(BP3) are satisfied, but, in the subsequent part of the proof, we will no longer be able to show that there is an $\bar{x} \in V_{\hat{\alpha},\hat{\beta},B} \cap V_{\alpha^*,\beta^*,B^*} \setminus V_{\hat{\alpha},\hat{\beta}+1,B}$. Indeed, with the above definition of $V_{\alpha,\beta,B}$, we will be forced to set

$$\mathscr{M}' = \left\{M_{\hat{\alpha},\beta} \,\middle|\, \hat{\alpha} \le \beta \le \hat{\beta}\right\} \cup \left\{M_{\alpha^*,\beta} \,\middle|\, \alpha^* \le \beta \le \beta^*\right\} \cup \left\{M_{\alpha,\beta} \,\middle|\, \alpha \le \beta < \max\{\hat{\alpha},\alpha^*\}\right\},$$

and then we cannot exclude that $M_{\hat{\alpha},\hat{\beta}+1} \in \mathscr{M}'$ (in particular, this happens whenever $\alpha^* > \hat{\beta}+1$).

On the other hand, suppose we define

$$V_{\alpha,\beta,B} = \{x_\alpha\} \cup \Bigg(B \cap \Big(\bigcap \big\{ M_{\alpha',\beta'} \,\big|\, \alpha' \leq \beta' < \alpha,\, x_{\alpha'} \in B \big\} \Big) \cap \Bigg(\bigcap_{\alpha \leq \beta' \leq \beta} M_{\alpha,\beta'} \Bigg) \setminus \big\{ x_{\alpha'} \,\big|\, \alpha' \leq \beta \big\} \Bigg).$$

Then (BP2) is satisfied, and we could also arrange the subsequent part of the proof to show that the "presumed" topology τ generated by such an assignment is not upper. Nevertheless, τ is not a topology at all, because we cannot prove (BP3)! For example, the reader may easily check that given V_{α,β_1,B_1} and V_{α,β_2,B_2} basic neighborhoods of x_α, neither $V_{\alpha,\max\{\beta_1,\beta_2\},B_1\cap B_2}$ nor $V_{\alpha,\max\{\beta_1,\beta_2\},B_1\cup B_2}$ is included, in general, in $V_{\alpha,\beta_1,B_1} \cap V_{\alpha,\beta_2,B_2}$.

4. Examples of upper topologies

The example produced in the previous section may look rather twisted, and it is natural to wonder whether it can be simplified. In particular, is the use of ω_1-independent families really essential for the final topology τ to have the desired properties, or could we obtain an analogous result just by means of a well-order on the unit interval $[0,1]$?

In this section, we show that several modifications of the Euclidean topology on $[0,1]$, defined by referring—in a very natural way—only to a well-order on $[0,1]$ itself, are upper in $\Sigma_2([0,1])$, and this might somehow justify the introduction of further combinatorial tools for our construction in §3.

Henceforth, for the sake of simplicity, given a topology σ on a set X and a point $x \in X$, we will denote by $\sigma(x)$ the collection of all $V \in \sigma$ such that $x \in V$ (i.e., $\sigma(x)$ is the collection of all open σ-neighborhoods of x).

Lemma 4.1. *Let τ be a T_2 topology on a set X. Suppose there are a point $\bar{x} \in X$, a fundamental system $\widetilde{\mathscr{V}}_{\bar{x}}$ of open τ-neighborhoods of $\bar{x}$, a τ-neighborhood V^* of $\bar{x}$, and a set $M \subseteq X$ such that*

(1) $V^* \cap M = \emptyset$;
(2) $\forall\, V \in \widetilde{\mathscr{V}}_{\bar{x}} \colon \mathrm{Int}_\tau \mathrm{Cl}_\tau V \cap M \neq \emptyset$;
(3) $\forall\, V \in \widetilde{\mathscr{V}}_{\bar{x}} \colon V \cup (\mathrm{Int}_\tau \mathrm{Cl}_\tau V \cap M) \in \tau$.

Then τ is upper in $\Sigma_2(X)$.

Proof: Consider a new topology τ' on X such that $\tau'|_{X\setminus\{\bar{x}\}} = \tau|_{X\setminus\{\bar{x}\}}$, while the point $\bar{x}$ has a fundamental system of (open) neighborhoods given by

$$\big\{V \cup (\mathrm{Int}_\tau \mathrm{Cl}_\tau V \cap M) \,\big|\, V \in \widetilde{\mathscr{V}}_{\bar{x}}\big\}.$$

Using property (3), together with the facts that τ is T_2 (hence, in particular, T_1) and that $\tau'|_{X\setminus\{\bar{x}\}} = \tau|_{X\setminus\{\bar{x}\}}$, it is immediate to see that condition (BP2) (see §3) is fulfilled for $x = \bar{x}$. Thus, τ' is clearly a topology on X, coarser than τ. Moreover, notice that τ' is T_2, as it is finer than (or equal to) the semiregularization σ of τ—where σ is T_2 by Proposition 2.1. Indeed, at every $x \neq \bar{x}$, the topology τ' coincides with τ (which is finer than σ), while at the point $\bar{x}$, the basic σ-neighborhoods are of the form $\mathrm{Int}_\tau \mathrm{Cl}_\tau V$ with $V \in \tau(\bar{x})$; hence, each of them includes any set of the kind $V' \cup (\mathrm{Int}_\tau \mathrm{Cl}_\tau V' \cap M)$, with $V' \in \widetilde{\mathscr{V}}_{\bar{x}}$ and $V' \subseteq V$, which is a τ'-neighborhood of $\bar{x}$. Finally, we see that $\tau' \subsetneqq \tau$, as V^* is a τ-neighborhood of $\bar{x}$ which includes no τ'-neighborhood of $\bar{x}$; indeed, every τ'-neighborhood of $\bar{x}$ includes a set of the kind $V \cup (\mathrm{Int}_\tau \mathrm{Cl}_\tau V \cap M)$ with $V \in \widetilde{\mathscr{V}}_{\bar{x}}$, and by (2), such a set has nonempty intersection with M, while $V^* \cap M = \emptyset$.

Now, we prove that $\langle \tau' \cup \{V^*\}\rangle = \tau$, and this will entail, by Theorem 4.3 [5], that τ is upper in $\Sigma_2(X)$. Of course, we just have to show that for every $V \in \tau(\bar{x})$, there is a $\langle \tau' \cup \{V^*\}\rangle$-neighborhood of $\bar{x}$ included in τ. Actually, given $V \in \tau(\bar{x})$, take $V' \in \widetilde{\mathscr{V}}_{\bar{x}}$ with $V' \subseteq V$; then the set $\big(V' \cup (\mathrm{Int}_\tau \mathrm{Cl}_\tau V' \cap M)\big) \cap V^* = (V' \cap V^*) \cup (\mathrm{Int}_\tau \mathrm{Cl}_\tau V' \cap M \cap V^*) = V' \cap V^*$ (we have used (1)) is a $\langle \tau' \cup \{V^*\}\rangle$-neighborhood of $\bar{x}$ and is included in V. □

In the statement of the next three results, we will assume to have fixed a one-to-one indexing $\{x_\alpha \,|\, \alpha \in \mathfrak{c}\}$ of the interval $I = [0, 1]$.

Example 4.2. *Let σ be the Euclidean topology on $I = [0, 1]$ and define a new topology τ_1 on I by assigning to every $x_\alpha \in I$ the fundamental system of (open) neighborhoods*

$$\mathscr{V}_\alpha = \big\{\{x_\alpha\} \cup (W \cap \{x_\gamma \,|\, \beta < \gamma < \mathfrak{c}\}) \,\big|\, W \in \sigma(x_\alpha),\, \beta \in \mathfrak{c}\big\}.$$

(It is easy to check that (BP1)–(BP3) are fulfilled.) Then τ_1 is an upper topology in $\Sigma_2(I)$.

Proof: It is clear that $\sigma \subseteq \tau_1$, so that τ_1 is T_2. Now, let $\alpha_0 \in \mathfrak{c}$ be such that $x_{\alpha_0} = 0$, and for every $n \in \mathbb{N}$, take $\alpha_n \in \mathfrak{c}$ such that $x_{\alpha_n} = \frac{1}{n}$. Since the cofinality of $\mathfrak{c}$ is greater than ω, we see that $\hat{\alpha} = \sup_{n\in\mathbb{N}} \alpha_n \in \mathfrak{c}$. We want to apply Lemma 4.1 with $\bar{x} = 0$, $\widetilde{\mathscr{V}}_{\bar{x}} = \bigl\{\{0\} \cup ([0,\varepsilon[\, \cap \{x_\beta \,|\, \beta < \gamma < \mathfrak{c}\}) \bigm| 0 < \varepsilon < 1, \beta \in \mathfrak{c}\bigr\}$, $V^* = \{0\} \cup \{x_\gamma \,|\, \hat{\alpha} < \gamma < \mathfrak{c}\}$, and $M = \{x_{\alpha_n} \,|\, n \in \mathbb{N}\}$. Clearly, (1) holds, since V^* cannot contain any x_{α_n} with $n \in \mathbb{N}$. As for properties (2) and (3), consider first that for every $0 < \varepsilon < 1$ and $\beta \in \mathfrak{c}$, we have the equality $\mathrm{Cl}_{\tau_1}\bigl(\{0\} \cup ([0,\varepsilon[\, \cap \{x_\gamma \,|\, \beta < \gamma < \mathfrak{c}\})\bigr) = [0,\varepsilon]$; indeed, for every $x_\alpha \in [0,\varepsilon]$, $W \in \sigma(x_\alpha)$, and $\beta' \in \mathfrak{c}$, we see that since $[0,\varepsilon[\, \cap W$ must include in any case a nonempty open subset of (I,σ), which will then have cardinality $\mathfrak{c}$, $[0,\varepsilon[\, \cap W$ cannot be included in $\{x_\gamma \,|\, \gamma \le \max\{\beta,\beta'\}\}$; hence, $\bigl(\{0\} \cup ([0,\varepsilon[\, \cap \{x_\gamma \,|\, \beta < \gamma < \mathfrak{c}\})\bigr) \cap \bigl(\{x_\alpha\} \cup (W \cap \{x_\gamma \,|\, \beta' < \gamma < \mathfrak{c}\})\bigr) \supseteq [0,\varepsilon[\, \cap W \cap \bigl\{x_\gamma \bigm| \max\{\beta,\beta'\} < \gamma < \mathfrak{c}\bigr\} \neq \emptyset$. Since $\mathrm{Int}_{\tau_1}[0,\varepsilon] \supseteq \mathrm{Int}_\sigma[0,\varepsilon]$ and since $\varepsilon \notin \mathrm{Int}_{\tau_1}[0,\varepsilon]$ (using again the fact that every $W \in \sigma(\varepsilon)$ meets the set $]\varepsilon,1]$ in a nonempty σ-open set, which has the cardinality of the continuum), we conclude that $\mathrm{Int}_{\tau_1}[0,\varepsilon] = \mathrm{Int}_\sigma[0,\varepsilon] = [0,\varepsilon[$. Therefore, on the one hand, we see that (2) is fulfilled (as for every $0 < \varepsilon < 1$, the set $[0,\varepsilon[$ clearly contains some $x_{\alpha_n} = \frac{1}{n}$ with $n \in \mathbb{N}$). On the other hand, to prove (3), it is sufficient to show that for every $\beta \in \mathfrak{c}$ and for every $0 < \varepsilon < 1$, each element $x_{\alpha_n} = \frac{1}{n}$ belonging to $[0,\varepsilon[\, \cap M$ has a τ_1-neighborhood V' included in $([0,\varepsilon[\, \cap M) \cup ([0,\varepsilon[\, \cap \{x_\gamma \,|\, \beta < \gamma < \mathfrak{c}\})$; and this is immediate, since we may choose $V' = \bigl\{\frac{1}{n}\bigr\} \cup ([0,\varepsilon[\, \cap \{x_\gamma \,|\, \beta < \gamma < \mathfrak{c}\})$. □

Example 4.3. *For every $\alpha \in \mathfrak{c}$, let*

$$\mathscr{V}_\alpha = \bigl\{W \cap \{x_\gamma \,|\, \alpha \le \gamma < \mathfrak{c}\}\bigr) \bigm| W \in \sigma(x_\alpha)\bigr\}.$$

Then $x_\alpha \mapsto \mathscr{V}_\alpha$ is an assignment of open neighborhoods for a T_2 topology τ_2 on I, which is upper in $\Sigma_2(I)$.

Proof: Properties (BP1)–(BP3) are still straightforward to check, as well as the fact that $\sigma \subseteq \tau_2$ (so that τ_2 is T_2). Letting $S = \{\alpha \in \mathfrak{c} \,|\, x_\alpha \in \mathbb{Q} \cap I\}$ and $\hat{\alpha} = \sup S + 1\ (\in \mathfrak{c})$, the fact that $\{x_\alpha \,|\, \alpha < \hat{\alpha}\}$ is dense in I allows us to find, for every $n \in \omega$, an $\alpha_n < \hat{\alpha}$ such that $\bigl|x_{\alpha_n} - x_{\hat{\alpha}}\bigr| < \frac{1}{n}$.

Let τ' be the topology on I such that $\tau'|_{X\setminus\{x_{\hat{\alpha}}\}} = \tau_2|_{X\setminus\{x_{\hat{\alpha}}\}}$, while, at the point $x_{\hat{\alpha}}$, τ' coincides with σ (it is straightforward to show that τ' is actually a topology). Let $A = \{x_\alpha \,|\, \hat{\alpha} \le \alpha < \mathfrak{c}\}$.

Then A is easily seen to be τ_2-open but not τ'-open (as every τ'-neighborhood of $x_{\hat{\alpha}}$ must contain some x_{α_n}). Thus, in particular, we see that $\tau' \subsetneqq \tau_2$. We want to prove that $\langle \tau' \cup \{A\}\rangle = \tau_2$, which will imply, by Theorem 4.3 [5], that τ_2 is upper. Clearly, since $\langle \tau' \cup \{A\}\rangle$ agrees with τ_2 on $X \setminus \{x_{\hat{\alpha}}\}$ (as $\tau' \subseteq \langle \tau' \cup \{A\}\rangle \subseteq \tau_2$), we need only to show that for every $V \in \tau_2(x_{\hat{\alpha}})$, there exists $W \in \sigma(x_{\hat{\alpha}})$ such that $W \cap A \subseteq V$. And this is obviously true, as every basic τ_2-neighborhood of $x_{\hat{\alpha}}$ is exactly of the form $W \cap \{x_\gamma \,|\, \hat{\alpha} \leq \gamma < \mathfrak{c}\}$. □

Notice that even "mixing" in some possible way the topologies σ, τ_1, and τ_2 on the set I, we cannot obtain a non-upper topology τ (provided that τ does not agree with σ itself, which is non-upper inasmuch as it is minimal).

Proposition 4.4. *Let τ be a topology on I such that, for every $x \in I$, τ coincides at x with (at least) one of the topologies σ, τ_1, τ_2, where σ is the Euclidean topology and τ_1 and τ_2 are the topologies defined in examples 4.2 and 4.3, respectively. Then, if $\tau \neq \sigma$, τ is upper in $\Sigma_2(I)$.*

Proof: Suppose first that there is an $x_{\hat{\alpha}} \in I$ such that, at that point, τ coincides with τ_2 but not with σ. This implies, in particular, that $x_{\hat{\alpha}} \in \mathrm{Cl}_\sigma\{x_\gamma \,|\, \gamma < \hat{\alpha}\}$. Otherwise, letting $\widetilde{W} \in \sigma(x_{\hat{\alpha}})$ to be such that $\widetilde{W} \cap \{x_\gamma \,|\, \gamma < \hat{\alpha}\} = \emptyset$ (whence $\widetilde{W} = \widetilde{W} \cap \{x_\gamma \,|\, \hat{\alpha} \leq \gamma < \mathfrak{c}\}$), it would follow that for every basic τ-neighborhood (equivalently, τ_2-neighborhood) of $x_{\hat{\alpha}}$ of the form $W \cap \{x_\gamma \,|\, \hat{\alpha} \leq \gamma < \mathfrak{c}\}$, with $W \in \sigma(x_{\hat{\alpha}})$, the set $W \cap \widetilde{W} = W \cap \widetilde{W} \cap \{x_\gamma \,|\, \hat{\alpha} \leq \gamma < \mathfrak{c}\}$ would be a σ-neighborhood of $x_{\hat{\alpha}}$ included in $W \cap \{x_\gamma \,|\, \hat{\alpha} \leq \gamma < \mathfrak{c}\}$, thus contradicting the fact that τ does not agree with σ at $x_{\hat{\alpha}}$. Now, let τ' be the topology on I such that $\tau'|_{X \setminus \{x_{\hat{\alpha}}\}} = \tau|_{X \setminus \{x_{\hat{\alpha}}\}}$, while τ' coincides with σ at $x_{\hat{\alpha}}$. Thus, $\tau' \subsetneqq \tau$, and, as in the proof of Example 4.3, it is easily checked that $A = \{x_\gamma \,|\, \hat{\alpha} \leq \gamma < \mathfrak{c}\}$ is an element of $\tau \setminus \tau'$ such that $\langle \tau' \cup \{A\}\rangle = \tau$, so that τ is upper by Theorem 4.3 [5].

If, on the contrary, the above situation does not take place, then, for every $x \in I$, we see that τ agrees at x with either σ or τ_1. Letting

$$\Omega = \{x \in I \,|\, \tau \text{ agrees with } \tau_1 \text{ at } x\},$$

we notice that $\Omega \in \tau_1$. Indeed, if $\Omega \notin \tau_1$, then consider an $x_{\alpha^*} \in \Omega$ such that $x_{\alpha^*} \in \Omega \cap \mathrm{Cl}_{\tau_1}(I \setminus \Omega)$, and let $S = \{\alpha \in \mathfrak{c} \,|\, x_\alpha \in \mathbb{Q} \cap I\}$.

Setting $\tilde{\alpha} = \sup S + 1$, the fact that $x_{\alpha^*} \in \Omega$ implies that there exists $V \in \tau(x_{\alpha^*})$ with $V \subseteq \{x_{\alpha^*}\} \cup \{x_\gamma \,|\, \tilde{\alpha} < \gamma < \mathfrak{c}\}$. Pick any $x_{\alpha^\sharp} \in V \cap (I \setminus \Omega)$. Then there must be some $W \in \sigma(x_{\alpha^\sharp})$ such that $W \subseteq V \subseteq \{x_{\alpha^*}\} \cup \{x_\gamma \,|\, \tilde{\alpha} < \gamma < \mathfrak{c}\}$. But this is impossible because the set $\{x_\alpha \,|\, \alpha \in S\}$—hence also the set $\big\{x_\alpha \,\big|\, \alpha \in S \setminus \{\alpha^*\}\big\}$—is σ-dense in I, so that, taking an $x_\alpha \in W$ with $\alpha \in S \setminus \{\alpha^*\}$, we see that $x_\alpha \notin \{x_{\alpha^*}\} \cup \{x_\gamma \,|\, \tilde{\alpha} < \gamma < \mathfrak{c}\}$ (as $\tilde{\alpha} = \sup S + 1$). Therefore, $\Omega \in \tau_1$.

Since $\Omega \neq \emptyset$ (otherwise, τ would coincide with σ), we may fix $x_{\alpha_0} \in \Omega$; moreover, for the sake of simplicity, we may also assume $x_{\alpha_0} \notin \{0, 1\}$. (Clearly, Ω is infinite, as a nonempty element of τ_1.) Let $\hat{\varepsilon} = \min\{x_{\alpha_0}, 1 - x_{\alpha_0}\}$; then, for every ε with $0 < \varepsilon < \hat{\varepsilon}$ and for every $\alpha \in \mathfrak{c}$, we have the equalities

$$\mathrm{Cl}_\tau\big(\{x_{\alpha_0}\} \cup (\,]x_{\alpha_0} - \varepsilon, x_{\alpha_0} + \varepsilon[\, \cap \{x_\gamma \,|\, \alpha < \gamma < \mathfrak{c}\})\big) = [x_{\alpha_0} - \varepsilon, x_{\alpha_0} + \varepsilon]$$

and

$$\mathrm{Int}_\tau \mathrm{Cl}_\tau\big(\{x_{\alpha_0}\} \cup (\,]x_{\alpha_0} - \varepsilon, x_{\alpha_0} + \varepsilon[\, \cap \{x_\gamma \,|\, \alpha < \gamma < \mathfrak{c}\})\big) = \,]x_{\alpha_0} - \varepsilon, x_{\alpha_0} + \varepsilon[. \tag{4.1}$$

The proofs of these facts go essentially as the proofs that

$$\mathrm{Cl}_{\tau_1}\big(\{0\} \cup ([0, \varepsilon[\, \cap \{x_\gamma \,|\, \beta < \gamma < \mathfrak{c}\})\big) = [0, \varepsilon]$$

and

$$\mathrm{Int}_{\tau_1}\mathrm{Cl}_{\tau_1}\big(\{0\} \cup ([0, \varepsilon[\, \cap \{x_\gamma \,|\, \beta < \gamma < \mathfrak{c}\})\big) = [0, \varepsilon[$$

in Example 4.2, taking into account that, when working locally with any $x \in I$, we always have to distinguish the case where $x \in \Omega$ from that where $x \in I \setminus \Omega$ (this does not raise problems, as the latter case is easier than the former). Also, using the fact that $\Omega \in \tau_1$, we may easily associate to every $n \in \mathbb{N}$ an $\alpha_n \in \mathfrak{c}$ such that $x_{\alpha_n} \in (\Omega \setminus \{x_{\alpha_0}\}) \cap \big]x_{\alpha_0} - \frac{1}{n}, x_{\alpha_0} + \frac{1}{n}\big[$; then we let

$$\hat{\alpha} = \sup\{\alpha_n \,|\, n \in \mathbb{N}\}.$$

Now we are in a position to apply Lemma 4.1, with $\bar{x} = x_{\alpha_0}$, $\widetilde{\mathscr{V}}_{\bar{x}} = \big\{\{x_{\alpha_0}\} \cup (\,]x_{\alpha_0} - \varepsilon, x_{\alpha_0} + \varepsilon[\, \cap \{x_\beta \,|\, \beta < \gamma < \mathfrak{c}\}) \,\big|\, 0 < \varepsilon < \hat{\varepsilon},\, \beta \in \mathfrak{c}\big\}$, $V^* = \{x_{\alpha_0}\} \cup \{x_\gamma \,|\, \hat{\alpha} < \gamma < \mathfrak{c}\}$, and $M = \{x_{\alpha_n} \,|\, n \in \mathbb{N}\}$. Indeed, (1) is clear, while (2) follows from (4.1) and the fact that $(x_{\alpha_n})_{n\in\mathbb{N}}$ σ-converges to x_{α_0}. As for (3), as in the proof of Example 4.2, it is sufficient to show that, for every $\beta \in \mathfrak{c}$ and every $0 < \varepsilon < \hat{\varepsilon}$, each element x_{α_n} belonging to $]x_{\alpha_0} - \varepsilon, x_{\alpha_0} + \varepsilon[\, \cap M$ has a τ-neighborhood (equivalently, a τ_1-neighborhood) V' included in $(\,]x_{\alpha_0}-\varepsilon, x_{\alpha_0}+\varepsilon[\cap M) \cup (\,]x_{\alpha_0}-\varepsilon, x_{\alpha_0}+\varepsilon[\cap\{x_\gamma \,|\, \beta<\gamma<\mathfrak{c}\})$, and this is done simply by taking $V' = \{x_{\alpha_n}\} \cup (\,]x_{\alpha_0}-\varepsilon, x_{\alpha_0}+\varepsilon[\cap\{x_\gamma \,|\, \beta<\gamma<\mathfrak{c}\})$. □

We have preferred to formulate and prove examples 4.2 and 4.3 before Proposition 4.4, even if the two former results are actually special cases of the latter one, because this has seemed much more natural to us.

Of course, there are other possible ways to define topologies on I which are finer than σ, by using a well-order on I itself, but we will not get into a more sophisticated investigation of them. Instead, we want to show that using the kind of constructions above, it is possible to produce a topology which is non-upper in a weaker sense. Let us introduce the following definition.

Definition 4.5. A pair (σ,τ) of elements of a poset $(\Sigma,\leq)$, with $\sigma<\tau$, is said to be an *upwards anti-gap* [a *downwards anti-gap*] if for every $\eta\in\Sigma$ with $\sigma\leq\eta<\tau$ [$\sigma<\eta\leq\tau$] there exists $\vartheta\in\Sigma$ such that $\eta<\vartheta<\tau$ [$\sigma<\vartheta<\eta$]. The pair (σ,τ) is said to be an *anti-gap* if it is both an upwards and a downwards anti-gap.

An element $\sigma\in\Sigma$ is said to be *strongly upper* [*strongly lower*] if it is not minimal [not maximal] and for every $\sigma'\in\Sigma$ with $\sigma'<\sigma$ [$\sigma<\sigma'$], the pair (σ',σ) is not an upwards anti-gap [the pair (σ,σ') is not a downwards anti-gap].

Of course, it follows from the above definition that every strongly upper element in a poset Σ is upper. Now we will give an example of an upper, not strongly upper topology in $\Sigma_2(I)$. To this end, we need to start with an indexing of I satisfying some supplementary property besides that of being one-to-one.

In the statements and proofs of the next two results, "+" will always denote the ordinal sum operation.

Lemma 4.6. *There exists a one-to-one indexing $\{x_\alpha \,|\, \alpha\in\mathfrak{c}\}$ of I such that for every limit ordinal $\lambda\in\mathfrak{c}$, the set $\{x_{\lambda+m} \,|\, m\in\omega\}$ is dense in I.*

Proof: Let $\{B_m \,|\, m\in\omega\}$ be a countable base for I consisting of nonempty sets, and for every $m\in\omega$, let $\{y_{m,\gamma} \,|\, \gamma\in\mathfrak{c}\}$ be an indexing of B_m. Let $\sqsubseteq$ be the lexicographic order on $\mathfrak{c}\times\omega$ and j be the (unique) strictly increasing function from $(\mathfrak{c} \times \omega, \sqsubseteq)$ onto $\mathfrak{c}$ (so that, in particular, $\{j(\beta, 0) \,|\, \beta \in \mathfrak{c}\}$ is the set of all limit ordinals of $\mathfrak{c}$, and $j(\beta, m) = j(\beta, 0) + m$ for every $\beta \in \mathfrak{c}$ and $m \in \omega$). By transfinite induction, define $\tilde{\beta}(\beta, m)$ for every $(\beta, m) \in \mathfrak{c} \times m$ as

$$\tilde{\beta}(\beta, m) = \min \big\{\beta' \in \mathfrak{c} \,\big|\, y_{m,\beta'} \notin \{y_{m'',\tilde{\beta}(\beta'',m'')} \,|\, (\beta'', m'') \sqsubset (\beta, m)\}\big\}. \tag{4.2}$$

Clearly, this definition is correct because every B_m has cardinality $\mathfrak{c}$; also, $(\beta, m) \mapsto y_{m,\tilde{\beta}(\beta,m)}$ is easily seen to be one-to-one. We want to prove that $(\beta, m) \mapsto y_{m,\tilde{\beta}(\beta,m)}$ is onto I. First of all, notice that for every fixed $\bar{m} \in \omega$, the function $\beta \mapsto \tilde{\beta}(\beta, \bar{m})$ is strictly increasing, which follows from (4.2); indeed, for $\beta_1 < \beta_2$, we see that

$$\{y_{m'',\tilde{\beta}(\beta'',m'')} \,|\, (\beta'', m'') \sqsubset (\beta_1, \bar{m})\} \subseteq \{y_{m'',\tilde{\beta}(\beta'',m'')} \,|\, (\beta'', m'') \sqsubset (\beta_2, \bar{m})\},$$

whence

$$\big\{\beta' \in \mathfrak{c} \,\big|\, y_{\bar{m},\beta'} \notin \{y_{m'',\tilde{\beta}(\beta'',m'')} \,|\, (\beta'', m'') \sqsubset (\beta_1, \bar{m})\}\big\} \supseteq \big\{\beta' \in \mathfrak{c} \,\big|\, y_{\bar{m},\beta'} \notin \{y_{m'',\tilde{\beta}(\beta'',m'')} \,|\, (\beta'', m'') \sqsubset (\beta_1, \bar{m})\}\big\},$$

and that $\tilde{\beta}(\beta_2, \bar{m}) \neq \tilde{\beta}(\beta_1, \bar{m})$ because

$$y_{\bar{m},\tilde{\beta}(\beta_1,\bar{m})} \in \{y_{m'',\tilde{\beta}(\beta'',m'')} \,|\, (\beta'', m'') \sqsubset (\beta_2, \bar{m})\}.$$

As a consequence, the set $\big\{\tilde{\beta}(\beta, \bar{m}) \,\big|\, \beta \in \mathfrak{c}\big\}$ must be cofinal in $\mathfrak{c}$. Now, given an arbitrary $z \in I$, there must be $(\bar{\beta}, \bar{m}) \in \mathfrak{c} \times \omega$ such that $z = y_{\bar{m},\bar{\beta}}$ (as $\{B_m \,|\, m \in \omega\}$ is, in particular, a cover of I); then, letting $\beta \in \mathfrak{c}$ be such that $\tilde{\beta}(\beta, \bar{m}) > \bar{\beta}$, we see, by (4.2), that $z = y_{\bar{m},\bar{\beta}} \in \{y_{m'',\tilde{\beta}(\beta'',m'')} \,|\, (\beta'', m'') \sqsubset (\beta, m)\}$, so that, in particular, $z = y_{m'',\tilde{\beta}(\beta'',m'')}$ for some $(\beta'', m'') \in \mathfrak{c} \times \omega$.

Thus, letting $z_{(\beta,m)} = y_{m,\tilde{\beta}(\beta,m)}$ for every $(\beta, m) \in \mathfrak{c} \times \omega$, we have just proved that $\{z_{(\beta,m)} \,|\, (\beta, m) \in \mathfrak{c} \times \omega\}$ is a one-to-one indexing of I. Notice that for every $\bar{\beta} \in \mathfrak{c}$, the set $\{z_{(\bar{\beta},m)} \,|\, m \in \omega\} = \{y_{m,\tilde{\beta}(\bar{\beta},m)} \,|\, m \in \omega\}$ is dense in I, as it contains at least one point from any B_m. Therefore, letting $x_\alpha = z_{j^{-1}(\alpha)}$ for every $\alpha \in \mathfrak{c}$, we conclude that $\{x_\alpha \,|\, \alpha \in \mathfrak{c}\}$ is a one-to-one indexing of I, and that for every limit $\lambda \in \mathfrak{c}$ the set $\{x_{\lambda+m} \mid m \in \omega\} = \{z_{(\beta_\lambda,m)} \mid m \in \omega\}$—where β_λ is the unique element of $\mathfrak{c}$ such that $j(\beta_\lambda, 0) = \lambda$—is dense in I. □

Example 4.7. *Let $\{x_\alpha \mid \alpha \in \mathfrak{c}\}$ be a one-to-one indexing of I satisfying the condition of Lemma 4.6, and let $\alpha_0 \in \mathfrak{c}$ be such that $x_{\alpha_0} = 0$. Also, let τ_1 and τ_2 be the topologies of Example 4.2 and Example 4.3, respectively, corresponding to the indexing of I we have just fixed, and let τ be the topology on I such that $\tau|_{I\setminus\{0\}} = \tau_2|_{I\setminus\{0\}}$, while τ coincides with τ_1 at the point 0; i.e., a fundamental system of (open) τ-neighborhoods for 0 is given by*

$$\big\{\{0\} \cup ([0,\varepsilon[\cap\{x_\gamma \mid \beta < \gamma < \mathfrak{c}\}) \,\big|\, 0 < \varepsilon < 1,\, \beta \in \mathfrak{c}\big\}.$$

Then τ is not strongly upper in $\Sigma_2(I)$.

Proof: It is easy to realize that (BP2) is fulfilled, so that τ is actually a topology on I. Moreover, we see that $\tau_2 \subsetneqq \tau$; indeed, on the one hand, it is clear that τ_1 is finer than τ_2, so that $\tau_2 \subseteq \tau$, too. On the other hand, by an argument already used in the proofs of examples 4.2 and 4.3, we may associate to every $n \in \mathbb{N}$ an ordinal α_n with $\alpha_0 < \alpha_n < \mathfrak{c}$, such that $0 < x_{\alpha_n} < \frac{1}{n}$; then, letting $\hat{\alpha} = \sup_{n\in\mathbb{N}} \alpha_n$, it is easily checked that $\{0\} \cup \{x_\gamma \mid \hat{\alpha} < \gamma < \mathfrak{c}\}$ is an element of $\tau(0)$ which includes no τ_2-neighborhood of 0.

Now, we prove that (τ_2, τ) is an upwards anti-gap in $\Sigma_2(I)$. Indeed, let $\eta \in \Sigma_2(I)$ be such that $\tau_2 \subseteq \eta \subsetneqq \tau$. The strict inclusion $\eta \subsetneqq \tau$ and the equalities $\tau_2|_{I\setminus\{0\}} = \eta|_{I\setminus\{0\}} = \tau|_{I\setminus\{0\}}$ imply that there exist $\hat{\beta} \in \mathfrak{c}$ and $0 < \hat{\varepsilon} \leq 1$, such that no η-neighborhood of 0 is included in $[0,\hat{\varepsilon}[\cap(\{0\} \cup \{x_\gamma \mid \hat{\beta} < \gamma < \mathfrak{c}\})$. Notice that

$$M = \{0\} \cup \big\{x_\gamma \,\big|\, \hat{\beta} < \gamma < \mathfrak{c}\big\}$$

is an element of τ (in particular, of $\tau(0)$) which is not in η; otherwise, since $\sigma \subseteq \tau_2 \subseteq \eta$, $[0,\hat{\varepsilon}[\cap(\{0\} \cup \{x_\gamma \mid \hat{\beta} < \gamma < \mathfrak{c}\})$ would be in η (and in $\eta(0)$, as well). Thus, letting $\vartheta = \langle \eta \cup \{M\} \rangle$, we see that $\eta \subsetneqq \vartheta \subseteq \tau$; we claim that actually $\vartheta \subsetneqq \tau$—more precisely, that there is a $\beta \in \mathfrak{c}$ such that no ϑ-neighborhood of 0 is included in $\{0\} \cup \{x_\gamma \mid \gamma > \beta\}$—and this will complete the proof that (τ_2, τ) is an upwards anti-gap.

Towards a contradiction, suppose that for every $\beta \in \mathfrak{c}$ there is a $W_\beta \in \vartheta(0)$ which is included in $\{0\} \cup \{x_\gamma \mid \beta < \gamma < \mathfrak{c}\}$. Thanks to

Proposition 2.3, we may assume W_β to be of the form $V_\beta \cap M$, with $V_\beta \in \eta(0)$; therefore, we have the relation

$$\forall \beta \in \mathfrak{c} \colon V_\beta \cap M \subseteq \{0\} \cup \{x_\gamma \mid \beta < \gamma < \mathfrak{c}\}. \tag{4.3}$$

Now, since no $V' \in \eta(0)$ can be included in $M = \{0\} \cup \{x_\gamma \mid \hat\beta < \gamma < \mathfrak{c}\}$ (otherwise, $V' \cap [0, \hat\varepsilon[$ would be an element of $\eta(0)$ included in $[0, \hat\varepsilon[\cap (\{0\} \cup \{x_\gamma \mid \hat\beta < \gamma < \mathfrak{c}\})$, contrary to our initial choice of $\hat\beta$ and $\hat\varepsilon$), we may associate to every $\beta \in \mathfrak{c}$ an ordinal $\tilde\alpha(\beta) \in \mathfrak{c}$ such that

$$x_{\tilde\alpha(\beta)} \in V_\beta \setminus M = V_\beta \setminus \big(\{0\} \cup \big\{x_\gamma \,\big|\, \hat\beta < \gamma < \mathfrak{c}\big\}\big). \tag{4.4}$$

This implies, in particular, that $\tilde\alpha(\beta) \leq \hat\beta$ and that $\tilde\alpha(\beta) \neq \alpha_0$ for every $\beta \in \mathfrak{c}$. Let $\zeta = \max\big\{|\hat\beta|, |\alpha_0|, \omega\big\}$, and for every $\gamma \leq \hat\beta$, set $S_\gamma = \big\{\beta \in \mathfrak{c} \,\big|\, \tilde\alpha(\beta) = \gamma\big\}$; then $\bigcup_{\gamma \in \hat\beta + 1} S_\gamma = \mathfrak{c}$ (and $S_\gamma \cap S_{\gamma'} = \emptyset$ for $\gamma \neq \gamma'$), so that, since $\mathfrak{c} > \zeta$, there must exist at least one $\hat\gamma \leq \hat\beta$ such that $|S_{\hat\gamma}| > \zeta$. (Notice that we cannot infer the existence of a $\hat\gamma \leq \hat\beta$ with $|S_{\hat\gamma}| = \mathfrak{c}$, because the continuum could be a singular cardinal.) We also see that $\hat\gamma \neq \alpha_0$ (indeed, either $\alpha_0 > \hat\beta$ or, if $\alpha_0 \leq \hat\beta$, then $S_{\alpha_0} = \emptyset$); therefore, for every $\beta \in S_{\hat\gamma}$, the facts that $x_{\hat\gamma} = x_{\tilde\alpha(\beta)} \in V_\beta$ (by (4.4)) and that $V_\beta \in \eta(0) \subseteq \eta$, imply the existence of an $\tilde n(\beta) \in \mathbb{N}$ such that $\big]x_{\hat\gamma} - \frac{1}{\tilde n(\beta)}, x_{\hat\gamma} + \frac{1}{\tilde n(\beta)}\big[\cap \{x_\gamma \mid \hat\gamma \leq \gamma < \mathfrak{c}\} \subseteq V_\beta$ (remembering that $\tau|_{I\setminus\{0\}} = \tau_2|_{I\setminus\{0\}}$). Since $|S_{\hat\gamma}| > \zeta \geq \omega$, there will be an $S^* \subseteq S_{\hat\gamma}$ with $|S^*| > \zeta$, and an $n^* \in \mathbb{N}$, such that $\tilde n(\beta) = n^*$ for every $\beta \in S^*$; thus, in particular,

$$\forall \beta \in S^* \colon \Big]x_{\hat\gamma} - \frac{1}{n^*}, x_{\hat\gamma} + \frac{1}{n^*}\Big[\cap \{x_\gamma \mid \hat\gamma \leq \gamma < \mathfrak{c}\} \subseteq V_\beta. \tag{4.5}$$

Let $\lambda = \max\big\{\alpha_0, \hat\beta\big\} + \omega$; then the ordinal $\lambda + \omega$ ($= \max\big\{\hat\beta, \alpha_0\big\} + \omega + \omega$) has cardinality $\zeta = \max\big\{|\hat\beta|, |\alpha_o|, \omega\big\}$. Since $|S^*| > \zeta$, we may pick

$$\beta^* \in S^* \setminus (\lambda + \omega). \tag{4.6}$$

Now, λ is a limit ordinal; hence, from the properties of the indexing $\{x_\alpha \mid \alpha \in \mathfrak{c}\}$, it follows that $\{x_{\lambda+m} \mid m \in \omega\}$ is dense in I; in particular, there exists $m^* \in \omega$ such that $x_{\lambda+m^*} \in \big]x_{\hat\gamma} - \frac{1}{n^*}, x_{\hat\gamma} + \frac{1}{n^*}\big[$. Then, since $\lambda + m^* \geq \lambda > \hat\beta \geq \hat\gamma$, we deduce from (4.5) that $x_{\lambda+m^*} \in V_\beta$ for every $\beta \in S^*$; in particular, $x_{\lambda+m^*} \in V_{\beta^*}$. On the other hand, since $\lambda + m^* > \hat\beta$, we see that $x_{\lambda+m^*} \in M$; hence, by (4.3), we conclude that $x_{\lambda+m^*} \in \{0\} \cup \{x_\gamma \,|\, \beta^* < \gamma < \mathfrak{c}\}$, i.e., $\lambda + m^* = \alpha_0$ or $\lambda + m^* > \beta^*$. This leads to a contradiction, because, on the one hand, we see that $\alpha_0 < \lambda \leq \lambda + m^*$, and, on the other hand, from (4.6), it follows that $\beta^* \geq \lambda + \omega > \lambda + m^*$. □

Notice that if we assume the continuum to be regular, then the above example holds whatever is the one-to-one indexing $\{x_\alpha \,|\, \alpha \in \mathfrak{c}\}$ of I that has been fixed at the beginning.

Observe also that the topology produced in §3 is not strongly upper in $\Sigma_2(I)$, as it is not even upper; however, its construction cannot be carried out in ZFC. On the other hand, as was pointed out to the author by R. G. Wilson, we see that whenever τ is a non-minimal, non-upper topology in $\Sigma_2(X)$ (for a certain set X) and Y is an infinite set disjoint from X, then the topology σ, given by the disjoint sum of τ and the discrete topology on Y, is an upper, not strongly upper topology in $\Sigma_2(X \cup Y)$.

Remark 4.8. The topologies τ_1 and τ_2 are actually strongly upper in $\Sigma_2(I)$. Let us sketch the proof for τ_2. If τ is any T_2 topology on I with $\tau \subsetneqq \tau_2$, then there exists a point $x_{\hat{\alpha}} \in I$ such that τ is strictly coarser than τ_2 at $x_{\hat{\alpha}}$. Consider a new (T_2) topology τ' on I such that $\tau'|_{X\setminus\{x_{\hat{\alpha}}\}} = \tau_2|_{X\setminus\{x_{\hat{\alpha}}\}}$, while τ' coincides with τ at the point $x_{\hat{\alpha}}$ ((BP2) is fulfilled at the point $x_{\hat{\alpha}}$ simply because τ_2 is finer than τ). Then τ' is still strictly coarser than τ_2, and, in particular, it is strictly coarser at the point $x_{\hat{\alpha}}$; thus, there must exist $W \in \sigma(x_{\hat{\alpha}})$ such that no τ'-neighborhood of $x_{\hat{\alpha}}$ is included in $W \cap \{x_\gamma \,|\, \hat{\alpha} \leq \gamma < \mathfrak{c}\}$. This implies that no $V \in \tau'(x_{\hat{\alpha}})$ is included in $\{x_\gamma \,|\, \hat{\alpha} \leq \gamma < \mathfrak{c}\}$ (otherwise, since σ is easily seen to be the semiregularization of τ', it follows from Proposition 2.2 that $W \cap V$ is a τ'-neighborhood of $x_{\hat{\alpha}}$ included in $W \cap \{x_\gamma \,|\, \hat{\alpha} \leq \gamma < \mathfrak{c}\}$). Then $A = \{x_\gamma \,|\, \hat{\alpha} \leq \gamma < \mathfrak{c}\} \in \tau \setminus \tau'$, and we can prove, as in Example 4.3, that $\langle \tau' \cup \{A\} \rangle = \tau_2$; this implies, as in the proof of Theorem 4.3 [5], that there exists a topology τ'' on I such that $\tau' \subseteq \tau''$ and (τ'', τ_2) is a gap. Since $\tau'' \supseteq \tau' \supseteq \tau$, we are done.

The proof that τ_1 is strongly upper in $\Sigma_2(I)$ is rather more complicated, and we do not give it here so as not to make this section too long.

5. Lower topologies in $\mathscr{L}_1$ and Σ_2

The study of lower topologies appears to be rather different from that of upper topologies. Thanks to a basic characterization provided by [3], we can somehow "recognize" the lower topologies in $\mathscr{L}_1$ or in Σ_2 among the other ones; moreover, the notion turns out to be independent of either of the above structures we are considering, in the sense that if a Hausdorff topology is lower in $\mathscr{L}_1$, then it is lower in Σ_2, too.

Let us recall the notion of maximal point, introduced in [3].

Definition 5.1. A point p in a topological space X is said to be *maximal* if p is not isolated in X, and for every open $U \subseteq X$ such that $p \in \overline{U}$, the set $U \cup \{p\}$ is also open.

It is to be observed that a point $p \in X$ is maximal if and only if the set

$$\{A \setminus \{p\} \mid A \text{ is an open neighborhood of } p \text{ in } X\}$$

is a maximal filter of open sets (i.e., an *open ultrafilter* in the sense of [23]) in $X \setminus \{p\}$.

Then, Theorem 1.4 [3] states that a topology τ on X is a lower topology if and only if there exists a closed subset C of (X, τ) which contains a maximal point (of C). Such a strong result allows the authors to state, among other things, that if (X, τ) is a topological space with certain properties (for example, a compact T_2 space, or a sequential T_2 space, or a radial T_2 space), then τ cannot be lower on X. They leave open other questions of a similar kind. We solve two of them in this section.

First of all, we give a negative answer to Question 2.12 [3]. We recall that a space X is said to be *pseudoradial* if for every non-closed subset M of X there exists a cardinal ν and a ν-sequence $\{x_\alpha\}_{\alpha\in\nu}$ of elements of M, which converges to some point $y \in X \setminus M$ (i.e., for every neighborhood V of y there exists $\alpha \in \nu$ such that $\{x_{\alpha'} \mid \alpha' \geq \alpha\} \subseteq V$). The following result is folklore, but we give it a proof for the reader's convenience.

Lemma 5.2. *If M is a non-closed subset of a pseudoradial T_1 space X, then there exist a point $y \in X \setminus M$, a* regular *cardinal number ξ, and a ξ-sequence $\{x_\alpha\}_{\alpha\in\xi}$ which converges to y,* with $\alpha \mapsto x_\alpha$ one-to-one*.*

Proof: By pseudoradiality, there exist a cardinal ν, a $\bar{y} \in X \setminus M$, and a ν-sequence $\{y_\gamma\}_{\gamma\in\nu}$ in M which converges to $\bar{y}$. Let ξ be the cofinality of ν; then we know by general facts that there exists a strictly increasing $\vartheta\colon \xi \to \nu$ such that $\{\vartheta(\alpha) \mid \alpha \in \xi\}$ is cofinal in ξ. As a consequence, the ξ-sequence $\left\{y_{\vartheta(\alpha)}\right\}_{\alpha\in\xi}$ clearly converges to $\bar{y}$. Set, for the sake of simplicity, $x_\alpha = y_{\vartheta(\alpha)}$ for $\alpha \in \xi$.

Consider that for every $\alpha \in \xi$, the set $S_\alpha = \{\alpha' \in \xi \mid x_{\alpha'} = x_\alpha\}$ cannot be cofinal in ξ; otherwise, every neighborhood of $\bar{y}$ (which belongs to $X \setminus M$) would contain x_α (which belongs to M), while X is T_1. Thus, $|S_\alpha| < \xi$ for every $\alpha \in \xi$; this allows us to define by transfinite induction a function $j\colon \xi \to \xi$ by

$$j(\beta) = \min\Big(\xi \setminus \bigcup_{\beta'<\beta} S_{j(\beta')}\Big).$$

(Consider that a union of less than ξ many sets, each of cardinality less than ξ, cannot have cardinality ξ, because ξ is regular.) It is clear from the definitions of j and of the sets S_α that $x_{j(\beta)} \neq x_{j(\beta')}$ for $\beta' < \beta$, i.e., $\beta \mapsto x_{j(\beta)}$ is one-to-one; clearly, this implies that j itself is one-to-one. We prove that $\left\{x_{j(\beta)}\right\}_{\beta\in\xi}$ converges to $\bar{y}$: Given an arbitrary neighborhood V of $\bar{y}$, let $\hat{\alpha} \in \xi$ be such that $\{x_\alpha \mid \alpha \geq \hat{\alpha}\} \subseteq V$. Since j is one-to-one, we have the inequalities $|\{\beta \in \xi \mid j(\beta) < \hat{\alpha}\}| \leq |\hat{\alpha}| < \xi$; hence, there exists $\hat{\beta} \in \xi$ such that $j(\beta) \geq \hat{\alpha}$ for every $\beta \in \xi$ with $\beta \geq \hat{\beta}$, which implies that $\left\{x_{j(\beta)} \,\middle|\, \beta \geq \hat{\beta}\right\} \subseteq V$. □

Theorem 5.3. *A pseudoradial T_2 space cannot contain a maximal point.*

Proof: Towards a contradiction, suppose $\bar{x}$ is a maximal point in a pseudoradial T_2 space X. Then $\bar{x} \in \overline{X \setminus \{\bar{x}\}}$, so that (by the above lemma) there must exist a regular cardinal ξ and a ξ-sequence $\{x_\alpha\}_{\alpha\in\xi}$ in $X \setminus \{\bar{x}\}$, with $\alpha \mapsto x_\alpha$ one-to-one, which converges to a point outside $X \setminus \{\bar{x}\}$, i.e., to $\bar{x}$. By transfinite induction, we can associate to every $\beta \in \xi$ an ordinal $j(\beta) \in \xi$ and an open set A_β such that

(1) $x_{j(\beta)} \in A_\beta$;
(2) $\bar{x} \notin \overline{A_\beta}$;
(3) $A_{\beta'} \cap A_\beta = \emptyset$ for $\beta' < \beta$;
(4) $j(\beta') < j(\beta)$ for $\beta' < \beta$.

Indeed, suppose we have defined $j(\beta)$ and A_β for $\beta < \hat\beta$, in such a way that (1)–(4) are satisfied. Due to (2), it is clear that for every $\beta < \hat\beta$ there must exist $\alpha^*(\beta)$ such that

$$\forall \alpha \in \xi \colon \big(\alpha > \alpha^*(\beta) \Longrightarrow x_\alpha \notin A_\beta\big).$$

(Consider that every neighborhood of $\bar x$ contains a tail of the ξ-sequence $\{x_\alpha\}_{\alpha\in\xi}$.) By regularity of ξ, the ordinal

$$\alpha^\sharp = \sup\{\alpha^*(\beta) \mid \beta < \hat\beta\}$$

must belong to ξ; this implies that $\bar x \notin \overline{\bigcup_{\beta<\hat\beta} A_\beta}$. (Otherwise, by maximality, $\{\bar x\} \cup \big(\bigcup_{\beta<\hat\beta} A_\beta\big)$ would be an open neighborhood of $\bar x$ containing no x_α for $\alpha > \alpha^\sharp$, which contradicts the fact that $\{x_\alpha\}_{\alpha\in\xi}$ converges to $\bar x$.) Therefore, take an open neighborhood V of $\bar x$, missing the set $\bigcup_{\beta<\hat\beta} A_\beta$; then $\{x_\alpha\}_{\alpha\in\xi}$ is eventually in V, and we may define $j\big(\hat\beta\big)$ to be an element of ξ such that $\big\{x_\alpha \,\big|\, j\big(\hat\beta\big) \le \alpha < \xi\big\} \subseteq V$. Thus, (4) is satisfied for $\beta = \hat\beta$; indeed, for every $\beta' < \hat\beta$, it follows from (1) that $x_{j(\beta')} \in A_{\beta'}$, and hence $x_{j(\beta')} \notin V$. Now, let $A_{\hat\beta}$ be an open neighborhood of $x_{j(\hat\beta)}$ such that $A_{\hat\beta} \subseteq V$ and $\bar x \notin \overline{A_{\hat\beta}}$; then (1), (2), and (3) are satisfied for $\beta = \hat\beta$.

Therefore, j is well defined and fulfills (1)–(4) for every $\beta \in \xi$. In particular, it follows from (4) that j is strictly increasing, so that $\big\{x_{j(\beta)}\big\}_{\beta\in\xi}$ converges to $\bar x$. Letting S' and S'' be the set of all odd and all even ordinals, respectively, in ξ, we see that $\bar x \in \overline{\{x_{j(\beta)} \mid \beta \in S'\}}$ and $\bar x \in \overline{\{x_{j(\beta)} \mid \beta \in S''\}}$, which implies by (1) that $\bar x \in \overline{\bigcup_{\beta\in S'} A_\beta}$ and $\bar x \in \overline{\bigcup_{\beta\in S''} A_\beta}$. By maximality, both $\{\bar x\} \cup \bigcup_{\beta\in S'} A_\beta$ and $\{\bar x\} \cup \bigcup_{\beta\in S''} A_\beta$ are open neighborhoods of $\bar x$, and hence, the same holds for their intersection. But by (3), we have the equality $\big(\{\bar x\} \cup \bigcup_{\beta\in S'} A_\beta\big) \cap \big(\{\bar x\} \cup \bigcup_{\beta\in S''} A_\beta\big) = \{\bar x\}$, so that $\bar x$ is isolated in X, which is, of course, a contradiction. $\square$

Notice that pseudoradiality is not a hereditary property, in general (actually, one can easily prove that a space is hereditarily pseudoradial if and only if it is radial); however, it is clear that closed subspaces of pseudoradial spaces are pseudoradial. Thus, the above result, combined with Proposition 1.4 [3], gives the following corollary.

Corollary 5.4. *No pseudoradial Hausdorff topology on a set X can be lower.*

Our second result in this section solves the second part of Question 2.6 [3]; namely, we will prove that a countably compact regular topology cannot be lower. As in the previous problem, since closed subspaces of countably compact spaces are countably compact, it will be sufficient to show that a regular countably compact space cannot have a maximal point. Actually, we will deduce this fact from a slightly more general result, where we assume regularity only at a single point. In accordance with the beginning of §2 [5], we will say that a point $\bar{x}$ in a topological space X is a *regular point* if

$$\text{(5.1)} \quad \forall V \text{ neighborhood of } \bar{x}\colon \exists W \text{ neighborhood of } \bar{x}\colon \overline{W} \subseteq V.$$

Theorem 5.5. *If τ is a Hausdorff countably compact topology on a set X, then (X, τ) cannot have a regular point which is maximal.*

Proof: Towards a contradiction, suppose a countably compact T_2 space X to have a maximal point $\bar{x}$ satisfying (5.1). Let Γ be the collection of all collections $\mathscr{A}$ of nonempty open subsets of X (actually, of $X \setminus \{\bar{x}\}$) satisfying the following conditions:

(1) $\mathscr{A}$ is infinite;
(2) $\forall A \in \mathscr{A}: \bar{x} \notin \overline{A}$;
(3) the elements of $\mathscr{A}$ are pairwise disjoint.

Consider that Γ is not empty. Indeed, define by induction for every $n \in \omega$ a nonempty open subset A_n of $X \setminus \{\bar{x}\}$, such that

$$\text{(5.2)} \qquad \bar{x} \notin \overline{A_n} \ \text{ and } \ A_{n'} \cap A_n = \emptyset \ \text{ for } \ n' < n,$$

in the following way. Since $X \setminus \{\bar{x}\}$ is nonempty (otherwise, $\bar{x}$ would not be adherent to it), we may consider a point $\bar{y} \in X \setminus \{\bar{x}\}$ and (as X is T_2) an open neighborhood A_0 of $\bar{y}$ such that $\bar{x} \notin \overline{A_0}$. Then, supposing we have defined A_n for every $n < \bar{n} \in \omega$ in such a way that (5.2) is fulfilled, the set $V = X \setminus \left(\bigcup_{n<\bar{n}} \overline{A_n}\right)$ will be an open neighborhood of $\bar{x}$; using again the fact that $\bar{x}$ is not isolated in X, we may consider $\tilde{y} \in V$ and an open neighborhood $A_{\bar{n}}$ of $\tilde{y}$ such that $A_{\bar{n}} \subseteq V$ and $\bar{x} \notin \overline{A_{\bar{n}}}$. Then (5.2) is satisfied for $n = \bar{n}$.

Now, since the collection Γ is easily seen to be also inductive, by Zorn's lemma, it contains a maximal element $\mathscr{A}^*$. Then $\bigcup \mathscr{A}^*$ is dense in $X \setminus \{\bar{x}\}$ (otherwise, it would admit a proper extension in

Γ); thus, $\bar{x}$, being adherent to $X \setminus \{\bar{x}\}$, is adherent to $\bigcup \mathscr{A}^*$, too. By maximality, it follows that

$$V = \{\bar{x}\} \cup \bigcup \mathscr{A}^* \tag{5.3}$$

is an open neighborhood of $\bar{x}$ in X; hence, we can use (5.1) to obtain an open neighborhood W of $\bar{x}$ such that $\overline{W} \subseteq V$. Let

$$\tilde{\mathscr{A}} = \{A \in \mathscr{A}^* \mid W \cap A \neq \emptyset\}, \tag{5.4}$$

and notice that $\tilde{\mathscr{A}}$ must be infinite; otherwise, using (2) for $\mathscr{A} = \mathscr{A}^*$, $W \cap \left(X \setminus \bigcup_{A \in \tilde{\mathscr{A}}} \overline{A}\right)$ would be an open neighborhood of $\bar{x}$ missing the set $\bigcup \mathscr{A}^*$, while we have proved above that $\bar{x} \in \overline{\bigcup \mathscr{A}^*}$. Thus, we may partition $\tilde{\mathscr{A}}$ into two infinite sets $\tilde{\mathscr{A}_0}$ and $\tilde{\mathscr{A}_1}$; letting $\Omega_i = \bigcup \tilde{\mathscr{A}_i}$ for $i = 0, 1$, we see that $\bar{x}$ cannot be adherent to both Ω_0 and Ω_1; otherwise, by maximality, $\{\bar{x}\} \cup \Omega_i$ would be a neighborhood of $\bar{x}$ for $i = 0, 1$, and the same would hold for $\left(\{\bar{x}\} \cup \Omega_0\right) \cap \left(\{\bar{x}\} \cup \Omega_1\right) = \{\bar{x}\}$ (we have used (3) again for $\mathscr{A} = \mathscr{A}^*$), while $\bar{x}$ is not isolated in X. Therefore, let $\hat{\imath} \in \{0, 1\}$ be such that $\bar{x} \notin \overline{\Omega_{\hat{\imath}}}$; for every $A \in \tilde{\mathscr{A}_{\hat{\imath}}}$, pick a point

$$x_A \in W \cap A$$

(taking (5.4) into account), so that, in particular, the set

$$M = \{x_A \mid A \in \tilde{\mathscr{A}_{\hat{\imath}}}\}$$

is infinite (still by (3)). By countable compactness, there is in X an accumulation point $\bar{y}$ of M, and we see that $\bar{y} \in \overline{M} \subseteq \overline{W} \subseteq V$. Then, by (5.3), we have either $\bar{y} = \bar{x}$ or $\bar{y} \in A^*$ for a (unique) $A^* \in \mathscr{A}^*$. But the former case is impossible, as $M \subseteq \bigcup \tilde{\mathscr{A}} = \Omega_{\hat{\imath}}$ and hence, $\bar{y} \in \overline{M} \subseteq \overline{\Omega_{\hat{\imath}}}$, while $\bar{x} \notin \overline{\Omega_{\hat{\imath}}}$; and in the latter case, we have a contradiction with the fact that $\bar{y}$ is an accumulation point for M, as A^* is an open neighborhood of $\bar{y}$ which meets M in at most one point (consider that, since $\tilde{\mathscr{A}_{\hat{\imath}}} \subseteq \tilde{\mathscr{A}} \subseteq \mathscr{A}^*$, for every $A \in \tilde{\mathscr{A}_{\hat{\imath}}}$ with $A \neq A^*$, we have the equality $A \cap A^* = \emptyset$, so that $x_A \notin A^*$). □

As pointed out to the author by O. T. Alas, the same argument used for the above proof shows, in fact, that Hausdorff feebly compact topologies cannot have a regular point which is maximal.

However, this *does not* imply that feebly compact (nor pseudocompact) regular topologies cannot be lower in Σ_2, since such properties are not inherited, in general, by closed subsets.

It is to be mentioned that the first part of Question 2.6 [3] was solved by Nathan Carlson, who exhibited a Hausdorff countably compact H-closed lower topology with countable tightness. See [5, Theorem 3.7].

6. The posets of regular and of locally compact topologies

In this section, we deal with the posets $\Sigma_3(X)$ and $\Sigma_{lc}(X)$ of all regular and of all locally compact T_2 topologies, respectively, on a set X. First, we show that there can be two locally compact (T_2) topologies on X, which differ at more than one point of X and form a gap in $\Sigma_{lc}(X)$; this gives a negative answer to an implicit question raised in [1]. Actually, the authors show in Lemma 3.1, that if two locally compact topologies σ, τ on X are such that (X, τ) is totally separated (i.e., totally disconnected, according to the terminology recommended in §6.2, Historical and bibliographic notes [6]), and (σ, τ) is a gap in $\Sigma_{lc}(X)$, then σ and τ must differ at exactly one point. Then they state that they do not know whether the assumption of total separation may be dropped in such a result. Notice that, both in $\Sigma_3(X)$ and in the poset $\Sigma_t(X)$ of all Tychonoff topologies on a set X, it holds that if two topologies form a gap, then they must differ at exactly one point of X (see [1, Lemma 2.1 and Corollary 2.3], where the former result, as the authors point out, is just a reformulation of Lemma 22 [12]).

The example we provide here takes as σ the Euclidean topology on the upper half-plane X of $\mathbb{R}^2$ (including the coordinate x-axis), and as τ a refinement of σ which makes X the disjoint topological sum of a real line and of an open half-plane of $\mathbb{R}^2$ itself. In the proof, we will take d to be the Euclidean metric on X, and for every $\mathbf{x} \in X$ and $\varepsilon > 0$, we will denote by $S_d(\mathbf{x}, \varepsilon)$ the open ball centered at $\mathbf{x}$ and of radius ε, i.e., $S_d(\mathbf{x}, \varepsilon) = \{\mathbf{y} \in X \mid d(\mathbf{x}, \mathbf{y}) < \varepsilon\}$.

Example 6.1. *Let* $X = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$*, and set* $Y = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ *and* $Z = \{(x, 0) \mid x \in \mathbb{R}\}$*. Let* σ *be the Euclidean topology on* X *and* τ *be the topology on* X *such that* $\tau|_Y = \sigma|_Y$*, while each* $(x, 0) \in Z$ *has a fundamental system of (open)* τ*-neighborhoods* *given by*

$$\{\,]x-\varepsilon, x+\varepsilon[\,\times\{0\}\,|\,\varepsilon>0\}.$$

Then (σ,τ) *is a gap in* Σ_{lc}.

Proof: Let γ be an arbitrary topology in Σ_{lc} such that $\sigma\subseteq\gamma\subseteq\tau$. We want to prove that either $\gamma=\sigma$ or $\gamma=\tau$.

First of all, we will show that for every $\mathbf{x}\in Z$, the topology γ at the point $\mathbf{x}$ coincides with either σ or τ. Indeed, given an arbitrary $\bar{\mathbf{x}}\in Z$, if there exists $W\in\gamma(\bar{\mathbf{x}})$ with $W\subseteq Z$, then for every $V\in\gamma(\bar{\mathbf{x}})$ we may assume that $V\subseteq Z$. Since $\sigma|_Z=\tau|_Z$, there will be $U\in\sigma(\bar{\mathbf{x}})\subseteq\gamma(\bar{\mathbf{x}})$ such that $U\cap Z=V$. Then $U\cap W\subseteq U\cap Z=V$, and $U\cap W\in\gamma(\bar{\mathbf{x}})$. Therefore, we have proved that for every (open) γ-neighborhood V of $\bar{\mathbf{x}}$ in X, there is a γ-neighborhood of $\bar{\mathbf{x}}$ in X which is included in V; since $\gamma\subseteq\tau$, γ and τ coincide at $\bar{\mathbf{x}}$.

Suppose now that for every $W\in\gamma(\bar{\mathbf{x}})$, we have the inequality $W\cap Y\neq\emptyset$. We want to prove that, in this case, γ coincides with σ at $\bar{\mathbf{x}}$. Indeed, towards a contradiction, assume there is a $V\in\gamma(\bar{\mathbf{x}})$ such that for every $n\in\mathbb{N}\,(=\omega\setminus\{0\})$, we have the inequality $S_d\left(\bar{\mathbf{x}},\frac{1}{n}\right)\setminus V\neq\emptyset$; since (X,γ) is locally compact, we may further assume that $\mathrm{Cl}_\gamma V$ is a compact subspace of (X,γ). Also, using the fact that $\gamma|_Z=\sigma|_Z$, it is easily seen that we may obtain the (formally) stronger inequality

$$\forall n\in\mathbb{N}\colon\ \Big(S_d\Big(\bar{\mathbf{x}},\frac{1}{n}\Big)\setminus V\Big)\cap Y\neq\emptyset. \tag{6.1}$$

On the other hand, notice that we have, as well, the inequality

$$\forall n\in\mathbb{N}\colon\ S_d\Big(\bar{\mathbf{x}},\frac{1}{n}\Big)\cap V\cap Y\neq\emptyset; \tag{6.2}$$

indeed, since $\sigma\subseteq\gamma$, every $S_d\left(\bar{\mathbf{x}},\frac{1}{n}\right)\cap V$ is an element of γ containing $\bar{\mathbf{x}}$, and we have assumed before that every γ-neighborhood of $\bar{\mathbf{x}}$ meets Y. Now, for every $n\in\mathbb{N}$, since $S_d\left(\bar{\mathbf{x}},\frac{1}{n}\right)\cap Y$ is clearly connected (with respect to each of σ, γ, and τ), and since $S_d\left(\bar{\mathbf{x}},\frac{1}{n}\right)\cap V\cap Y$ is γ-open (in X, hence also in $S_d\left(\bar{\mathbf{x}},\frac{1}{n}\right)\cap Y$), (6.1) and (6.2) combine to show that $S_d\left(\bar{\mathbf{x}},\frac{1}{n}\right)\cap V\cap Y$ is not γ-closed in

$S_d\left(\bar{\mathbf{x}}, \frac{1}{n}\right) \cap Y$; i.e., there exists

$$\begin{aligned}\mathbf{x}_n \in \mathrm{Cl}_\gamma &\left(S_d\Big(\bar{\mathbf{x}}, \frac{1}{n}\Big) \cap V \cap Y\right) \\ &\cap \left(\Big(S_d\Big(\bar{\mathbf{x}}, \frac{1}{n}\Big) \cap Y\Big) \setminus \Big(S_d\Big(\bar{\mathbf{x}}, \frac{1}{n}\Big) \cap V \cap Y\Big)\right).\end{aligned} \tag{6.3}$$

In particular, (6.3) implies that

$$\forall n \in \mathbb{N}\colon \mathbf{x}_n \in \Big(S_d\Big(\bar{\mathbf{x}}, \frac{1}{n}\Big) \cap \mathrm{Cl}_\gamma V\Big) \setminus V. \tag{6.4}$$

Since we have assumed $\mathrm{Cl}_\gamma V$ to be γ-compact, the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ must have a γ-cluster point (hence, also a σ-cluster point) $\mathbf{x}^* \in \mathrm{Cl}_\gamma V\,(\subseteq X)$. But in (X, σ), the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ converges to $\bar{\mathbf{x}}$, so that $\bar{\mathbf{x}}$ is also its unique cluster point; therefore, $\bar{\mathbf{x}} = \mathbf{x}^*$. Now, by (6.4), V is a γ-neighborhood of $\bar{\mathbf{x}}$ missing each $\mathbf{x}_n$, and this contradicts the fact that $\bar{\mathbf{x}}$ is a γ-cluster point of $(\mathbf{x}_n)_{n\in\mathbb{N}}$.

Thus, letting

$$S_1 = \{x \in \mathbb{R} \mid \gamma \text{ coincides with } \sigma \text{ at } (x, 0)\}$$

and

$$S_2 = \{x \in \mathbb{R} \mid \gamma \text{ coincides with } \tau \text{ at } (x, 0)\},$$

it follows that $\{S_1, S_2\}$ is a partition of $\mathbb{R}$. We want to show that either $S_1 = \mathbb{R}$ or $S_2 = \mathbb{R}$; in this case, since $\sigma|_Y = \gamma|_Y = \tau|_Y$ and $Y \in \sigma\,(\subseteq \gamma \subseteq \tau)$, the equality $S_1 = \mathbb{R}$ will imply that $\gamma = \sigma$, while the equality $S_2 = \mathbb{R}$ will imply that $\gamma = \tau$.

Towards a contradiction, suppose $S_1 \neq \mathbb{R}$ and $S_2 \neq \mathbb{R}$; since $\mathbb{R}$ is connected (when talking about the real line, we always think of it as endowed with the Euclidean topology), we see that S_1 or S_2 is not closed in $\mathbb{R}$.

Case 1: There is $x_2 \in \overline{S}_1 \setminus S_1$.

Consider the τ-neighborhood $W =]x_2 - 1, x_2 + 1[\, \times \{0\}$ of $(x_2, 0)$ in X; since $x_2 \in S_2$, there exists a $V \in \gamma\big((x_2, 0)\big)$ such that $V \subseteq W$. In particular, from $V \subseteq Z$, $\gamma|_Z = \sigma|_Z$, and $x_2 \in \overline{S_1}$, we infer that there exists $s_1 \in S_1$ such that $(s_1, 0) \in V$. Thus, V turns out to be a γ-neighborhood of $(s_1, 0)$ entirely included in Z, which is impossible as γ coincides with σ at $(s_1, 0)$.

Case 2: There is $x_1 \in \overline{S}_2 \setminus S_2$.

Fix a $V \in \sigma\big((x_1, 0)\big)$ such that $\mathrm{Cl}_\gamma V$ is compact in (X, γ). Since $x_1 \in S_1$, there will be an $\varepsilon > 0$ such that $]x_1 - \varepsilon, x_1 + \varepsilon[\, \times [0, \varepsilon[\, \subseteq V$; then, by $x_1 \in \overline{S_2}$, we may consider an $s_2 \in\,]x_1 - \varepsilon, x_1 + \varepsilon[\, \cap S_2$. Letting $\mathbf{z}_n = \big(s_2, \frac{\varepsilon}{n+1}\big)$ for every $n \in \mathbb{N}$, we see that the sequence $(\mathbf{z}_n)_{n\in\mathbb{N}}$ is entirely contained in $]x_1 - \varepsilon, x_1 + \varepsilon[\, \times [0, \varepsilon[\, \subseteq V \subseteq \mathrm{Cl}_\gamma V$ and σ-converges to $(s_2, 0)$. Since $\mathrm{Cl}_\gamma V$ is γ-compact, the above sequence will have a γ-cluster point (hence, a σ-cluster point) $\mathbf{z}$, and from the fact that $(\mathbf{z}_n)_{n\in\mathbb{N}} \longrightarrow (s_2, 0)$ in (X, σ), we obtain the equality $\mathbf{z} = (s_2, 0)$. This leads to a contradiction, as $s_2 \in S_2$ implies that there is a γ-neighborhood W of $(s_2, 0)$ with $W \subseteq Z$; hence, W does not contain any $\mathbf{z}_n$. □

Now we consider the poset $\Sigma_3(X)$ of all regular topologies on a set X. In [1, Theorem 2.14], it is proved that every T_3 topology on a set X, which is not feebly compact, is upper in $\Sigma_3(X)$. Then Question 2.15 [1] asks whether every non-minimal regular topology on a set X is upper in $\Sigma_3(X)$. We give here a partial answer, showing that every Tychonoff topology on X, which is non-minimal in $\Sigma_3(X)$, is upper (actually, strongly upper) there. As a consequence, a possible counterexample to the above question should have rather peculiar properties, i.e., should be a feebly compact regular space which is not completely regular.

First of all, we prove a result which constitutes a kind of analogue of Theorem 4.3 [5] for the poset Σ_3.

Lemma 6.2. *Let $\tau \in \Sigma_3(X)$, and suppose there are a point $\bar{x} \in X$, a $V \in \tau$ with $\bar{x} \in V$, and a $\sigma \in \Sigma_3(X)$, such that*

(1) $\sigma \subsetneqq \tau$;
(2) $\sigma|_{X\setminus\{\bar{x}\}} = \tau|_{X\setminus\{\bar{x}\}}$;
(3) $\langle \sigma \cup \{V\}\rangle = \tau$.

Then there exists a topology $\hat{\gamma} \in \Sigma_3(X)$ such that $\sigma \subseteq \hat{\gamma} \subsetneqq \tau$, and $(\hat{\gamma}, \tau)$ is a gap in $\Sigma_3(X)$ (in particular, τ upper in $\Sigma_3(X)$).

Proof: Let $\Gamma = \big\{\gamma \in \Sigma_3(X) \,\big|\, \sigma \subseteq \gamma \subsetneqq \tau\big\}$; then $\Gamma \neq \emptyset$, as $\sigma \in \Gamma$. We prove that Γ is inductive. Indeed, let Θ be a chain in Γ and set $\vartheta = \langle \bigcup \Theta\rangle$. Notice that $\vartheta \neq \tau$, in particular, that

$$\forall W \in \vartheta\colon (\bar{x} \in W \Longrightarrow W \setminus V \neq \emptyset).$$

Indeed, towards a contradiction, suppose there is a $W \in \vartheta(\bar{x})$ with $W \subseteq V$. Since a base for $\langle \bigcup \Theta\rangle$ is given by all finite intersections of elements of $\bigcup\Theta$, there will be sets $T_1,\dots,T_n$ and topologies $\vartheta_1,\dots,\vartheta_n\in\Theta$, with $T_i\in\vartheta_i$ for every $i=1,\dots,n$, such that $\bar{x}\in T_1\cap\cdots\cap T_n\subseteq W$. Let $\hat{\imath}\in\{1\dots,n\}$ be such that $\vartheta_{\hat{\imath}}=\max\{\vartheta_1,\dots,\vartheta_m\}$ (with respect to inclusion), so that $T_1\cap\cdots\cap T_n\in\vartheta_{\hat{\imath}}$; also, for every $x\in V\setminus\{\bar{x}\}$, let $S_x\in\vartheta_{\hat{\imath}}$ be such that $x\in S_x\subseteq V\setminus\{\bar{x}\}$ (an S_x like this does exist because $\tau|_{X\setminus\{\bar{x}\}}=\vartheta_{\hat{\imath}}|_{X\setminus\{\bar{x}\}}$ by the definition of Γ, $V\in\tau$ and $X\setminus\{\bar{x}\}$ is open both in $\vartheta_{\hat{\imath}}$ and in τ). Then clearly, $\left(\bigcup_{x\in V\setminus\{\bar{x}\}}S_x\right)\cup(T_1\cap\cdots\cap T_n)=V$, and since $T_1\cap\cdots\cap T_n$ and S_x are in $\vartheta_{\hat{\imath}}$, it follows that $V\in\vartheta_{\hat{\imath}}$. But then, by (3), $\tau=\langle\sigma\cup\{V\}\rangle\subseteq\vartheta_{\hat{\imath}}$, contradicting the definition of Γ.

Now let us prove that the space (X,ϑ) is regular. Of course, (X,ϑ) is T_1. Given any $x\in X$ and $U\in\vartheta(x)$, if $x\neq\bar{x}$, then we may assume without loss of generality that $\bar{x}\notin U$—so that $U\in\vartheta|_{X\setminus\{\bar{x}\}}$, too. Since, by (2), $\sigma|_{X\setminus\{\bar{x}\}}=\vartheta|_{X\setminus\{\bar{x}\}}=\tau|_{X\setminus\{\bar{x}\}}$ and $X\setminus\{\bar{x}\}$ is σ-open, we have $U\in\vartheta|_{X\setminus\{\bar{x}\}}=\sigma|_{X\setminus\{\bar{x}\}}\subseteq\sigma$. Therefore, there exists $W\in\sigma\,(\subseteq\vartheta)$ such that $x\in W\subseteq\mathrm{Cl}_\sigma W\subseteq U$, which implies that $\mathrm{Cl}_\vartheta W\subseteq\mathrm{Cl}_\sigma W\subseteq U$. Consider now the case where $x=\bar{x}$; then, as before, we know that there are sets $T_1,\dots,T_n$ and topologies $\vartheta_1,\dots,\vartheta_n\in\Theta$, with $T_i\in\vartheta_i$ for every $i=1,\dots,n$, such that $\bar{x}\in T_1\cap\cdots\cap T_n\subseteq U$. Then, for every $i\in\{1,\dots,n\}$, there must exist an $S_i\in\vartheta_i$ such that $\bar{x}\in S_i\subseteq\mathrm{Cl}_{\vartheta_i}S_i\subseteq T_i$. As a consequence, $S_1\cap\cdots\cap S_n$ is an open ϑ-neighborhood of $\bar{x}$ such that $\mathrm{Cl}_\vartheta(S_1\cap\cdots\cap S_n)\subseteq\mathrm{Cl}_\vartheta S_1\cap\cdots\cap\mathrm{Cl}_\vartheta S_n\subseteq\mathrm{Cl}_{\vartheta_1}S_1\cap\cdots\cap\mathrm{Cl}_{\vartheta_n}S_n\subseteq T_1\cap\cdots\cap T_n\subseteq U$. (Notice, in passing, that each of the inclusions $\mathrm{Cl}_{\vartheta_i}S_i\subseteq\mathrm{Cl}_\vartheta S_i$ is, in fact, an equality.)

Thus, we have proved that $\vartheta\in\Gamma$, and hence, that Γ is inductive. Therefore, by Zorn's lemma, there is a maximal element $\hat{\gamma}$ in Γ. Clearly, $(\hat{\gamma},\tau)$ forms a gap in $\Sigma_3(X)$, as every $\gamma\in\Sigma_3(X)$ with $\hat{\gamma}\subseteq\gamma\subsetneqq\tau$ would automatically be finer than or equal to σ and hence, would belong to Γ, so that $\gamma=\hat{\gamma}$ by maximality. ☐

Theorem 6.3. *Every Tychonoff topology τ on a set X, which is not minimal in $\Sigma_3(X)$, is strongly upper there.*

Proof: Let σ' be an arbitrary element of $\Sigma_3(X)$ such that $\sigma'\subsetneqq\tau$. Fix a point $\bar{x}\in X$ such that σ' is strictly coarser than τ at $\bar{x}$, and define a new topology σ^* such that $\sigma^*|_{X\setminus\{\bar{x}\}}=\tau|_{X\setminus\{\bar{x}\}}$, while a fundamental system of (open) neighborhoods for σ^* at $\bar{x}$ is given by $\{A\in\sigma'\,|\,\bar{x}\in A\}$. Of course, σ^* is still T_1 and is finer than (or equal to) σ'; we prove that σ^* is regular. Indeed, given $x\in X$, if $x = \bar{x}$, then for every $V \in \sigma^*(\bar{x})$ there exists $W \in \sigma'$ with $\bar{x} \in W \subseteq V$; therefore, taking $T \in \sigma'$ with $x \in T \subseteq \mathrm{Cl}_{\sigma'}T \subseteq W$, we also see that T is a σ^*-neighborhood of $\bar{x}$ such that $\mathrm{Cl}_{\sigma^*}T = \mathrm{Cl}_\tau T \subseteq \mathrm{Cl}_{\sigma'}T \subseteq W \subseteq V$. If, on the contrary, $x \neq \bar{x}$, then consider an arbitrary $V \in \sigma^*(x)$; we may assume that $\bar{x} \notin V$, so that $V \in \tau$, too. Since σ' is T_2, there is $W \in \sigma'(x)\,(\subseteq \tau(x))$ such that

$$\bar{x} \notin \mathrm{Cl}_{\sigma'}W; \tag{6.5}$$

and since τ is regular, there is $T \in \tau$ such that

$$x \in T \subseteq \mathrm{Cl}_\tau T \subseteq V. \tag{6.6}$$

Then $W \cap T$ is an open τ-neighborhood of x, and $\bar{x} \notin \mathrm{Cl}_{\sigma^*}(W \cap T)$ because of (6.5) and the fact that σ^* and σ' coincide at $\bar{x}$. As a consequence, and using (6.6), we conclude that $\mathrm{Cl}_{\sigma^*}(W \cap T) = \mathrm{Cl}_\tau(W \cap T) \subseteq \mathrm{Cl}_\tau T \subseteq V$.

Now, we want to modify σ^* so that we can apply Lemma 6.2. Since σ^* is strictly coarser than τ at $\bar{x}$ (as it coincides there with σ'), there exists a $V^* \in \tau$ with $\bar{x} \in V^*$ such that $T \setminus V^* \neq \emptyset$ for every σ^*-neighborhood T of $\bar{x}$. By complete regularity, there exists a τ-continuous $f\colon X \to [0,1]$ such that $f(\bar{x}) = 0$ and $f(x) = 1$ for every $x \in X \setminus V^*$. Set $V = f^{-1}\big(\big[0, \frac{1}{3}\big[\big)$ and $V_n = f^{-1}\big(\big[0, 1 - \frac{1}{n+1}\big[\big)$ for every $n \in \mathbb{N}$, so that, in particular, $\mathrm{Cl}_\tau V_n = \mathrm{Cl}_{\sigma^*}V_n \subseteq V_{n+1} \subseteq V^*$ and

$$\begin{aligned} \mathrm{Cl}_\tau\big(X \setminus \mathrm{Cl}_\tau V_{n+1}\big) &\subseteq \mathrm{Cl}_\tau(X \setminus V_{n+1}) \\ &= X \setminus V_{n+1} \subseteq X \setminus \mathrm{Cl}_\tau V_n \end{aligned} \tag{6.7}$$

for every $n \in \mathbb{N}$. Then let σ be the topology on X such that $\sigma|_{X\setminus\{\bar{x}\}} = \tau|_{X\setminus\{\bar{x}\}}$, while the point $\bar{x}$ is given the fundamental system of (open) neighborhoods

$$\big\{W_{S,T,n} \,\big|\, S \in \sigma^* \text{ with } \bar{x} \in S, \quad T \in \tau \text{ with } \bar{x} \in T \text{ and } T \subseteq V,\ n \in \mathbb{N}\big\},$$

where, for every σ^*-open neighborhood S of $\bar{x}$, for every τ-open neighborhood T of $\bar{x}$ included in V, and for every $n \in \mathbb{N}$, we set

$$W_{S,T,n} = T \cup \big(S \setminus \mathrm{Cl}_\tau V_n\big) = T \cup \big(S \setminus \mathrm{Cl}_{\sigma^*}V_n\big).$$

It is easy to check that σ is actually a topology on X. Clearly, $\sigma^* \subseteq \sigma \subseteq \tau$; moreover, the fact that $W_{S,T,n} \setminus V^* = (T \setminus V^*) \cup (S \setminus \mathrm{Cl}_\tau V_n) \setminus V^* \supseteq S \setminus V^* \neq \emptyset$ for every basic σ-neighborhood $W_{S,T,n}$ of $\bar{x}$ implies that σ is strictly coarser than τ. Also, we see that $V \cap W_{S,T,n} = T$ for every basic σ-neighborhood $W_{S,T,n}$ of $\bar{x}$; therefore, letting $\tau' = \langle \sigma \cup \{V\} \rangle$, it follows that for every open τ-neighborhood T^* of $\bar{x}$, $V \cap W_{X,T^* \cap V,1} = T^* \cap V$ is an (open) τ'-neighborhood of $\bar{x}$ included in T^*. Thus, $\tau' = \tau$, so that condition (3) of Lemma 6.2 is satisfied; hence, to apply such a result, it remains only to prove that the space (X, σ) is regular.

Since σ is finer than σ^*, and hence is T_2, we will show that given any $x \in X$ and any $Z \in \sigma(x)$, there is another σ-neighborhood of x whose σ-closure is included in Z. The case $x \neq \bar{x}$ is proved in the same way as for σ^*; therefore, suppose $x = \bar{x}$ and $Z = W_{S,T,n}$ for some $S \in \sigma^*(\bar{x})$, for some $T \in \tau(\bar{x})$ with $T \subseteq V^*$, and for some $n \in \mathbb{N}$. Then there are $S' \in \sigma^*(\bar{x})$ and $T' \in \tau(\bar{x})$, such that $\mathrm{Cl}_{\sigma^*} S' = \mathrm{Cl}_\tau S' \subseteq S$ and $\mathrm{Cl}_\tau T' \subseteq T \,(\subseteq V^*)$. Therefore, by (6.7),

$$\begin{aligned}\mathrm{Cl}_\sigma W_{S',T',n+1} = \mathrm{Cl}_\tau W_{S',T',n+1} &= (\mathrm{Cl}_\tau T') \cup \mathrm{Cl}_\tau(S' \setminus \mathrm{Cl}_\tau V_{n+1}) \\ &\subseteq T \cup \big((\mathrm{Cl}_\tau S') \cap \mathrm{Cl}_\tau(X \setminus \mathrm{Cl}_\tau V_{n+1})\big) \\ &\subseteq T \cup \big(S \cap (X \setminus \mathrm{Cl}_\tau V_n)\big) = W_{S,T,n}.\end{aligned}$$

Thus, σ is regular, and applying Lemma 6.2, we conclude that there exists a $\hat{\gamma} \in \Sigma_3(X)$ with $(\sigma' \subseteq \sigma^* \subseteq)\, \sigma \subseteq \hat{\gamma} \subsetneqq \tau$, such that $(\hat{\gamma}, \tau)$ is a gap in $\Sigma_3(X)$. $\square$

It is to be mentioned that in the poset $\Sigma_t(X)$ of all Tychonoff topologies on a set X, it holds that every non-minimal topology is upper [1, Corollary 2.6]. However, it does not seem that such a result can be useful to give a shorter proof of Theorem 6.3, since in between two Tychonoff topologies there may lie a regular non-Tychonoff topology. As an example, consider the regular non-Tychonoff space M produced in Example 1.5.9 [6]. Call τ the topology of M and modify it to get a new topology σ, such that $\sigma|_{M_0} = \tau|_{M_0}$, while the basic σ-neighborhoods of the point z_0 are all sets of the form $M \setminus \big(F \cup \bigcup_{z \in G}(A_1(z) \cup A_2(z))\big)$, where F and G are finite subsets of $M_0 \setminus L$ and L, respectively. Then σ is easily seen to be a compact (T_2) topology on M, and $\sigma \subseteq \tau \subseteq \delta$, where δ is the discrete topology on M.

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