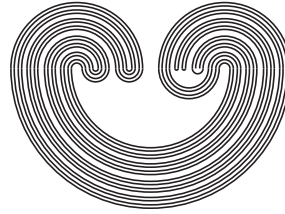

TOPOLOGY PROCEEDINGS



Volume 32, 2008

Pages 227–237

<http://topology.auburn.edu/tp/>

A NOTE ON
LINEARLY LINDELÖF SPACES
AND DUAL PROPERTIES

by

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Electronically published on July 8, 2008

Topology Proceedings

Web: <http://topology.auburn.edu/tp/>

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E-mail: topolog@auburn.edu

ISSN: 0146-4124

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**A NOTE ON
LINEARLY LINDELÖF SPACES
AND DUAL PROPERTIES**

LIANG-XUE PENG AND FRANKLIN D. TALL

ABSTRACT. In the first part of this note, we discuss some sufficient conditions for linearly Lindelöf spaces to be Lindelöf spaces. We prove that if X is a linearly Lindelöf space and for any discrete subspace A of X , $\overline{A} \setminus A$ is a G_δ -set of X , then X is a Lindelöf space. We also discuss some sufficient conditions for discretely Lindelöf spaces to be Lindelöf. Additionally, we draw some conclusions concerning σ -discretely Lindelöf spaces. In the second part of this note, we make some remarks on some questions related to “dually property \mathcal{P} ” and D -spaces.

1. INTRODUCTION

D -spaces were introduced by Eric K. van Douwen in his thesis [11]. A *neighborhood assignment* for a space X is a function ϕ from X to the topology of the space X , such that $x \in \phi(x)$ for any $x \in X$. A space X is called a D -space, if for any neighborhood assignment ϕ for X there exists a closed discrete subspace D of X , such that $X = \bigcup\{\phi(d) : d \in D\}$ (see [11] and [12]). There has been much work on D -spaces (see [6], [7], [13], [14], and [26]).

2000 *Mathematics Subject Classification.* Primary 54D20; Secondary 54A25, 54A35.

Key words and phrases. D -space, discretely Lindelöf, dually \mathcal{P} , Lindelöf, linearly Lindelöf.

Research of the first author was supported by a PHR(IHLB) grant.

Research of the second author was supported by NSERC grant A-7354.

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In [22], J. van Mill, V. V. Tkachuk, and R. G. Wilson further developed ideas related to D -spaces by defining for a topological property \mathcal{P} , a space X to be *dually* \mathcal{P} if for each neighborhood assignment $\{\phi(x) : x \in X\}$, there is a subspace $Y \subseteq X$ with property \mathcal{P} such that $X = \bigcup\{\phi(x) : x \in Y\}$. This concept was then further investigated by Ofelia T. Alas, Tkachuk, and Wilson in [3]. We shall make some remarks related to this latter paper in the second part of this note.

A space X is called *linearly Lindelöf* if every open cover that is totally ordered by \subseteq has a countable subcover (see [5]). It is natural to consider the general question: Which additional conditions entail that a linearly Lindelöf space is Lindelöf? We know that every countably metacompact linearly Lindelöf space is Lindelöf (see [16]). In the first part of the note, we obtain some sufficient conditions for linearly Lindelöf spaces to be Lindelöf.

A space X is *strongly discretely Lindelöf* if the closure of every discrete subspace of X is Lindelöf (see [4] and [5]). In [8], such a space is just called discretely Lindelöf, so we will also call such a space *discretely Lindelöf*. Every discretely Lindelöf space is linearly Lindelöf (see [4]). In [1], Alas proved that if X is a discretely Lindelöf space, and the tightness $t(X)$ is less than \aleph_ω , then X is Lindelöf. In this note, we also get some sufficient conditions for discretely Lindelöf or linearly Lindelöf spaces to be Lindelöf. We also draw some conclusions concerning σ -discretely Lindelöf spaces.

Let N be the set of all positive natural numbers, and $\omega = N \cup \{0\}$. Let X be a space. The concepts of $s(X)$ and $t(X)$ can be found in [15]. All the spaces in this note are assumed to be T_1 -spaces. In notation and terminology, we will follow [10], [15], and [19].

2. ON LINEARLY LINDELÖF SPACES AND DISCRETELY LINDELÖF SPACES

Let's recall that a space X is *scattered* if every subspace A of X has at least one isolated point of A . The proof of Lemma 2.1 follows the argument of B. É. Šapirovsĭiĭ (see [15, Proposition 4.8]).

Lemma 2.1. *Let X be a space. If ϕ is a neighborhood assignment for X , then there is a discrete subspace A of X and an open family $\{V(x) : x \in A\}$ of X , such that $X = \bigcup\{\phi(x) : x \in \overline{A}\}$, and*

$X \setminus \bigcup\{V(x) : x \in A\} = \overline{A} \setminus A$, $V(x) \cap A = \{x\}$, and $x \in V(x) \subseteq \phi(x)$ for each $x \in A$.

Proof: Let ϕ be any neighborhood assignment for X . We can construct sequences $\{x_\alpha : \alpha < \gamma\}$ of points of X and $\{V(x_\alpha) : \alpha < \gamma\}$ of open sets of X for some ordinal γ , such that

- (1) $x_0 \in V(x_0) \subseteq \phi(x_0)$, and
- (2) $x_\alpha \in V(x_\alpha) \subseteq \phi(x_\alpha) \setminus \overline{\{x_\beta : \beta < \alpha\}}$, and
- (3) $x_\alpha \in X \setminus (\bigcup\{V(x_\beta) : \beta < \alpha\} \cup \overline{\{x_\beta : \beta < \alpha\}})$ for each $\alpha < \gamma$, and also
- (4) $X = \bigcup\{V(x_\alpha) : \alpha < \gamma\} \cup \overline{\{x_\alpha : \alpha < \gamma\}}$.

Let $A = \{x_\alpha : \alpha < \gamma\}$. So we have that $X = \bigcup\{\phi(x) : x \in \overline{A}\}$. For each $\alpha < \gamma$, $V_{x_\alpha} \cap A = \{x_\alpha\}$. Thus, $X \setminus \bigcup\{V(x) : x \in A\} = \overline{A} \setminus A$. \square

Lemma 2.2. *Let X be a space. If ϕ is a neighborhood assignment for X , then there is a family $\{P_\alpha : \alpha \in \Lambda\}$ of decreasing closed subsets of X and a family $\{D_\alpha : \alpha \in \Lambda\}$ of discrete subspaces of X , where $D_\alpha \subseteq P_\alpha$, and for each $x \in D_\alpha$, there is an open subset $V(x)$ of X , such that $x \in V(x) \subseteq \phi(x)$ satisfying:*

- (1) $\bigcap\{P_\alpha : \alpha \in \Lambda\} = \emptyset$, and
- (2) $P_\gamma = \bigcap\{P_\alpha : \alpha < \gamma\}$ for each limit ordinal $\gamma \in \Lambda$, and
- (3) $(P_\alpha \setminus \bigcup\{V(x) : x \in D_\alpha\}) = P_{\alpha+1} = \overline{D_\alpha} \setminus D_\alpha$,
- (4) and $|V(x) \cap D_\alpha| = 1$ for each $x \in D_\alpha$, such that $X = \bigcup\{\phi(x) : x \in A\}$, where $A = \bigcup\{D_\alpha : \alpha \in \Lambda\}$, and
- (5) A is a scattered subspace of X .

Proof: Let $P_0 = X$. Then we can get a discrete subspace D_0 which satisfies the conditions appearing in Lemma 2.1. Assume we have sequences $\{P_\alpha : \alpha < \lambda\}$ and $\{D_\alpha : \alpha < \lambda\}$ which satisfy the conditions. If $\bigcap\{P_\alpha : \alpha < \lambda\} = \emptyset$, then we are done. We may assume that $\bigcap\{P_\alpha : \alpha < \lambda\} \neq \emptyset$. We let $P_\lambda = \bigcap\{P_\alpha : \alpha < \lambda\}$. Thus, $P_\lambda \subseteq \bigcup\{\phi(x) : x \in P_\lambda\}$. By the argument of Lemma 2.1, we can get a discrete subspace $D_\lambda \subseteq P_\lambda$ and an open family $\{V_x : x \in D_\lambda\}$ of X , such that $P_\lambda \setminus \bigcup\{V(x) : x \in D_\lambda\} = \overline{D_\lambda} \setminus D_\lambda = P_{\lambda+1}$ and $V(x) \cap D_\lambda = \{x\}$ for each $x \in D_\lambda$.

For any $B \subseteq A$, let $\beta = \min\{\alpha : B \cap D_\alpha \neq \emptyset, \alpha \in \Lambda\}$. For each $x \in B \cap P_\beta$, we have $V(x) \cap B = \{x\}$. Thus, A is a scattered subspace of X . \square

By Lemma 2.2, we have the following theorem.

Theorem 2.3. *Any space X is dually scattered.*

Theorem 2.4. *If X is a linearly Lindelöf space and for any discrete subspace A of X , $\overline{A} \setminus A$ is a G_δ -set of X , then X is a Lindelöf space.*

Proof: Let \mathcal{U} be any open cover of X . For any $x \in X$, there is some $U(x) \in \mathcal{U}$, such that $x \in U(x)$. Let $\phi(x) = U(x)$ for each $x \in X$. Thus, $\phi = \{\phi(x) : x \in X\}$ is a neighborhood assignment for X . By Lemma 2.2, we know that there is a family $\{P_\alpha : \alpha \in \Lambda\}$ of decreasing closed subsets of X , and there is a family $\{D_\alpha : \alpha \in \Lambda\}$ of discrete subspaces of X , satisfying the conditions in Lemma 2.2. For each $\alpha \in \Lambda$, $P_{\alpha+1} = \overline{D_\alpha} \setminus D_\alpha$. Thus, $P_{\alpha+1}$ is a G_δ -set of X . So $X \setminus P_{\alpha+1}$ is an F_σ -set of X . Thus, D_α is an F_σ -set of X . X is a linearly Lindelöf space. So $|D_\alpha| \leq \omega$ for each $\alpha \in \Lambda$.

CLAIM. For any closed subset $F \subseteq X$, if there is some $\alpha \in \Lambda$, such that $F \subseteq X \setminus P_\alpha$, then there is a countable subfamily $\mathcal{U}_F \subseteq \mathcal{U}$, such that $F \subseteq \bigcup \mathcal{U}_F$.

Proof of Claim: (1) If $F \subseteq X \setminus P_1$, then the claim is obviously true following from $|D_0| \leq \omega$.

Let $\beta \in \Lambda$. Suppose we have proved the claim for each $\alpha < \beta$. We will prove that it is true for β .

(2) $\beta = \alpha + 1$ for some ordinal α . So $F \subseteq X \setminus P_{\alpha+1}$. Thus, $F \cap (P_\alpha \setminus P_{\alpha+1}) \subseteq \bigcup \{\phi(x) : x \in D_\alpha\}$ and $|D_\alpha| \leq \omega$. By induction, we know that $F \setminus \bigcup \{\phi(x) : x \in D_\alpha\}$ can be covered by a countable subfamily of \mathcal{U} . Then so can F .

(3) If β is a limit ordinal and $F \subseteq X \setminus P_\beta$, then we know that $F \subseteq \bigcup \{X \setminus P_\alpha : \alpha < \beta\}$. So there is a sequence $\{\alpha_n : n \in \omega\}$, $\alpha_n < \beta$ for each $n \in \omega$, such that $F \subseteq \bigcup \{X \setminus P_{\alpha_n} : n \in \omega\}$.

(i) If $cf(\beta) > \omega$, then there is some $\alpha < \beta$, such that $\alpha_n < \alpha$ for each $n \in \omega$. Thus, $F \subseteq X \setminus P_\alpha$. By induction, we know that there is some countable subfamily of \mathcal{U} which covers F .

(ii) If $cf(\beta) = \omega$, then we may find an increasing sequence $\{\alpha_n + 1 : n \in \omega\}$, such that $\sup\{\alpha_n + 1 : n \in \omega\} = \beta$. Thus, $F \subseteq \bigcup \{X \setminus P_{\alpha_n+1} : n \in \omega\}$. For each $n \in \omega$, $X \setminus P_{\alpha_n+1}$ is an F_σ -set of X . Thus, $X \setminus P_{\alpha_n+1}$ can be covered by a countable subfamily of \mathcal{U} for each $n \in \omega$. Then so can F . This concludes the proof of the claim.

In the following, we will finish the proof of the theorem. We prove it by induction.

(1) If $\Lambda = 1$, then $X = \bigcup\{\phi(x) : x \in D_0\}$ and $|D_0| \leq \omega$.

(2) If $\Lambda = \alpha + 1$ for some ordinal α , then $P_\alpha \subseteq \bigcup\{\phi(x) : x \in D_\alpha\}$ and $|D_\alpha| \leq \omega$. Thus, $X \setminus \bigcup\{\phi(x) : x \in D_\alpha\}$ can be covered by a countable subfamily of \mathcal{U} by the claim. Then so can X .

(3) If $\Lambda = \beta$ and β is a limit ordinal, then there is an increasing sequence $\{\alpha_n + 1 : n \in \omega\}$, such that $X = \bigcup\{X \setminus P_{\alpha_n+1} : n \in \omega\}$ by the linearly Lindelöf property of X . $X \setminus P_{\alpha_n+1}$ is an F_σ -set of X , for each $n \in \omega$. Thus, $X \setminus P_{\alpha_n+1}$ can be covered by a countable subfamily of \mathcal{U} . So X is covered by a countable subfamily of \mathcal{U} . \square

Theorem 2.5. *Let X be a space. X is a Lindelöf space if and only if X is linearly Lindelöf and every closed nowhere dense subset of X is Lindelöf.*

Proof: The forward implication is obvious.

We prove the reverse implication. Let \mathcal{U} be any open cover of X . Let \mathcal{V} be a maximal disjoint family of open sets inscribed in \mathcal{U} . Thus, $A = X \setminus \bigcup \mathcal{V}$ is nowhere dense. By hypothesis, there exists a countable subfamily $\mathcal{U}_1 \subseteq \mathcal{U}$ that covers A . Since X is linearly Lindelöf, all elements of \mathcal{V} , except maybe countably many of them, are subsets of $\bigcup \mathcal{U}_1$. Since \mathcal{V} is inscribed in \mathcal{U} , \mathcal{U} includes a countable subcover of $X \setminus \bigcup \mathcal{U}_1$. \square

In the following, we summarize which additional conditions make a linearly Lindelöf space Lindelöf. We know that every linearly Lindelöf D -space is Lindelöf. So we have the following corollary.

Corollary 2.6. *If X is a linearly Lindelöf space, then any of the following conditions imply X is Lindelöf.*

- (1) X is metalindelöf (in fact, any condition which, with countable extent, implies Lindelöf will do, e.g., subparacompact);
- (2) X is a D -space;
- (3) the boundary of every discrete subspace of X is a G_δ -set of X ;
- (4) every closed nowhere dense subspace of X is Lindelöf.

By Lemma 2.1, we have the following conclusion.

Theorem 2.7. *Every discretely Lindelöf space is dually Lindelöf.*

Proof: Let ϕ be a neighborhood assignment for X . Then there is a discrete subspace A of X , such that $X = \bigcup\{\phi(x) : x \in \overline{A}\}$ by Lemma 2.1. Since X is discretely Lindelöf, \overline{A} is Lindelöf. Thus, X is dually Lindelöf. \square

Let's discuss what additional conditions entail that a discretely Lindelöf space is Lindelöf.

The following definition is due to A. V. Arhangel'skii [4]. Let κ be an ordinal number. A κ -long free sequence in a space X is a transfinite sequence $S = \{x_\alpha : \alpha < \kappa\}$ of elements of X , such that, for every $\alpha < \kappa$, the closures in X of the sets $L_s(\alpha) = \{x_\beta : \beta < \alpha\}$ and $R_s(\alpha) = \{x_\beta : \alpha \leq \beta < \kappa\}$ are disjoint. Let $F(X) = \sup\{\kappa : S \text{ is a } \kappa\text{-long free sequence in } X\}$. We know that $|F(X)| \leq s(X)$.

Lemma 2.8 (see [1]). *If X is a discretely Lindelöf space and $F(X)$ ($t(X)$ or $s(X)$) is less than \aleph_ω , then X is Lindelöf.*

Theorem 2.9. *If $2^{\aleph_0} < 2^{\aleph_n}$ for some $n \in \omega$ and X is a discretely Lindelöf, locally separable, hereditarily normal space, then X is Lindelöf.*

Proof: Let A be any discrete subspace of X ; therefore, \overline{A} is Lindelöf. Thus, by the locally separable property, there is a separable open set V of X , such that $\overline{A} \subseteq V$. So V is a separable and hereditarily normal subspace of X . A is a discrete subspace of V . So $|A| < \aleph_n$. Thus, $s(X) \leq \aleph_n < \aleph_\omega$. Hence, X is Lindelöf by Lemma 2.8. \square

It is fact (noted, e.g., in [9]) that $2^{\aleph_0} < \aleph_\omega$ implies separable, regular, linearly Lindelöf spaces are Lindelöf. Similarly, we have the following theorem.

Theorem 2.10. *If $2^{2^{\aleph_0}} < \aleph_\omega$ and X is a T_2 linearly Lindelöf space with no uncountable discrete subspace, then X is Lindelöf.*

Proof: T_2 -spaces with countable spread have cardinality $\leq 2^{2^{\aleph_0}}$. \square

Theorem 2.11. *PFA implies a linearly Lindelöf space with countable spread is hereditarily Lindelöf.*

Proof: Suppose not, then there is a right-separated subspace of size \aleph_1 (see [27]). By countable spread, that subspace is hereditarily

separable. But then it's hereditarily Lindelöf by PFA (see [32]), a contradiction. \square

An S -space is a regular hereditarily separable space which is not hereditarily Lindelöf (see [27] and [29]). A Lindelöf S -space which exists under \diamond (see [27]), for example, shows the independence of Lindelöf spaces with countable spread being hereditarily Lindelöf.

Here is another variation. Recall that a space is of *pointwise countable type* if each point is included in a compact set of countable character.

Theorem 2.12. *A discretely Lindelöf, hereditarily normal space of pointwise countable type satisfying the countable chain condition is Lindelöf provided either $2^{\aleph_0} < \aleph_\omega$ or $2^{\aleph_0} < 2^{\aleph_1}$.*

Proof: In [31], it is shown that $2^{\aleph_0} < 2^{\aleph_1}$ implies normal spaces of pointwise countable type are weakly \aleph_1 -collectionwise Hausdorff. So the space is of spread $< \aleph_\omega$. In order to prove the second half of the theorem, it suffices to show that pointwise countable type is inherited by open subspaces. But this is an easy exercise (see [10]). For the first half, we observe that the weakly collectionwise Hausdorff technique of [30] enables us to prove that in a normal space of pointwise countable type, for each closed discrete subspace B of size 2^{\aleph_0} , there is an uncountable subspace A of B which is separated by disjoint open sets. Thus, if 2^{\aleph_0} were less than \aleph_ω and the space's spread were $\geq \aleph_\omega$, then there would be a discrete subspace of size 2^{\aleph_0} , and hence, \aleph_1 disjoint open sets. \square

In the following, we will discuss some properties of σ -discretely Lindelöf spaces. A space X is σ -discretely Lindelöf if the closure of every σ -discrete subspace of X is Lindelöf.

Theorem 2.13. *If X is discretely Lindelöf and the closure of every Lindelöf subspace of X is Lindelöf, then X is σ -discretely Lindelöf.*

Proof: Let $Y = \bigcup\{Y_n : n \in \omega\}$, each Y_n is discrete. Then $\bigcup\{\overline{Y_n} : n \in \omega\}$ is dense in \overline{Y} . But $\bigcup\{\overline{Y_n} : n \in \omega\}$ is Lindelöf. So \overline{Y} is Lindelöf. \square

Corollary 2.14. *A locally separable, discretely Lindelöf space is σ -discretely Lindelöf if and only if closures of Lindelöf subspaces are Lindelöf.*

Proof: By local separability, every Lindelöf subspace Y is included in a separable open set U . Let D be countable—hence, σ -discrete, and dense in U . Then $\overline{D} \supseteq \overline{Y}$ and so \overline{Y} is Lindelöf. \square

In [9], the Axiom CC is introduced and the following two results proved.

Lemma 2.15. *CC implies that if X is a locally compact Hausdorff space which is either normal or countably tight, then either*

- (1) *X is the countable union of countably compact closed subspaces, or*
- (2) *X has an uncountable closed discrete subspace, or*
- (3) *X has a countable subset with non-Lindelöf closure.*

Proposition 2.16. *$CC + 2^{\aleph_0} < \aleph_\omega$ implies that locally compact, normal, linearly Lindelöf spaces are Lindelöf.*

Theorem 2.17. *CC implies that locally compact, normal, σ -discretely Lindelöf spaces are Lindelöf.*

Proof: Alternatives (2) and (3) of Lemma 2.15 are excluded, leaving us with (1), which yields countable metacompactness and hence, Lindelöfness. \square

CC follows from PFA and is consistent with CH (see [9]). Since PFA implies $2^{\aleph_0} = \aleph_2$ (see [17, p. 609]), we also have the following theorem.

Theorem 2.18. *PFA implies that locally compact, countably tight, linearly Lindelöf spaces are Lindelöf.*

We know that every σ -discretely Lindelöf space is discretely Lindelöf and every discretely Lindelöf space is linearly Lindelöf. The space constructed by Oleg Pavlov ([24]), by assuming $MA + \aleph_\omega < 2^{\aleph_0}$, is first countable, linearly Lindelöf and not Lindelöf; thus, it is not discretely Lindelöf by Lemma 2.8. We don't have an example of a discretely Lindelöf space that is not σ -discretely Lindelöf.

3. SOME REMARKS ON “DUALLY \mathcal{P} ” AND D-SPACES

In [3], Alas, Tkachuk, and Wilson note that A. J. Ostaszewski's space [23] is a consistent example of a weakly Lindelöf, perfectly normal space which is not Lindelöf. There is actually a ZFC example: Heikki J. K. Junnila [18] gave an example of a countable chain

condition (hence, weakly Lindelöf), perfectly normal, metacompact space with an uncountable closed discrete subspace (hence, not Lindelöf).

Problem 4.6 of [3] asks whether every dually hereditarily separable space is weakly Lindelöf. As noted earlier, under PFA, hereditarily separable regular spaces are hereditarily Lindelöf (see [32]), so a dually hereditarily separable regular space is dually hereditarily Lindelöf. But then by Theorem 2.8 of [22], such a space is Lindelöf.

In an earlier paper [2], Alas, Tkachuk, and Wilson note that a compact Souslin line is an example of a compact space in which the closure of every discrete subspace is hereditarily separable, but the space is not hereditarily separable. Assuming $MA + \sim CH$, however, this cannot happen. If the closure of every discrete subspace is hereditarily separable, then the space has countable spread. By Šapirovskii [28], $MA + \sim CH$ implies such compact spaces are hereditarily Lindelöf and hereditarily separable.

Theorem 3.1. *If it is consistent there is a supercompact cardinal, it is consistent that every locally compact, perfectly normal space is a D -space.*

Proof: In [21], Paul B. Larson and Franklin D. Tall prove that if it is consistent that there is a supercompact cardinal, it is consistent that every locally compact perfectly normal space is paracompact. They also remark that it is likely that the large cardinal assumption can be removed. A locally compact paracompact T_2 -space is a D -space (see [25]), so it follows that so are locally compact, perfectly normal spaces in the model used by Larson and Tall. \square

On the other hand, Ostaszewski's space is locally compact, perfectly normal, countably compact, but not compact, so it is not a D -space.

Acknowledgment. The authors thank the referee for his/her valuable suggestions which greatly improved the paper.

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