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HOMOGENEITY AND COMPLETE ACCUMULATION POINTS

ALEXANDER ARHANGEL'SKII

ABSTRACT. A space X is uncountably compact if every uncountable subset of X has a point of complete accumulation in X. The following facts are established. If X is an uncountably compact homogeneous Tychonoff space then either the cardinality of X is less than $\aleph_\omega$ or X is compact. If X is an uncountably compact Tychonoff space in which every point is a G_δ-set, then the cardinality of X does not exceed $\aleph_\omega \cdot 2^\omega$. We also clarify the structure of uncountably compact Tychonoff spaces.

Our terminology is very close to that in [4]. A topological space is *homogeneous* if for any points x and y of X there exists a homeomorphism h of X onto itself such that $h(x) = y$. Recall also that a point x of a space X is a point of *complete accumulation* for an infinite subset A of X if for every open neighborhood Ox of x the intersection of Ox with A has the same cardinality as A. It is well known that a topological space X is compact if and only if every infinite subset of X has a point of complete accumulation in X [1]. However, the Lindelöf property cannot be characterized in a similar way: It is not true that in every (regular T_1) Lindelöf space each uncountable set has a complete accumulation point. Indeed, let Y_n be a Lindelöf space of the cardinality $\aleph_n$ for each $n \in \omega$; then the free topological sum of the spaces Y_n is a Lindelöf space Y such

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that the set Y does not have a point of complete accumulation in the space Y.

A somewhat weaker condition is satisfied by each Lindelöf space X: Every uncountable subset of X of regular cardinality has a point of complete accumulation in X. However, the last condition is not sufficiently strong to characterize Lindelöfness; it characterizes so called *linearly Lindelöf* spaces (see [3] and [5]).

Let us call a space X *uncountably compact* if every uncountable subset of X has a point of complete accumulation in X. Clearly, every Lindelöf space of cardinality less than $\aleph_\omega$ is uncountably compact. On the other hand, there are uncountably compact non-compact Tychonoff spaces of as large cardinality as we wish, since the one-point Lindelöfication of an arbitrary uncountable discrete space is, obviously, uncountably compact.

Our main result is the following statement.

Theorem 1. *Suppose that X is an uncountably compact, homogeneous, regular T_1-space. Then either $|X| < \aleph_\omega$ or X is compact.*

The requirement of homogeneity in the above statement can be considerably weakened in the following way. Let us call a space X *weakly point-to-open homogeneous* if for each $x \in X$ and each non-empty open set V there exists a one-to-one continuous mapping f of X into itself such that $f(x) \in V$.

Theorem 2. *Suppose that X is a weakly point-to-open homogeneous regular T_1-space which is uncountably compact. Then either $|X| < \aleph_\omega$ or X is compact.*

In the proof of this theorem, we use the following statement which is a part of the folklore and was known to Alexandroff and Urysohn. A proof of it is provided below for the sake of completeness.

Theorem 3. *Every uncountably compact space X is Lindelöf.*

Proof: Assume that X is not Lindelöf, and let τ be the smallest cardinal number such that there exists an open covering $\gamma = \{U_\alpha : \alpha < \tau\}$ containing no countable subcover of X. Then, by minimality of τ, a stronger condition is satisfied:

(m) *If $M \subset \tau$ and $|M| < \tau$, then $\{U_\alpha : \alpha \in M\}$ does not cover X.*

Clearly, $\aleph_0 < \tau$. Fix a function $g : X \to \tau$ such that $x \in U_{g(x)}$ for each $x \in X$. By the property (m) of γ, there exists a transfinite sequence $\{x_\alpha : \alpha < \tau\}$ satisfying the following recursive conditions:

$$x_\alpha \in X \setminus \cup\{U_\beta : \beta < \alpha\},$$

and

$$x_\alpha \in X \setminus \cup\{U_{g(x_\beta)} : \beta < \alpha\},$$

for each $\alpha < \tau$. Clearly, $x_\alpha \neq x_\beta$ for any $\beta \neq \alpha$. Therefore, the cardinality of the set $A = \{x_\alpha : \alpha < \tau\}$ is exactly τ. On the other hand, each U_α contains less than τ points of A. Since γ covers X, it follows that the uncountable set A does not have a point of complete accumulation in X. □

Proof of Theorem 2: Assume that $|X| \geq \aleph_\omega$, and fix a subset A of X such that $|A| = \aleph_\omega$. Since X is uncountably compact, some $c \in X$ is a point of complete accumulation for A in X. By Theorem 3, X is Lindelöf. Observe that X is Tychonoff, since it is also a regular T_1-space.

Assume that X is not compact. Then it is not pseudocompact, since it is Lindelöf [4]. Therefore, there exists an infinite discrete family $\{V_n : n \in \omega\}$ of non-empty open sets in X. Since X is weakly point-to-open homogeneous and the cardinality of any open neighborhood of c is not less than $\aleph_\omega$, we have $|V_n| \geq \aleph_\omega$ for each $n \in \omega$. Now we can fix a subset B_n of V_n of the cardinality $\aleph_n$ for each $n \in \omega$. Put $B = \cup\{B_n : n \in \omega\}$. Clearly, the cardinality of B is $\aleph_\omega$ and B doesn't have a point of complete accumulation in X. Therefore, X is not uncountably compact. This contradiction completes the proof. □

Corollary 4. *A homogeneous regular T_1-space X of cardinality $\geq \aleph_\omega$ is uncountably compact if and only if it is compact.*

This result should be compared to the following statement, the proof of which is now obvious.

Corollary 5. *A regular T_1-space X of cardinality $< \aleph_\omega$ is uncountably compact if and only if it is Lindelöf.*

Theorem 6. *Suppose that X is an uncountably compact regular T_1-space. Then there exists a compact subspace F of X such that* *the cardinality of each closed subset P disjoint from F is less than $\aleph_\omega$.*

Proof: The space X is a regular Lindelöf T_1-space. Therefore, X has a Hausdorff compactification bX.

Let F be the set of all points $x \in X$ such that the cardinality of each open neighborhood of x is not less than $\aleph_\omega$. Clearly, F is closed in X.

Assume now that F is not compact. Then F is not closed in bX. Therefore, there exists $z \in \overline{F} \setminus F$. Since X is Lindelöf and bX is a Hausdorff compactification of X, there is a sequence $\xi = \{U_n : n \in \omega\}$ of open neighborhoods U_n of z in bX such that $\overline{U_{n+1}} \subset U_n$ for each $n \in \omega$ and $(\cap\{\overline{U_n} : n \in \omega\}) \cap X = \varnothing$. We may also assume that if $V_n = U_n \setminus \overline{U_{n+1}}$, then $V_n \cap F \neq \varnothing$ for each $n \in \omega$. Now it follows from the definition of F that the cardinality of each $V_n \cap X$ is not less than $\aleph_\omega$. Thus, we can fix a subset A_n of $V_n \cap X$ such that the cardinality of A_n is $\aleph_n$. Put $A = \cup\{A_n : n \in \omega\}$. Then, clearly, $|A| = \aleph_\omega$, and no point of X is a complete accumulation point for A, since the complement of $\overline{U_n}$ in X contains less than $\aleph_\omega$ points of A. However, this is impossible, since X is uncountably compact. This contradiction shows that F is compact.

Now let U be any open neighborhood of F in X. The cardinality of the set $B = X \setminus U$ must be smaller than $\aleph_\omega$, since otherwise every point of complete accumulation of B has to be in F, by the definition of F. Obviously, the closure of B does not intersect F (note also that B, if uncountable, must have at least one point of complete accumulation in X, since X is uncountably compact). □

We now apply the above result to prove a cardinal invariants inequality.

Theorem 7. *If X is a regular uncountably compact T_1-space such that each point in X is a G_δ-point, then the cardinality of X does not exceed $\aleph_\omega \cdot 2^\omega$.*

Proof: By Theorem 6, we can fix a compact subspace F of X such that the cardinality of the complement of an arbitrary open neighborhood of F in X is smaller than $\tau = \aleph_\omega$. Then F is first countable and hence, the cardinality of F is not greater than 2^ω. Since each point of F is a G_δ-point in X and F is compact, it follows that there exists a family γ of open neighborhoods of F in

X such that $|\gamma| \leq 2^\omega$ and $\cap\gamma = F$ [2]. Then, clearly, the cardinality of $X \setminus F$ does not exceed $\aleph_\omega \cdot 2^\omega$. $\square$

Corollary 8. *If $\aleph_\omega < 2^\omega$, then the cardinality of every regular uncountably compact T_1-space of countable pseudocharacter does not exceed 2^ω.*

Corollary 9. *Every uncountably compact Tychonoff space is the union of a compact subspace and an open subspace the local cardinality of which at every point is less than $\aleph_\omega$.*

It is easy to see that Theorem 1 could be derived from the last statement. The assumption in Corollary 8 that $\aleph_\omega < 2^\omega$ cannot be dropped, since the known consistent examples of Lindelöf regular T_1-spaces of countable pseudocharacter and cardinality greater than 2^ω are uncountably compact.

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