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EVOLUTION OF THE McMULLEN DOMAIN FOR SINGULARLY PERTURBED RATIONAL MAPS

ROBERT L. DEVANEY AND SEBASTIAN M. MAROTTA

Abstract. In this paper we study the dynamics of the two parameter family of complex maps given by $f_{\lambda,a}(z) = z^n + \lambda/(z-a)^d$ where $n \geq 2$ and $d \geq 1$ are integers and a and λ are complex parameters. It is known that if $a = 0$ and $|\lambda| \neq 0$ is sufficiently small, the Julia set of $f_{\lambda,a}$ is a Cantor set of simple closed curves; this is the McMullen domain in the λ-plane. As soon as $a \neq 0$ (and $|a| \neq 1$), the Julia sets are very different. We show that if $|a| \neq 0, 1$ and $|\lambda|$ is sufficiently small, the Julia set of $f_{\lambda,a}$ now consists of only a countable collection of simple closed curves together with uncountably many point components. Moreover, if $0 < |a| < 1$ ($|a| > 1$, resp.), then any two such maps are conjugate on their Julia sets provided $|\lambda|$ is sufficiently small. We also describe the four-dimensional a, λ-parameter plane for $|a| \neq 1$ and $|\lambda| << 1$. In particular, there is no McMullen domain in the λ-plane containing $\lambda = 0$ when $a \neq 0$. Rather, the McMullen domain moves away from $\lambda = 0$ and surrounds the region containing the Julia sets of the above type.

1. Introduction

In the last few years, a number of papers have appeared that deal with the dynamics of functions obtained by perturbing the complex function z^n by adding a pole at the origin [1], [4], [3], [6]. These rational functions are of the form $f_\lambda(z) = z^n + \lambda/z^d$. When $|\lambda| << 1$, we consider this function as a *singular perturbation* of

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z^n. The reason for this terminology is that, when $\lambda = 0$, the degree of the map is n and the dynamical behavior is well understood. When $\lambda \neq 0$, however, the degree jumps to $n+d$ and the dynamical behavior changes significantly.

The interest in this type of perturbation arises from the application of Newton's method to find the roots of a family of polynomials that, at one particular parameter value, has a multiple root. At this parameter value, the Newton iteration function undergoes a similar type of singular perturbation.

In this paper, we study a more general class of functions for which the pole is not located at the origin but rather is located at some other point in the complex plane that does not lie on the unit circle. In particular, we consider the family of functions given by

$$f_{\lambda,a}(z) = z^n + \frac{\lambda}{(z-a)^d}$$

where $n \geq 2$ and $d \geq 1$ are integers and a and λ are complex parameters with $|a| \neq 0, 1$. Our goal is to describe the dynamics on the *Julia set* of $f_{\lambda,a}$, i.e., the set of points at which the family of iterates of $f_{\lambda,a}$ is not a normal family in the sense of Montel. Equivalently, the Julia set is the closure of the set of repelling periodic points of $f_{\lambda,a}$ as well as the boundary of the set of points whose orbits escape to ∞. We denote the Julia set by $J = J(f_{\lambda,a})$. The complement of the Julia set is called the *Fatou set*. We shall show that there is a dramatic change in the structure of these sets when a becomes nonzero.

When $\lambda = 0$, the dynamical behavior of $f_{\lambda,a}$ is well understood. In this case ∞ and 0 are superattracting fixed points and the Julia set is the unit circle. When we add the perturbation by setting $\lambda \neq 0$ but very small, several aspects of the dynamics remain the same, but others change dramatically. For example, when $\lambda \neq 0$, the point at ∞ is still a superattracting fixed point and there is an immediate basin of attraction of ∞ that we call $B = B_\lambda$. The function $f_{\lambda,a}$ takes B n-to-1 onto itself. On the other hand, there is a neighborhood of the pole a that is now mapped d to 1 onto B. Since the degree of $f_{\lambda,a}$ changes from n to $n+d$, $2d$ additional critical points are created. The set of critical points includes ∞ and a whose orbits are completely determined, so there are $n+d$

additional "free" critical points. The orbits of these points are of fundamental importance in characterizing the Julia set of $f_{\lambda,a}$.

Given a with $a \neq 0$, we may find $\delta_1 > 0$ such that, if $0 < |\lambda| < \delta_1$, $f_{\lambda,a}$ still has an attracting fixed point $q = q_\lambda$ near the origin. Let $Q = Q_\lambda$ denote the immediate basin of q. Also, if $|a| < 1$ ($|a| > 1$, resp.), we may find a $\delta_2 > 0$ such that, if $|\lambda| < \delta_2$, then the boundary of B (the boundary of Q, resp.) is a simple closed curve that lies near the unit circle and does not contain a. We denote this boundary by ∂B (∂Q, resp.). If $|a| < 1$, the pole lies inside this curve, whereas a lies outside the curve if $|a| > 1$. Let $\lambda_a = \min\{\delta_1, \delta_2\}$. Throughout this paper, we assume that $|\lambda| < \lambda_a$. If $|a| < 1$, the preimage of B that contains the pole a is called the *trap door* and we denote this set by $T = T_\lambda$. Note that T is disjoint from B since a lies inside ∂B. Every point that escapes to infinity and does not lie in B must do so by passing through T.

We next recall some known facts about the case $a = 0$. Let $\omega^{n+d} = 1$. Then the dynamics of $f_{\lambda,0}$ are symmetric under $z \mapsto \omega z$ since, in this case, $f_{\lambda,0}(\omega z) = \omega^n f_{\lambda,0}(z)$. One checks easily that the free critical points of $f_{\lambda,0}$ are given by $c_\lambda = (\lambda d/n)^{1/(n+d)}$ and the critical values by $v_\lambda = c_\lambda^n(1 + n/d)$. Since the free critical points are arranged symmetrically about the origin, their orbits also behave symmetrically and so there is essentially only one free critical orbit. In this case, we have the so-called Escape Trichotomy [4]. This result describes the three possible types of Julia sets that arise when the critical orbits escape.

Theorem 1.1 (The Escape Trichotomy)**.** *Let $a = 0$ and suppose that the free critical orbits tend to ∞.*

(1) *If the critical values lie in B, then $J(f_{\lambda,0})$ is a Cantor set.*
(2) *If the critical values lie in T, then $J(f_{\lambda,0})$ is a Cantor set of simple closed curves (quasi-circles).*
(3) *If the critical values do not lie in B or T, then $J(f_{\lambda,0})$ is a connected set called Sierpiński curve.*

We remark that (2) is due to Curt McMullen [7] and does not occur if $n = d = 2$ or if n is arbitrary and $d = 1$. When $n = d \geq 3$ and $a = 0$, the set of parameter values λ for which the critical values lie in T is a punctured open disk in the parameter plane that surrounds the origin and is bounded by a simple closed curve. This open set is called the *McMullen domain* (see Figure 1) [2], [5].

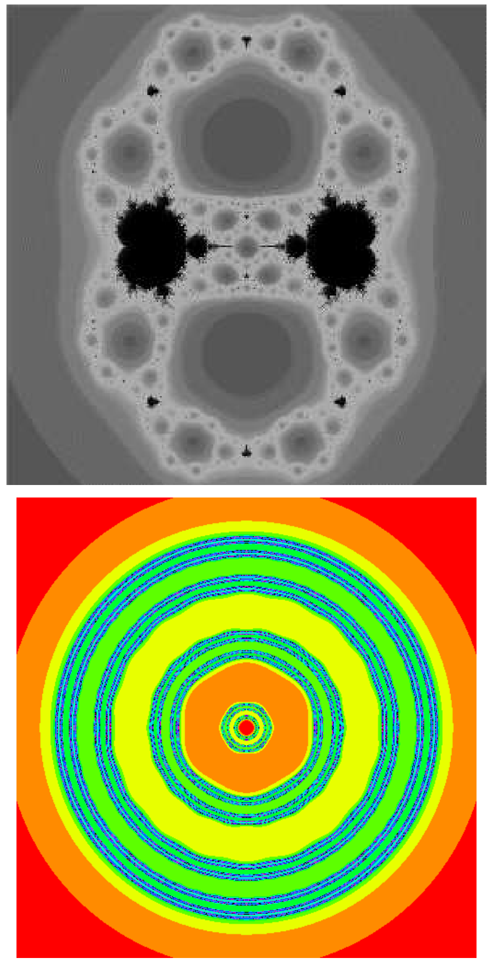

FIGURE 1. Top: the parameter λ-plane when $n = d = 3$ and $a = 0$. The McMullen domain is the small region in the center of the picture.
Bottom: the Julia set of $f_{\lambda,0}$ for $\lambda = 0.0001$ drawn from the McMullen domain. The Julia set is a Cantor set of simple closed curves.

When $a \neq 0$ and λ is sufficiently small, the situation is quite different. For one thing, unlike the case $a = 0$ and $\lambda \neq 0$, there is still the attracting fixed point q together with its basin Q close to the origin. The $n + d$ free critical points of $f_{\lambda,a}$ are no longer symmetrically located about the origin and their orbits no longer behave symmetrically. As we show below, the set of critical points may be divided into two groups: the first group consists of $n - 1$ critical points that are attracted to q. These are the critical points that bifurcate away from the origin which is a critical point of order $n - 1$ when $\lambda = 0$. The remaining $d + 1$ critical points surround a and are mapped close to a^n. It follows that the dynamics of this family of functions is determined by the behavior of this set of $d+1$ critical points and the position of a when λ is small.

One of our goals in this paper is to prove the following theorem.

Theorem 1.2 (Structure of J)**.** *Let $n \geq 2$, let $d \geq 1$, and suppose that $|a| \neq 0, 1$. Then, for λ sufficiently small, the Julia set of $f_{\lambda,a}$ is composed of a countable number of disjoint simple closed curves and uncountably many point components that accumulate on these curves. Only one of these simple closed curves surrounds the origin, while all others bound disjoint disks that are mapped to B (if $|a| < 1$) or to Q (if $|a| > 1$).*

We can also use symbolic dynamics to describe the behavior of $f_{\lambda,a}$ on its Julia set.

Theorem 1.3 (Dynamics on J)**.** *Suppose $0 < |a_j| < 1$ (or $|a_j| > 1$) for $j = 1, 2$. Then there exists ϵ such that, if $|\lambda| < \epsilon$, the maps f_{λ,a_1} and f_{λ,a_2} are conjugate on their Julia sets. Moreover, the dynamics are determined by a specific quotient of a subshift of finite type.*

These results indicate that a major change occurs in the parameter planes for these families when a becomes nonzero. Let P_a denote the parameter plane (the λ-plane) for a fixed value of a. When $a = 0$ (and $n = d \geq 3$), the origin in P_0 is surrounded by an open disk D_0 such that if $\lambda \in D_0$, then the Julia set of $f_{\lambda,a}$ consists of uncountably many disjoint simple closed curves, each of which surrounds the origin. D_0 is the McMullen domain. But when $a \neq 0$, the origin in P_a is now surrounded by a disk D_a such that if $\lambda \in D_a$, then the Julia set of $f_{\lambda,a}$ contains only countably many disjoint simple closed curves, only one of which surrounds the origin, and all other curves bound disjoint disks. However, the McMullen domain does continue to exist but its structure is quite different.

Theorem 1.4. *When $|a|$ is sufficiently small, there is a neighborhood of $\lambda = 0$ in the parameter plane in which the structure of each Julia set is as described in Theorem 1.2 (Structure of J). There is also an annular region in P_a that surrounds this neighborhood and in which the Julia set is a Cantor set of simple closed curves, each of which surrounds the origin and a.*

2. Preliminaries

A straightforward computation shows that when $\lambda \neq 0$, $f_{\lambda,a}$ has $n+d$ new critical points that satisfy the equation

$$z^{n-1}(z-a)^{d+1} = \lambda \; d/n. \tag{2.1}$$

When $\lambda = 0$, this equation has $n+d$ zeros, the origin with multiplicity $n-1$ and a with multiplicity $d+1$. By continuity, for small enough $|\lambda|$, these roots become simple zeros of $f'_{\lambda,a}$ that lie in the plane approximately symmetrically distributed around the fixed point q and the pole a. As a consequence, when $|\lambda|$ is small, $n-1$ of the critical points of $f_{\lambda,a}$ are grouped around q near the origin while $d+1$ of the critical points are grouped around the pole a.

Let $c = c_\lambda$ be any critical point of $f_{\lambda,a}$. Then we have $c^{n-1}(c-a)^{d+1} = \lambda \; d/n$. Dividing by $(c-a)^d$ on both sides, we find $c^{n-1}(c-a) = \lambda(c-a)^{-d}(d/n)$ and so the critical value v corresponding to c is given by

$$f_{\lambda,a}(c) = c^n + (n/d) \; c^{n-1}(c-a)$$

or, equivalently,

$$v = c^{n-1}(c \; (1+n/d) - a \; n/d). \tag{2.2}$$

It is easy to check that when $a = 0$, (2.1) and (2.2) reduce to the cases studied before. Note that as $\lambda \to 0$, the fixed point $q_\lambda \to 0$ as well. From (2.2), it follows that if $c \to 0$, then $v \to 0$. Similarly, if $c \to a$, then $v \to a^n$. Let $S_a = \{d+1 \text{ critical points around } a\}$ and $S_q = \{n-1 \text{ critical points around } q\}$. We have the following proposition.

Proposition 2.1 (Location of the critical values)**.** *For $|\lambda|$ sufficiently small, the set S_q is mapped arbitrarily close to q and the set S_a is mapped arbitrarily close to a^n.*

To describe the structure of the Julia set, we first need to give an approximate location for this set. To do this, recall that for $|\lambda| < \lambda_a$, the immediate basin of ∞ is bounded by a simple closed curve that lies close to the unit circle in the plane. We can actually say more.

Proposition 2.2. *Suppose $n \geq 2$, $d \geq 1$ and let $|z| = r$.*

(1) *Suppose $0 < r < \min\{1, |a|\}$. Then for sufficiently small λ, $z \in Q$.*
(2) *On the other hand, if $r > \max\{1, |a|\}$, then for sufficiently small λ, $z \in B$.*

Proof: Fix n, d, and a. In the first case, we have $|z - a| \geq |a| - |z| = |a| - r > 0$, so that

$$|f_{\lambda,a}(z)| \leq |z|^n + \frac{|\lambda|}{|z-a|^d} \leq r^n + \frac{|\lambda|}{(|a|-r)^d}.$$

Let $|\lambda| < (|a| - r)^d(r - r^n)$. Then

$$|f_{\lambda,a}(z)| < r^n + \frac{(|a|-r)^d(r-r^n)}{(|a|-r)^d} = r^n + r - r^n = r.$$

As a consequence, the disk of radius r is mapped strictly inside itself, and so, by the Schwarz Lemma, the orbit of any point in this disk converges to the fixed point q near the origin. So these points all lie in Q.

For case 2, we write $|z - a| \geq |z| - |a| = r - |a| > 0$. Then we have

$$|f_{\lambda,a}(z)| \geq |z|^n - \frac{|\lambda|}{|z-a|^d} \geq r^n - \frac{|\lambda|}{(r-|a|)^d}.$$

Let $|\lambda| < (r - |a|)^d(r^n - r)$. Then

$$|f_{\lambda,a}(z)| > r^n - \frac{(r-|a|)^d(r^n-r)}{(r-|a|)^d} = r^n - r^n + r = r.$$

Hence, $|f_{\lambda,a}(z)| > |z|$ and the orbit of z converges to ∞ so that $z \in B$. □

Corollary 2.3. *If $0 < |a| < 1$ and $|\lambda|$ is sufficiently small, then all of the critical values lie in Q. If $|a| > 1$ and $|\lambda|$ is sufficiently small, then the critical values corresponding to the critical points lying in S_q lie in Q, while the critical values corresponding to critical points in S_a lie in B.*

3. Structure of the Julia Set for $|a| < 1$

In this section, we shall prove that for any a with $0 < |a| < 1$, there exists ϵ_a such that if $|\lambda| < \epsilon_a$, then the Julia set of $f_{\lambda,a}$ consists of a countable collection of simple closed curves together with an uncountable collection of point components. Only one of these curves surrounds the origin, while all others bound disks that are eventually mapped onto B. Moreover, any two such maps are topologically conjugate on their Julia sets.

Proposition 3.1. *When $|\lambda|$ is sufficiently small and $0 < |a| < 1$, the trap door T and the immediate basin of infinity B are disjoint sets. Moreover, both of these sets are bounded by simple closed curves that are also disjoint. Only the boundary of B surrounds the origin.*

Proof: When $\lambda = 0$, we may choose an annular neighborhood $N_0 = \{z \mid \rho_1 \leq |z| \leq \rho_2\}$ of the unit circle on which $f_{\lambda,0}$ is an n to 1 expanding covering map that takes N_0 to a larger annulus that properly contains N_0. Call this image annulus N. The only points whose orbits under the map remain for all iterations in N are points on the unit circle. We may assume further that the annulus N does not contain a. For λ sufficiently small, we may then find a similar annular region N_λ that is mapped as an expanding covering map onto the same annulus N by $f_{\lambda,a}$. Since $f_{\lambda,a}$ is hyperbolic on N_λ, it again follows that the set of points whose orbits remain in N_λ under $f_{\lambda,a}$ is also a simple closed curve. In addition, all points in the exterior of this curve lie in B. Therefore, B is bounded by this simple closed curve surrounding 0.

Now the preimage of B, namely T, cannot intersect N_λ, for none of the points in N_λ that lie inside ∂B are mapped into B. By the results of the previous section, none of the critical points can lie in the closure of T, so ∂T is mapped as a d to 1 covering onto ∂B and so the boundaries of these two sets are also disjoint simple closed curves. Also, if $|\lambda|$ is sufficiently small, 0 lies in Q, so ∂T does not surround the origin. □

As shown in the previous section, we may assume that $|\lambda|$ is small enough so that all of the critical values corresponding to the free critical points lie in Q. As a consequence, none of the critical values lie in $\overline{T}$ or in any of the preimages of this set. Therefore, all of the preimages of $\overline{T}$ are closed disks that are mapped univalently onto $\overline{T}$ by some iterate of $f_{\lambda,a}$. Therefore, we have the following.

Proposition 3.2. *All of the preimages of $\overline{B}$ are closed disks that are disjoint from one another.*

Since the boundaries of these disks all map eventually to ∂B, each of these curves lies in the Julia set and we have produced a countable collection of simple closed curves in J.

Now choose r so that $r < |a| < r^{1/n} < 1$. By the results of the previous section, we may choose $|\lambda|$ sufficiently small so that

(1) all points in the closed disk of radius r centered at the origin lie in Q;
(2) the preimage of the circle of radius r is a simple closed curve τ close enough to the circle of radius $r^{1/n}$ so that a lies inside this curve;
(3) all critical points of $f_{\lambda,a}$ in the set S_a lie between the circle of radius r and the curve τ, but the images of these points lie close to a^n and hence inside the circle of radius r.

It follows that τ also lies in the basin of q and separates the boundary of T from that of B. Moreover, $f_{\lambda,a}$ is a covering map in the entire region outside τ.

Now choose $s > 1$ so that the circle of radius s centered at the origin lies in B. Denote the preimage of this circle in B by η. Note that η is a simple closed curve that is mapped n to 1 to the circle of radius s. Let $\mathcal{A}$ denote the annulus bounded by η and τ. Since there are no critical points in $\mathcal{A}$, $f_{\lambda,a}$ takes $\mathcal{A}$ as an n to 1 covering onto the annulus given by $\{z \,|\, r \leq |z| \leq s\}$. As in the proof of Proposition 3.1, the only points whose orbits lie for all time in $\mathcal{A}$ are those in ∂B.

Consider the annular region bounded by the circle of radius r and the curve τ. Since all of the critical points in this annulus are mapped inside the circle of radius r centered at the origin, it follows that there is an open disk about a that is mapped as a d to 1 covering (except at a) onto the exterior of the circle of radius r (excluding the point at ∞). Let $\mathcal{D}$ denote this region. Then all points inside the curve τ, except for those in $\mathcal{D}$, are mapped inside $|z| = r$. Note that all of the critical points belong to this set. Inside $\mathcal{D}$ there is another annulus $\mathcal{B}$ that is mapped d to 1 to $\mathcal{A}$ and a total

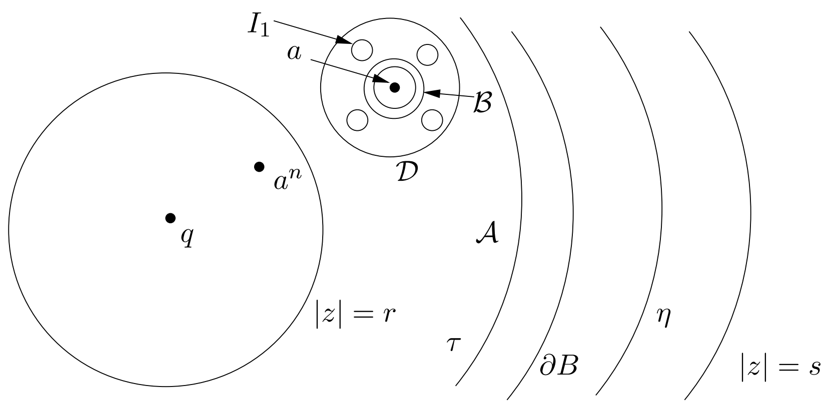

FIGURE 2. The regions $\mathcal{A}, \mathcal{B}, \mathcal{D}$, and the I_j's when $|a| < 1$. The annulus $\mathcal{A}$ is bounded by the curves τ and η.

of d open disks that are each mapped one-to-one onto $\mathcal{D}$. Denote these disks by $I_1, \ldots, I_d$. See Figure 2.

Since each I_k is mapped univalently over the union of all of the I_j, the set of points whose orbits remain for all iterations in the union of the I_j forms a Cantor set on which $f_{\lambda,a}$ is conjugate to the one-sided shift map on d symbols. This follows from standard arguments in complex dynamics [8]. This produces an uncountable number of point components in J. However, there are many other point components in J, as any point whose orbit eventually lands in the Cantor set is also in J. There are, however, still other points in J, as we show below.

To understand the complete structure of the Julia set, we show that J is homeomorphic to a quotient of a subset of the space of one-sided sequences of finitely many symbols. Moreover, we show that $f_{\lambda,a}$ on J is conjugate to a certain quotient of a subshift of finite type on this space. Since this is true for any a with $0 < |a| < 1$ and λ sufficiently small, this will prove our main results in the case $|a| < 1$.

To begin the construction of the sequence space, we first partition the annulus $\mathcal{A}$ into n "rectangles" that are mapped over $\mathcal{A}$ by $f_{\lambda,a}$.

Proposition 3.3. *There is an arc ξ lying in $\mathcal{A}$ and having the property that $f_{\lambda,a}$ maps ξ one-to-one onto a larger arc that properly* *contains* ξ *and connects the circles of radius* r *and* s *centered at the origin. Moreover,* ξ *meets* ∂B *at exactly one point, namely one of the fixed points in* ∂B*. With the exception of this fixed point, all other points on* ξ *lie in the Fatou set.*

Proof: Let $p = p_{\lambda,a}$ be one of the repelling fixed points on ∂B. Note that p varies analytically with both λ and a. As is well known, there is an invariant external ray in B extending from p to ∞. Define the portion of ξ in $B \cap \mathcal{A}$ to be the piece of this external ray that lies in $\mathcal{A}$.

To define the piece of ξ lying inside ∂B, let U be an open set that contains p and meets some portion of τ and also has the property that the branch of the inverse of $f_{\lambda,a}$ that fixes p is well-defined on U. Let $f_{\lambda,a}^{-1}$ denote this branch of the inverse of $f_{\lambda,a}$. Let $w \in \tau \cap U$ and choose any arc in U that connects w to $f_{\lambda,a}^{-1}(w)$. Then we let the remainder of the curve ξ be the union of the pullbacks of this arc by $f_{\lambda,a}^{-n}$ for all $n \geq 0$. Note that this curve limits on p as $n \to \infty$. □

We now partition $\mathcal{A}$ into n rectangles. Consider the n preimages of $f_{\lambda,a}(\xi)$ that lie in $\mathcal{A}$. Denote these preimages by $\xi_1, \ldots, \xi_n$ where $\xi_1 = \xi$ and the remaining ξ_j's are arranged counterclockwise around $\mathcal{A}$. Let A_j denote the closed region in $\mathcal{A}$ that is bounded by ξ_j and ξ_{j+1}, so that A_n is bounded by ξ_n and ξ_1. By construction, each A_j is mapped one-to-one over $\mathcal{A}$ except on the boundary arcs ξ_j and ξ_{j+1}, which are each mapped one-to-one onto $f_{\lambda,a}(\xi_1) \supset \xi_1$.

Now recall that the only points whose orbits remain for all iterations in $\mathcal{A}$ are those points on the simple closed curve ∂B. Let $z \in \partial B$. We may attach a symbolic sequence $S(z)$ to z as follows. Consider the n symbols $\alpha_1, \ldots, \alpha_n$. Define $S(z) = (s_0 s_1 s_2 \ldots)$ where each s_j is one of the symbols $\alpha_1, \ldots, \alpha_n$ and $s_j = \alpha_k$ if and only if $f_{\lambda,a}^j(z) \in A_k$. Note that there are two sequences attached to p, the sequences $(\overline{\alpha_1})$ and $(\overline{\alpha_n})$. Similarly, if $z \in \xi_k \cap \partial B$, then there are also two sequences attached to z, namely $(\alpha_{k-1}\overline{\alpha_n})$ and $(\alpha_k\overline{\alpha_1})$. Finally, if $f_{\lambda,a}^j(z) \in \xi_k$, then there are again two sequences attached to z, namely $(s_0 s_1 \ldots s_{j-1}\alpha_{k-1}\overline{\alpha_n})$ and $(s_0 s_1 \ldots s_{j-1}\alpha_k\overline{\alpha_1})$.

Note that if we make the above identifications in the space of all one-sided sequences of the α_j's, then this is precisely the same identifications that are made in coding the itineraries of the map

$z \mapsto z^n$ on the unit circle. So this sequence space with these identifications and the usual quotient topology is homeomorphic to the unit circle, and the shift map on this space is conjugate to $z \mapsto z^n$.

We now extend this symbolic dynamics to the set of points in the annulus $\mathcal{B}$ that are mapped to ∂B. Recall that $f_{\lambda,a}$ takes $\mathcal{B}$ onto $\mathcal{A}$ as a d-to-1 covering. So we let $\nu_1, \ldots, \nu_d$ be the d preimages of ξ_1 in $\mathcal{B}$ arranged in counterclockwise order beginning at some given preimage. Define the rectangles $B_j, j = 1, \ldots, d$ to be the region in $\mathcal{B}$ bounded by ν_j and ν_{j+1} where B_d is bounded by ν_d and ν_1. We introduce d new symbols $\beta_1, \ldots, \beta_d$ and define the itinerary $S(z)$ for a point $z \in \mathcal{B}$ that is mapped to ∂B as follows. $S(z)$ is a sequence that begins with a single β_k and then ends in any sequence of α_j's.

From the above, we need to make the following additional identifications in this larger set of sequences:

(1) the sequences $(\beta_1\overline{\alpha_1})$ and $(\beta_d\overline{\alpha_n})$,
(2) the sequences $(\beta_{k-1}\overline{\alpha_n})$ and $(\beta_k\overline{\alpha_1})$.

Finally, we extend the definition of $S(z)$ to any point in J that remains in the union of the I_j by introducing the symbols $1, \ldots, d$ and defining $S(z)$ in the usual manner. This defines the itinerary of any point in J whose orbit

(1) remains in $\mathcal{A}$ for all iterations,
(2) starts in $\mathcal{B}$ and then remains for all subsequent iterations in $\mathcal{A}$,
(3) remains in $\cup I_j$ for all iterations.

To extend this definition to all of J, we let Σ' denote the set of sequences involving the symbols $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_d, 1, \ldots, d$ subject to the following restrictions.

(1) Any symbol can follow α_j;
(2) the symbol β_j can only be followed by $\alpha_1, \ldots, \alpha_n$;
(3) symbols $1, \ldots, d$ can be followed only by $1, \ldots, d, \beta_1, \ldots, \beta_d$.

Let Σ denote the space Σ' where we extend the above identifications to any pair of sequences that ends in a pair of sequences identified earlier. For example, we identify the two sequences $s_0 s_1 \ldots s_n \alpha_{k-1}\overline{\alpha_n}$ and $s_0 s_1 \ldots s_n \alpha_k \overline{\alpha_1}$. We endow Σ with the quotient topology. Then, by construction, the Julia set of $f_{\lambda,a}$ is homeomorphic to Σ, and $f_{\lambda,a} \,|\, J$ is conjugate to the shift map on Σ. This proves the main results in the case $|a| < 1$.

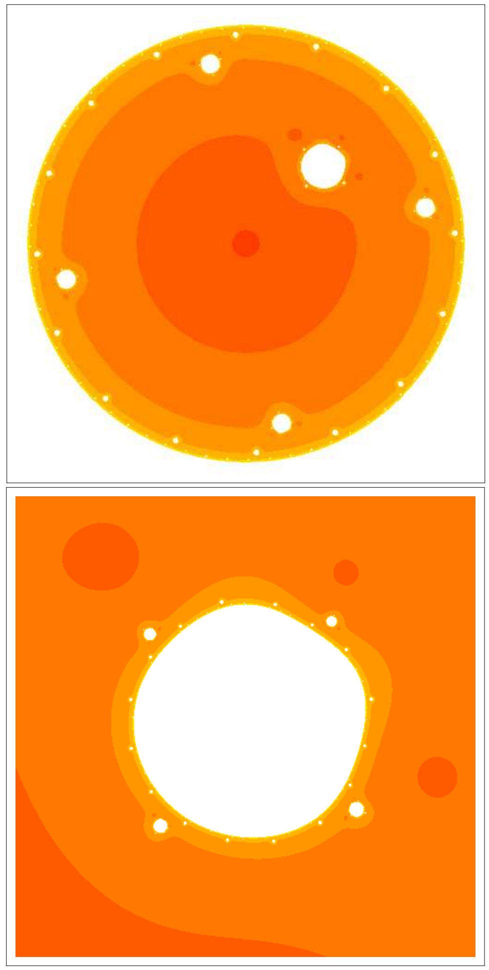

FIGURE 3. Top: the filled Julia set of $f_{\lambda,a}(z)$ for $n = d = 4$, $a = .5e^{i\pi/4}$, and $\lambda = 0.0001e^{i4\pi/3}$. The white regions are preimages of B. The different shades show the dynamics in Q. The picture is centered around the origin.
Bottom: magnification of a region centered around the pole a.

Figure 3 shows an example of the filled Julia set of $f_{\lambda,a}$ when $n = d = 4$ and $|a| < 1$.

4. Structure of the Julia Set for $|a| > 1$

There is a special kind of symmetry between the cases $0 < |a| < 1$ and $|a| > 1$. In Proposition 3.1, we proved that when $|\lambda| << 1$ and $|a| < 1$, the trap door T and the immediate basin of infinity B were disjoint. This implied that all the preimages of B were disjoint and also implied the existence of uncountably many components accumulating on the boundary of those preimages of B. Using the same argument, it is easy to see that for $|a| > 1$ and $|\lambda| << 1$, the immediate basin of q and its preimages are disjoint as well.

When $|a| > 1$, there are only $n - 1$ critical points in Q so there must be d preimages of Q somewhere else. By Proposition 2.2, these preimages cannot lie outside a circle of radius greater than $|a|$. We shall prove that B is infinitely connected and the Julia set is made up of a countable number of simple closed curves and uncountably many point components that accumulate on these curves.

By Corollary 2.3, the critical values corresponding to S_a are in B, so we next prove that the set S_a itself lies in B.

Proposition 4.1. *When $|a| > 1$ and $|\lambda|$ is sufficiently small, the critical points surrounding a, i.e., S_a, are contained in B and B is not simply connected.*

Proof: Fix $n \geq 2$, $d \geq 1$, and a with $|a| > 1$. Fix ρ with $|a| < \rho < |a|^n$ and let λ be sufficiently small so that

(1) $f_{\lambda,a}$ has an attracting fixed point q near the origin,
(2) the $n - 1$ critical points surrounding q are in Q,
(3) there is a preimage of the circle of radius ρ, a curve that we call γ, that lies completely inside the circle of radius $|a|$.

For λ small enough, S_a lies arbitrarily close to a. Thus, we have a function of degree $\delta = n + d$ that maps the exterior of γ (connectivity m) to the exterior of the circle of radius ρ (connectivity $h = 1$) and there are $N = (d+1)+(d-1)+(n-1) = 2d+n-1$ critical points in the domain (S_a, a and ∞). By the Riemann-Hurwitz formula, we get

$$m - 2 = \delta(h - 2) + N \quad \text{so that } m = d + 1\,. \tag{4.1}$$

Thus, there are d holes bounded by simple closed curves in the annulus bounded by γ and the circle of radius ρ that contain preimages of the interior of γ. It follows that these holes contain the d other preimages of Q so B is not simply connected. □

As before (see Proposition 3.1), it follows that there must be other components of the Julia set (uncountably many) accumulating on the boundaries of the preimages of Q. However, the symbolic dynamics on the Julia set is somewhat different.

To begin the construction of the sequence space, we first choose $|\lambda|$ sufficiently small so that

(1) all critical points in S_a lie in the annulus between the circle of radius ρ and the curve γ;
(2) all critical values that correspond to S_a lie outside the circle of radius ρ.

There are d preimages of the circle of radius ρ that we call $I_1, ..., I_d$ and which lie inside the annulus bounded by γ and the circle of radius ρ. Each I_j is mapped one-to-one to the circle of radius ρ. The set of points whose orbits remain for all iterations in the union of the I_j forms a Cantor set on which $f_{\lambda,a}$ is conjugate to the one-sided shift map on d symbols. This produces an uncountable number of point components in J. However, as in the case $|a| < 1$, there are many other point components in J.

Consider a circle of radius $s < 1$ that lies completely inside Q and contains all the critical points in S_q. Call $\mathcal{A}$ the annulus between this circle and the curve γ. Then $\partial Q \subset \mathcal{A}$ and notice that each I_j contains preimage of $\overline{Q}$ and a preimage of each one of the I_j's. See Figure 4.

As we did in the case $|a| < 1$ (Proposition 3.3), we can find an arc ξ lying in $\mathcal{A}$ and having the property that $f_{\lambda,a}$ maps ξ one-to-one onto a larger arc that properly contains ξ and connects the circle of radius s and γ. Moreover, ξ meets ∂Q at exactly one point, namely one of the fixed points in ∂Q. With the exception of this fixed point, all other points on ξ lie in the Fatou set.

Similarly, we now partition $\mathcal{A}$ into n rectangles using the n preimages of $f_{\lambda,a}(\xi)$ that lie in $\mathcal{A}$. Again denote these preimages by $\xi_1, ..., \xi_n$, where $\xi_1 = \xi$ and the remaining ξ_j's are arranged counterclockwise around $\mathcal{A}$. Let A_j denote the closed region in $\mathcal{A}$ that is bounded by ξ_j and ξ_{j+1}, so that A_n is bounded by ξ_n and ξ_1. By

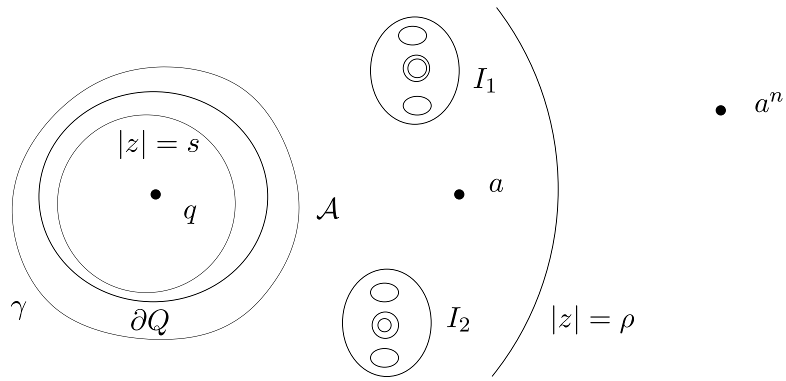

FIGURE 4. The region $\mathcal{A}$ and the I_j's when $|a| > 1$. The annulus $\mathcal{A}$ is bounded by the circle of radius $s < 1$ and the curve γ and contains the boundary of Q.

construction, each A_j is mapped one-to-one over $\mathcal{A}$ except on the boundary arcs ξ_j and ξ_{j+1}, which are each mapped one-to-one onto $f_{\lambda,a}(\xi_1) \supset \xi_1$. As before, the only points whose orbits remain for all iterations in $\mathcal{A}$ are those points on the simple closed curve ∂Q. Let $z \in \partial Q$. We may attach a symbol sequence $S(z)$ to z as follows. Consider the n symbols $\alpha_1, \ldots, \alpha_n$. Define $S(z) = (s_0 s_1 s_2 \ldots)$ where each s_j is one of the symbols $\alpha_1, \ldots, \alpha_n$ and $s_j = \alpha_k$ if and only if $f^j_{\lambda,a}(z) \in A_k$. We use the same identifications for points in ∂Q that we used for points in ∂B when $|a| < 1$. Then $f_{\lambda,a}|\partial Q$ is conjugate to $z \to z^n$ on the unit circle.

Finally, we extend the definition of $S(z)$ to any point in J that remains in the union of the I_j's by introducing the symbols $1, \ldots, d$ and defining $S(z)$ in the usual manner. We identify the sequences of the form $(j\overline{\alpha_1})$ and $(j\overline{\alpha_n})$ as well as $(j\alpha_k\overline{\alpha_n})$ and $(j\alpha_k\overline{\alpha_1})$.

Let Σ' denote the set of sequences $\alpha_1, \ldots, \alpha_n, 1, \ldots, d$. Let Σ denote the space Σ' with all of the identifications described above and endow Σ with the quotient topology. Then, by construction, the Julia set of $f_{\lambda,a}$ is homeomorphic to Σ and $f_{\lambda,a} \mid J$ is conjugate to the full shift map on Σ. This proves the main results in the case $|a| > 1$.

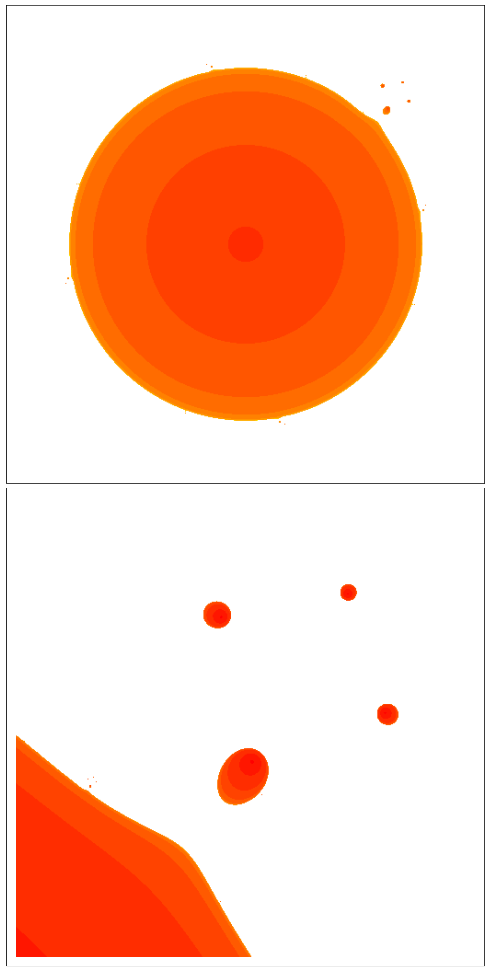

FIGURE 5. Top: the filled Julia set of $f_{\lambda,a}(z)$ for $n = d = 4$, $a = 1.2e^{i\pi/4}$, and $\lambda = 0.0001e^{i4\pi/3}$. The white region is B. The different shades show the dynamics in Q and its preimages. The picture is centered around the origin.
Bottom: magnification of a region around the pole a.

Figure 5 shows an example of the filled Julia set of $f_{\lambda,a}(z)$ with $n = d = 4$ when the pole a is outside the unit circle.

5. The $a\lambda$-plane

The theorems we have proved describe the structure of the $a\lambda$-plane when $|a| \neq 1$ and $|\lambda|$ is small. When $a = 0$, there is a McMullen domain surrounding $\lambda = 0$. But when $a \neq 0$, the McMullen domain is replaced by a region around $\lambda = 0$ where the Julia sets are quite different. Actually, the McMullen domain no longer surrounds the origin in the a-plane, but rather has evolved into a different type of space that contains an annulus surrounding 0. Figure 6 shows schematically two different regions in an $a\lambda$-slice of the parameter space when $n = d \geq 3$. The darker region around the λ axis (or plane) when $a = 0$ contains parameters in the McMullen domains for which the Julia sets of $f_{\lambda,a}$ are Cantor sets of simple closed curves. As we separate from the λ plane, that is,

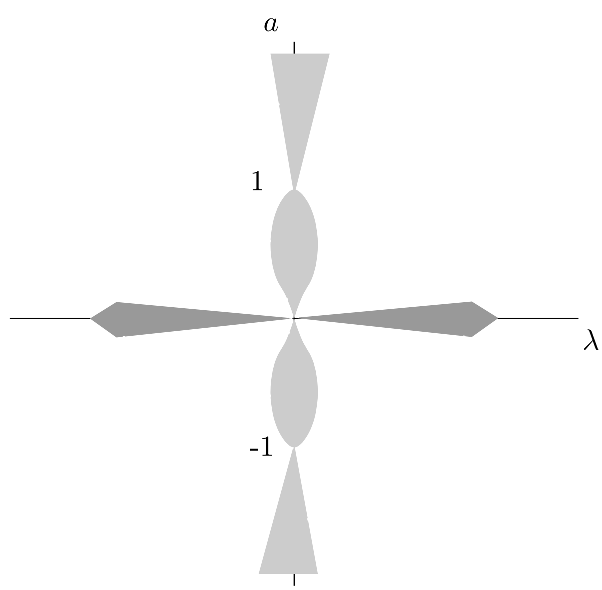

Figure 6. A schematic picture of a slice of the $a\lambda$-space where $|\lambda| << 1$ and $n = d \geq 3$. The darker regions along the λ axis represent McMullen domains in λ-plane. The shaded regions around the a axis show where we find the Julia sets studied in this paper.

as a becomes nonzero but $|\lambda|$ is very small, we find the region for which the Julia sets of $f_{\lambda,a}$ are the sets described in Theorem 1.2 (Structure of J).

Note that if $a = 0$ and λ lies in the McMullen domain, then the map $f_{\lambda,a}$ is hyperbolic on its Julia set. So the Cantor set of circles persists as we vary both a and λ starting at this parameter. In particular, there exists δ_λ such that if $|a| < \delta_\lambda$, then the Julia set of $f_{\lambda,a}$ is a Cantor set of circles. Now for any given λ^* in the McMullen domain, we may choose δ_{λ^*} so that this phenomenon persists for all a and λ in a neighborhood of $a = 0$ and $\lambda = \lambda^*$. Thus, for any simple closed curve γ in the McMullen domain for $a = 0$, there exists a_γ such that if $|a| < a_\gamma$ and λ lies on γ, then again $J(f_{\lambda,a})$ is a Cantor set of simple closed curves. Therefore, if we fix a with $|a| < a_\gamma$, then there is an annulus in the corresponding λ-plane with Julia sets of this type. Thus, we see that as a moves away from 0, the McMullen domain does not disappear. Rather, a hole opens up around $\lambda = 0$ in which the Julia set is as described in section 3 while the McMullen domain moves away from the origin. Between these two regions, the Julia sets clearly undergo major transformations, so the structure of the parameter planes in these intermediate regions is an open problem.

6. Conclusions

We have studied the two-parameter family of complex rational maps given by $f_{\lambda,a}(z) = z^n + \lambda/(z-a)^d$ where $n \geq 2$, $d \geq 1$, and a and λ are complex. We described the $a\lambda$-plane for the case $|\lambda| << 1$ and $|a| \neq 1$. We proved that if $|a| \neq 0, 1$ then the Julia sets consist of a countable number of simple closed curves and an uncountable number of point components that accumulate on those curves. It follows that when $n = d \geq 3$, the *typical* picture in parameter λ-plane (for $|\lambda| << 1$) does not feature a McMullen domain surrounding the origin. Finally, for $|\lambda|$ sufficiently small and for $|a| < 1$ ($|a| > 1$, resp.), any two $f_{\lambda,a}(z)$ are conjugate to each other.

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