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MUTUAL APOSYNDESIS AND PRODUCTS OF SOLENOIDS

JANUSZ R. PRAJS

ABSTRACT. This article is about the topological structure of the products of solenoids. It provides a characterization of the pairs of solenoids whose products are mutually aposyndetic.

The focus of this study is the topological structure of the products of solenoids. These products are fundamental spaces in the classification of compact (abelian) topological groups and important examples of homogeneous continua.

The concept of mutual aposyndesis was introduced by Charles Hagopian in [3]. Mutual aposyndesis, together with local connectedness, aposyndesis, indecomposability, and semi-indecomposability, belongs to the class of properties that differentiate spaces with respect to their subcontinua with interiors and the way these subcontinua separate points and sets. The structure of all solenoids with respect to these properties is the same. All solenoids are indecomposable and have only arcs for proper subcontinua. The product of two non-degenerate continua is aposyndetic, and the product of three non-degenerate continua is mutually aposyndetic [3, Theorem 2]. The question whether the product of two solenoids is mutually aposyndetic was partly answered in the affirmative by Alejandro Illanes in [4] for the product of p-adic and q-adic solenoids, where p and q are relatively prime integers.

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In a recent study [6], the author has shown that a Kelley continuum has a unique, minimal decomposition, called the *mutually aposyndetic decomposition*, such that the quotient space is a mutually aposyndetic continuum. Using this result, it has been proven in [6] that a homogeneous continuum with dense arc components is either mutually aposyndetic or semi-indecomposable. Since the product of (two) solenoids has dense arc components and is homogeneous, it belongs to exactly one of the two extreme classes of mutually aposyndetic or semi-indecomposable continua. According to Illanes's result, there are pairs of solenoids having mutually aposyndetic products. Here we show there also exist pairs of solenoids with semi-indecomposable products. Thus, even though all solenoids have the same structure in the sense discussed above, their products can manifest extremely opposite properties in that sense. The main result of this paper is a characterization of the class of pairs of solenoids having semi-indecomposable products. Using this characterization, we can easily identify the pairs with mutually aposyndetic products and the ones with semi-indecomposable products.

1. Preliminaries

All spaces are assumed to be metric and maps, continuous. The following concepts have been introduced in [7] and further studied in [8], [9], [10]. If X is a space, a subcontinuum K is called a *filament continuum* (in X) provided there exists a neighborhood N of K such that the component of N containing K has empty interior. A subcontinuum A of X is called *ample* if for every neighborhood U of A there is a continuum B such that $A \subset \text{Int}B \subset B \subset U$.

A space X is said to be *homogeneous* if for every $x, y \in X$ there is a homeomorphism $h : X \to X$ such that $h(x) = y$. In [7, Lemma 2.1 and Proposition 2.3], the following result was shown.

Theorem 1.1. *A subcontinuum K of a homogeneous continuum is ample if and only if K is non-filament.*

A continuum X is said to be *decomposable* if it has proper subcontinua Y and Z such that $X = Y \cup Z$. Continua that are not decomposable are called *indecomposable*.

The following concepts of *mutual aposyndesis* and *semi-indecomposability*[1] were introduced by Hagopian in [3]. A continuum X is *mutually aposyndetic at points x and y*, provided there are disjoint continua K and L in X containing x and y in their corresponding interiors. It is *mutually aposyndetic* if it is mutually aposyndetic at each pair of its distinct points. If X is not mutually aposyndetic at any pair of its distinct point, then X is called *semi-indecomposable.* Note that mutual aposyndesis and semi-indecomposability are properties opposite to each other, and a non-degenerate space cannot be simultaneously mutually aposyndetic and semi-indecomposable.

In a homogeneous continuum X (or, more generally, in Kelley spaces [6]), we can equivalently express mutual aposyndesis using ample continua. Indeed, X is mutually aposyndetic at x and y if and only if there are disjoint ample continua K and L in X containing x and y, respectively. Also, X is semi-indecomposable if and only if each two ample subcontinua of X have non-empty intersection.

If, for positive integers n, X_n are spaces and $g_n : X_{n+1} \to X_n$ are maps, the symbol $(\{X_n\}, \{f_n\})$ denotes the corresponding inverse system. We let $(X, \{f_n\}) = (\{X_n\}, \{f_n\})$ provided that $X_n = X$ for each n.

Two paths $p_1, p_2 : [0,1] \to X$ are called *path-homotopic* if there is a homotopy $H : [0,1] \times [0,1] \to X$ with $H(t,0) = p_1(t)$, $H(t,1) = p_2(t)$, $H(0,s) = p_1(0) = p_2(0)$, and $H(1,s) = p_1(1) = p_2(1)$ for all $s, t \in [0,1]$.

Throughout the paper, the symbol $\mathbb{S}^1$ denotes the unit circle in the complex plane, and $f_n : \mathbb{S}^1 \to \mathbb{S}^1$ denotes the function $f_n(z) = z^n$, for each positive integer n.

2. The torus and covering maps

In this section, we collect some old and new results about the 2-dimensional torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. Let $\pi_1, \pi_2 : \mathbb{T}^2 \to \mathbb{S}^1$ be the projection maps onto the first and second coordinates, respectively.

[1]Originally, Hagopian used the name *strictly non-mutually aposyndetic.* We use the name *semi-indecomposable* for brevity and also because the continua in question manifest properties similar to the properties of indecomposable continua.

First, we review some results about simple closed curves in the torus $\mathbb{T}^2$. A simple closed curve $S \subset \mathbb{T}^2$ can have exactly two different positions: (1) *separating*, meaning $\mathbb{T}^2 - S$ is disconnected, and (2) *non-separating*, with connected $\mathbb{T}^2 - S$ [11, Theorem 13, Exercise 14, and Exercise 25]. More precisely, we have the following.

Theorem 2.1. *Let S_1 and S_2 be simple closed curves in a torus $\mathbb{T}^2$. There exists a homeomorphism $h : \mathbb{T}^2 \to \mathbb{T}^2$ such that $h(S_1) = S_2$ if and only if S_1 and S_2 are either both separating or both non-separating simple closed curves in $\mathbb{T}^2$.*

These two classes can also be characterized as the classes of (1) inessential and (2) essential simple closed curves in $\mathbb{T}^2$, that is, curves having the embedding map into $\mathbb{T}^2$ nulhomotopic and non-nulhomotopic, respectively.

Proposition 2.2. *If K and L are two disjoint continua in the torus $\mathbb{T}^2$, then either $\mathbb{T}^2 - K$ or $\mathbb{T}^2 - L$ contains a non-separating simple closed curve.*

Proof: Let K_1 be the union of K and all components of $\mathbb{T}^2 - K$ except the one containing L and let L_1 be the union of L and all components of $\mathbb{T}^2 - L$ except the one containing K. The sets K_1 and L_1 are disjoint continua in $\mathbb{T}^2$ with connected complements. It suffices to show the conclusion for K_1 and L_1. Suppose $\mathbb{T}^2 - K_1$ contains only separating simple closed curves.

Case 1. There is a simple closed curve $S \subset \mathbb{T}^2 - K_1$ that is essential in $\mathbb{T}^2 - K_1$.

By the assumption, $\mathbb{T}^2 - S$ has two components: simply connected, C_1, and non-simply connected, C_2. In this case, C_1 is not contained in $\mathbb{T}^2 - K_1$, so $C_1 \cap K_1 \neq \emptyset$. Since S is the boundary of C_1 and $S \cap K_1 = \emptyset$, we have $K_1 \subset C_1$. Consequently, $C_2 \subset \mathbb{T}^2 - K_1$. Since C_2 contains non-separating simple closed curves, so does $\mathbb{T}^2 - K_1$, a contradiction.

Case 2. All simple closed curves in $\mathbb{T}^2 - K_1$ are inessential in $\mathbb{T}^2 - K_1$.

Then $\mathbb{T}^2 - K_1$ is a simply connected, non-compact 2-manifold, which is homeomorphic to $\mathbb{R}^2$. Consequently, there is a simple closed curve S_0 in $\mathbb{T}^2 - K_1$ such that L_1 is contained in the simply connected component $C_{0,1}$ of $\mathbb{T}^2 - S_0$. The other component, $C_{0,2}$, of $\mathbb{T}^2 - S_0$ contains simple closed curves that are essential in $\mathbb{T}^2$, and $C_{0,2} \subset \mathbb{T}^2 - L_1$. The conclusion follows. □

Proposition 2.3. *If $f = f_n \times f_n : \mathbb{T}^2 \to \mathbb{T}^2$ for some positive integer n and S is a simple closed curve in $\mathbb{T}^2$, then $f^{-1}(S)$ has at least n components.*

Proof: Let $y = (y_1, y_2) \in S$ and $h : \mathbb{S}^1 \to S$ be a homeomorphism with $h(1) = y$. Let $g_k : [0,1] \to \mathbb{S}^1$ be the loop $g_k(t) = (\cos 2\pi kt, \sin 2\pi kt)$ for each positive integer k, and note that g_k is the product of k maps g_1 (in the sense of the *product of paths*, as in the definition of the fundamental group). Define $\alpha_k, \beta_k : [0,1] \to \mathbb{S}^1$ and $\gamma_k : [0,1] \to \mathbb{T}^2$ to be the loops $\alpha_k = \pi_1 \circ h \circ g_k$, $\beta_k = \pi_2 \circ h \circ g_k$, and $\gamma_k(t) = (h \circ g_k)(t)$. Given an $x = (x_1, x_2) \in f^{-1}(y)$, we note $x_1 \in f_n^{-1}(y_1)$ and $x_2 \in f_n^{-1}(y_2)$. Let $\hat{\alpha}_n, \hat{\beta}_n : [0,1] \to \mathbb{S}^1$ be the corresponding lifted paths of α_n and β_n with respect to f_n such that $\hat{\alpha}_n(0) = x_1$ and $\hat{\beta}_n(0) = x_2$. In other words, $\hat{\alpha}_n$ and $\hat{\beta}_n$ are the paths that begin at x_1 and x_2, respectively, and satisfy $f_n \circ \hat{\alpha}_n = \alpha_n$ and $f_n \circ \hat{\beta}_n = \beta_n$. Since $f_n : \mathbb{S}^1 \to \mathbb{S}^1$ is the n-fold covering $f_n(z) = z^n$, and α_n and β_n are products of n loops α_1 and β_1, respectively, their corresponding lifted paths $\hat{\alpha}_n$ and $\hat{\beta}_n$ are loops. Note that the loop $\hat{\gamma}_n : [0,1] \to \mathbb{T}^2$, defined by $\hat{\gamma}_n(t) = (\hat{\alpha}_n(t), \hat{\beta}_n(t))$, is a path-lifting, with respect to the covering $f = f_n \times f_n$, of the product γ_n of n maps γ_1. It also satisfies $\hat{\gamma}_n(0) = \hat{\gamma}_n(1) = (x_1, x_2)$. Since f is a covering map, for sufficiently large m, the set $\hat{\gamma}_m([0,1])$ equals the component C of $f^{-1}(S)$ containing (x_1, x_2). Clearly, $\hat{\gamma}_m([0,1]) = \hat{\gamma}_{ln}([0,1])$ for an l such that $ln \geq m$. Thus, $C = \hat{\gamma}_m([0,1]) = \hat{\gamma}_{ln}([0,1]) = \hat{\gamma}_n([0,1])$ because $\hat{\gamma}_n$ is a loop. We have $\gamma_n(t) = y$ for exactly n values of t in $[0,1)$, and thus $|C \cap f^{-1}(y)| \leq n$. Since f is an n^2-fold covering, it follows $|f^{-1}(y)| = n^2$. Hence, there are at least n different components of $f^{-1}(S)$. □

Proposition 2.4. *If $f = f_n \times f_n : \mathbb{T}^2 \to \mathbb{T}^2$ for some positive integer n and S is a non-separating simple closed curve in $\mathbb{T}^2$, then $f^{-1}(\mathbb{T}^2 - S)$ has at least n components.*

Proof: Since the position of S in $\mathbb{T}^2$ is the same as the one of the meridian $\{1\} \times \mathbb{S}^1$, it follows that $\mathbb{T}^2 - S$ is homeomorphic to an open annulus with another simple closed curve S_0 being a strong deformation retract of $\mathbb{T}^2 - S$.

Suppose $p : [0,1] \to f^{-1}(\mathbb{T}^2 - S)$ is a path with $p(0) = a \in f^{-1}(S_0)$ and $p(1) = b \in f^{-1}(S_0)$. Since S_0 is a deformation retract of $\mathbb{T}^2 - S$, the path $p_0 = f \circ p$ is path-homotopic in $\mathbb{T}^2 - S$ to a path $p_1 : [0,1] \to \mathbb{T}^2 - S$ with $p_1([0,1]) \subset S_0$. Let $\hat{p}_1 : [0,1] \to \mathbb{T}^2$ be the path-lifting of p_1 with respect to f such that $\hat{p}_1(0) = a$. Since p_1 and p_0 are path-homotopic and the maps $\hat{p}_1$ and p are their corresponding liftings that begin at the same point a, it follows that $\hat{p}_1(1) = p(1) = b$. Note that $\hat{p}_1([0,1]) \subset f^{-1}(S_0)$, and thus a and b are in the same path component of $f^{-1}(S_0)$. We have shown that if $f^{-1}(\mathbb{T}^2 - S)$ is path connected between two points in $f^{-1}(S_0)$, then $f^{-1}(S_0)$ is path connected between these two points. Since the set $f^{-1}(\mathbb{T}^2 - S)$ is locally connected and locally compact, its path components and (connected) components are the same. By Proposition 2.3, the set $f^{-1}(S_0)$ has at least n components. Hence, $f^{-1}(\mathbb{T}^2 - S)$ has at least n components. □

Proposition 2.5. *If $f = f_m \times f_n : \mathbb{T}^2 \to \mathbb{T}^2$ for some relatively prime numbers m and n, and $\Delta = \{(z,z) \mid z \in \mathbb{S}^1\}$ is the diagonal of $\mathbb{T}^2$, then $f^{-1}(\Delta)$ is connected.*

Proof: For $k = mn$, let $g_k : [0,1] \to \mathbb{S}^1$ be the loop $g_k(t) = (\cos 2\pi kt, \sin 2\pi kt)$, and let $\hat{g}_{1,k}, \hat{g}_{2,k} : [0,1] \to \mathbb{S}^1$ be the path-liftings of g_k, with respect to f_m and f_n, respectively, such that $\hat{g}_{1,k}(0) = \hat{g}_{2,k}(0) = 1$. Consider the path $p_k : [0,1] \to \Delta \subset \mathbb{T}^2$, defined by $p_k(t) = (g_k(t), g_k(t))$, and the corresponding path lifting $\hat{p}_k : [0,1] \to \mathbb{T}^2$, with respect to $f = f_m \times f_n$, such that $\hat{p}_k(0) = (1,1)$. Note that $\pi_1 \circ \hat{p}_k = \hat{g}_{1,k}$, $\pi_2 \circ \hat{p}_k = \hat{g}_{2,k}$, and $\hat{p}_k(1) = (1,1)$.

Claim. If $\hat{p}_k(s) = \hat{p}_k(t)$ for some s and t with $0 \leq s < t \leq 1$, then $s = 0$ and $t = 1$.

Indeed, since $\hat{g}_{1,k}(s) = \hat{g}_{1,k}(t)$ and $\hat{g}_{1,k}$ is a path-lifting of g_k with respect to f_m, the number $kt - ks = mn(t-s)$ is a multiple of m. Similarly, using the definition of $\hat{g}_{2,k}$, we argue that $mn(t-s)$ is a multiple of n. The numbers m and n are relatively prime, and thus $mn(t-s)$ is a multiple of mn. Consequently, $t - s$ is a non-zero integer. Hence, $s = 0$ and $t = 1$.

For every $j \in \{0, 1, \ldots, k\}$, we have $\hat{p}_k(j/k) \in f^{-1}(1,1)$. Thus, by the Claim, $|\hat{p}_k([0,1]) \cap f^{-1}(1,1)| \geq mn$. Since f is an (mn)-fold covering map, it follows $f^{-1}(1,1) \subset \hat{p}_k([0,1])$. So $\hat{p}_k([0,1])$ is a simple closed curve in $f^{-1}(\Delta)$, and it contains an entire fiber, $f^{-1}(1,1)$, of f. The map $f|f^{-1}(\Delta) : f^{-1}(\Delta) \to \Delta$ is a covering for the simple closed curve Δ. Hence, $f^{-1}(\Delta) = \hat{p}_k([0,1])$, which completes the proof. □

3. The products of solenoids

A *solenoid* is the inverse limit of simple closed curves with covering bonding maps different from homeomorphisms[2]. Each solenoid can be represented as the limit of an inverse system $(\mathbb{S}^1, \{f_{r_k}\})$, where $R = (r_1, r_2, \dots)$ is a sequence of integers greater than 1. We denote this inverse limit by S_R. Each solenoid can also be represented as the inverse limit S_P, where $P = (p_1, p_2, \dots)$ is a sequence of primes (number 1 is not considered prime).

In the proof of the next theorem, which is the main result of the paper, we will use the following known and easy to proof fact. If $P = (p_1, p_2, \dots)$ and $Q = (q_1, q_2, \dots)$ are sequences of (not necessarily prime) integers, then the product $S_P \times S_Q$ is homeomorphic to the limit of the inverse sequence $(\mathbb{T}^2, \{f_{p_n} \times f_{q_n}\})$.

Theorem 3.1. *If $P = (p_1, p_2, \dots)$ and $Q = (q_1, q_2, \dots)$ are sequences of primes and S_P, S_Q are the corresponding solenoids, then the product $S_P \times S_Q$ is semi-indecomposable if and only if for every number M there are $i, j > M$ such that $p_i = q_j$.*

Proof: Let $\varphi_n : S_P \to \mathbb{S}^1$, $\psi_n : S_Q \to \mathbb{S}^1$ be the projection maps onto the n-th coordinate of the corresponding inverse limit space.

Suppose there exists a number M such that $p_i \neq q_j$ for all $i, j > M$. Since disregarding finitely many initial spaces and mappings in an inverse sequence leads to a homeomorphic inverse limit space, we can modify sequences P and Q so that $p_i \neq q_j$ for all i, j. Let $r_n = p_1 \cdot p_2 \cdots p_n$ and $s_n = q_1 \cdot q_2 \cdots q_n$. The product $S_P \times S_Q$ can be naturally represented as the limit of the inverse sequence $(\mathbb{T}^2, \{f_{p_n} \times f_{q_n}\})$. Note that $\varphi_n \times \psi_n$ are the projections of this inverse limit.

Let Δ be the diagonal of $\mathbb{T}^2$ as in Proposition 2.5, $\Delta_1 = \Delta$, and $\Delta_{n+1} = (f_{p_n} \times f_{q_n})^{-1}(\Delta_n)$. Since r_n and s_n are relatively prime

[2]Other authors also admit homeomorphisms for bonding maps in this definition, which results with additionally including the simple closed curves in the class of solenoids.

for every n and $\Delta_n = (f_{r_n} \times f_{s_n})^{-1}(\Delta)$, each set Δ_n is connected by Proposition 2.5. Consider the inverse limit K of the inverse sequence $(\{\Delta_n\}, \{(f_{p_n} \times f_{q_n})|\Delta_{n+1}\})$, which is naturally embedded in $S_P \times S_Q$. Since Δ_n's are connected, K is a continuum. By the definition of K, it follows $\varphi_1^{-1}(1) \times \psi_1^{-1}(1) = (\varphi_1 \times \psi_1)^{-1}(1,1) \subset K$. The solenoids S_P and S_Q have the local product structure of an arc times the topological Cantor set, and the sets $\varphi_1^{-1}(1)$ and $\psi_1^{-1}(1)$ are Cantor set cross sections of such structures for S_P and S_Q, respectively. Each point in $\varphi_1^{-1}(1)$ can be slightly enlarged to an arc in S_P so that the union of such arcs, a closed set C, has interior in S_P. Similarly, the set $\psi_1^{-1}(1)$ can be slightly enlarged to a closed set D with non-empty interior in S_Q. We note that $K_0 = K \cup (C \times D)$ is a continuum with interior in $S_P \times S_Q$. Moreover, such K_0 can be defined arbitrarily near, in the sense of the Hausdorff distance, to K. Therefore, K is non-filament in $S_P \times S_Q$. Consequently, K is an ample subcontinuum of $S_P \times S_Q$ by Theorem 1.1.

Let $\mathbf{z} = (z_1, z_2, \dots)$ be a fixed point in S_P with $z_1 \neq 1$. For every $\mathbf{x} = (x_1, x_2, \dots) \in S_P$ and $\mathbf{y} = (y_1, y_2, \dots) \in S_Q$, define $h(\mathbf{x}, \mathbf{y}) = (\mathbf{zx}, \mathbf{y})$, where $\mathbf{zx} = (z_1x_1, z_2x_2, \dots)$. Note that $h : S_P \times S_Q \to S_P \times S_Q$ is a homeomorphism and $(\varphi_1 \times \psi_1)(h(K)) \cap (\varphi_1 \times \psi_1)(K) = (\varphi_1 \times \psi_1)(h(K)) \cap \Delta = \emptyset$. Thus, K and $h(K)$ are disjoint ample subcontinua of $S_P \times S_Q$. Hence, $S_P \times S_Q$ is not semi-indecomposable.

Suppose for every M there are integers $i, j > M$ such that $p_i = q_j$. We inductively choose two subsequences k_n and l_n of positive integers as follows. Let $k_1 = 1$ and $l_1 = 1$. If k_n, l_n are already defined for some positive odd integer n, we choose integers k_{n+1} and l_{n+1} greater than $k_n + 1$ and $l_n + 1$, respectively, such that $p_{k_{n+1}} = q_{l_{n+1}}$. Further, define $k_{n+2} = k_{n+1} + 1$ and $l_{n+2} = l_{n+1} + 1$. Next, we define two sequences a_n and b_n of positive integers letting

$$a_1 = \prod_{j=1}^{k_2-1} p_j, \quad a_n = \prod_{j=k_n}^{k_{n+1}-1} p_j, \quad \text{and} \quad b_1 = \prod_{j=1}^{l_2-1} q_j, \quad b_n = \prod_{j=l_n}^{l_{n+1}-1} q_j.$$

Observe that $a_n = b_n = p_{k_n} = q_{l_n}$ for every even number n. We also have $a_1 \cdot a_2 \cdots a_n = p_1 \cdot p_2 \cdots p_{k_n-1}$ and $b_1 \cdot b_2 \cdots b_n = q_1 \cdot q_2 \cdots q_{l_n-1}$ for every n. Let $A = (a_1, a_2, \dots)$ and $B = (b_1, b_2, \dots)$. The expressions $(x_1, x_2, \dots) \mapsto (x_{k_1}, x_{k_2}, \dots)$ and $(x_1, x_2, \dots) \mapsto (x_{l_1}, x_{l_2}, \dots)$ establish homeomorphisms between solenoids S_P and

S_A and between S_Q and S_B, respectively. Therefore, it suffices to show that $S_A \times S_B$ is semi-indecomposable.

We view the space $S_A \times S_B$ as the limit of the inverse sequence $(\mathbb{T}^n, \{f_{a_n} \times f_{b_n}\})$. Let $\alpha_n : S_A \to \mathbb{S}^1$ and $\beta_n : S_B \to \mathbb{S}^1$ be the projections of S_A and S_B, respectively, onto the n-th coordinate. Then $\gamma_n = \alpha_n \times \beta_n : S_A \times S_B \to \mathbb{T}^2$ are the projections of $S_A \times S_B$.

Suppose K and L are disjoint subcontinua of $S_A \times S_B$ with non-empty interiors. Define $K_n = \gamma_n(K)$ and $L_n = \gamma_n(L)$. For sufficiently large number T, if $n > T$, we have $K_n \cap L_n = \emptyset$, and there are points $x_n \in K_n$ and $y_n \in L_n$ such that $\gamma_n^{-1}(x_n) \subset K$ and $\gamma_n^{-1}(y_n) \subset L$. Let n be such a number that, additionally, is even, and x_n and y_n be such points. Since K_n and L_n are disjoint subcontinua of $\mathbb{T}^2$, by Proposition 2.2, there is a non-separating simple closed curve S contained either in $\mathbb{T}^2 - K_n$ or in $\mathbb{T}^2 - L_n$. Assume $S \subset \mathbb{T}^2 - K_n$ (the proof in the other case is similar). Since $a_n = b_n$, by Proposition 2.4, it follows $(f_{a_n} \times f_{b_n})^{-1}(K_n) = (f_{a_n} \times f_{a_n})^{-1}(K_n)$ has at least a_n components. In particular, $(f_{a_n} \times f_{b_n})^{-1}(K_n)$ is not connected. Since $\gamma_n^{-1}(x_n) \subset K$ and the bonding maps are surjective, we have $(f_{a_n} \times f_{b_n})^{-1}(x_n) \subset \gamma_{n+1}(K) = K_{n+1}$. The bonding map $f_{a_n} \times f_{b_n}$ is a covering, and thus, $(f_{a_n} \times f_{b_n})^{-1}(x_n)$ intersects different components of $(f_{a_n} \times f_{b_n})^{-1}(K_n)$. Hence, the image, K_{n+1}, of the continuum K is not connected, an impossibility. □

4. Applications

In this section, using some known results, we show consequences that follow from Theorem 3.1. First, we recall the following result, which has been recently proven by the author in [6, Corollary 4.4].

Theorem 4.1. *If X is a homogeneous continuum with dense arc components, then X is either mutually aposyndetic or semi-indecomposable.*

Since solenoids and their products have dense arc components and are homogeneous, by Theorem 3.1 and Theorem 4.1, we have the following characterization of mutually aposyndetic products of solenoids.

Corollary 4.2. *If $P = (p_1, p_2, \dots)$ and $Q = (q_1, q_2, \dots)$ are sequences of primes, and S_P and S_Q are the corresponding solenoids, the three following conditions are equivalent.*

(a) *$S_P \times S_Q$ is mutually aposyndetic,*
(b) *$S_P \times S_Q$ is not semi-indecomposable, and*
(c) *there exists a number M such that $p_i \neq q_j$ for all $i, j > M$.*

Remark 4.3. From the view point of mutual aposyndesis, the product of two solenoids is the only non-trivial case. Indeed, each solenoid is indecomposable, and thus semi-indecomposable, and the product of more than two solenoids is mutually aposyndetic by [3, Theorem 2]. Thus, Corollary 4.2 yields a complete characterization of all mutually aposyndetic products of solenoids.

Let $\mathbb{N}$ be the set of positive integers and $\mathcal{P}$ be the class of sequences $P = (p_1, p_2, \dots)$ of primes. If $P = (p_1, p_2, \dots)$ and $Q = (q_1, q_2, \dots)$ are in $\mathcal{P}$, we write $P \preceq Q$ provided there are a set $\mathbb{N}_1$ of almost all positive integers and a one-to-one function $\alpha : \mathbb{N}_1 \to \mathbb{N}$ such that $q_{\alpha(n)} = p_n$ for each $n \in \mathbb{N}_1$. Clearly, $\preceq$ is a quasi-order in $\mathcal{P}$. The next theorem can be derived from [2, Theorem 6 and Theorem 8]. Note that the results of [2] are expressed in different terms.

Theorem 4.4 (Cook)**.** *Let S_P and S_Q be solenoids with $P, Q \in \mathcal{P}$. There exists a continuous surjection $f : S_P \to S_Q$ if and only if $Q \preceq P$.*

Expressing solenoids as inverse limits S_P, where $P \in \mathcal{P}$, can also be used to topologically classify solenoids. We formulate two more known results in this direction to develop a better view of the significance of the relation $\preceq$ on $\mathcal{P}$ for the class of solenoids. Indeed, two solenoids S_P and S_Q, where $P, Q \in \mathcal{P}$, are homeomorphic if and only if for almost all entries of P their order can be rearranged to agree with almost all entries of Q [5] (see also [1] for a more elementary proof). Using the relation $\preceq$, the above condition on P and Q can be expressed by $P \preceq Q$ and $Q \preceq P$.

Theorem 4.5 (McCord)**.** *Let S_P and S_Q be solenoids with $P, Q \in \mathcal{P}$. Then S_P and S_Q are homeomorphic if and only if $P \preceq Q$ and $Q \preceq P$.*

Two spaces X and Y are called *continuously equivalent* provided there exist continuous surjections $f : X \to Y$ and $g : Y \to X$. The next corollary, originally observed in [2, Theorem 9], follows from Theorem 4.4 and Theorem 4.5.

Corollary 4.6 (Cook)**.** *Two solenoids are homeomorphic if and only if they are continuously equivalent.*

Note that the negation of condition (c) in Corollary 4.2 is equivalent to the condition that there is a sequence $R = (r_1, r_2, \dots) \in \mathcal{P}$ such that $R \preceq P$ and $R \preceq Q$. The next corollary, which ends the paper, follows from Corollary 4.2 and Theorem 4.4.

Corollary 4.7. *If S_P and S_Q are solenoids, then their product $S_P \times S_Q$ is semi-indecomposable if and only if there exists a solenoid S_R such that both S_P and S_Q admit continuous surjections onto S_R.*

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