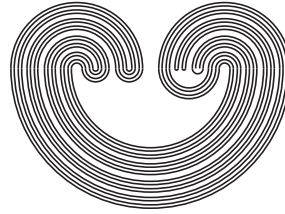


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MULTIFUNCTION SPACE  $L(X)$   
OF CUSCO MAPS

by

TANVI JAIN AND R. A. MCCOY

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**Mail:** Topology Proceedings  
Department of Mathematics & Statistics  
Auburn University, Alabama 36849, USA

**E-mail:** [topolog@auburn.edu](mailto:topolog@auburn.edu)

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## LINDELÖF PROPERTY OF THE MULTIFUNCTION SPACE $L(X)$ OF CUSCO MAPS

TANVI JAIN AND R. A. McCOY

**ABSTRACT.** A set-valued mapping  $F$  from a topological space  $X$  to a topological space  $Y$  is called a cusco map if  $F$  is upper semicontinuous and  $F(x)$  is a nonempty, compact, and connected subset of  $Y$  for each  $x \in X$ . We denote by  $L(X)$  the space of all subsets  $F$  of  $X \times \mathbb{R}$  such that  $F$  is the graph of a cusco map from the space  $X$  to the real line  $\mathbb{R}$ . In this paper, we find several necessary conditions and several sufficient conditions on  $X$  such that  $L(X)$  with the Vietoris topology is Lindelöf. We also study some conditions on  $X$  that are sufficient for  $L(X)$  with the Fell topology to be Lindelöf.

### 1. INTRODUCTION

For a function space, one of the topological properties that is difficult to characterize is the Lindelöf property. For example, an open problem (see [1]; [2, Chapter 1, Section 4]; [3, pp. 29–32]; [25, Exercise 3, p. 68]) is how can one characterize  $C_p(X)$  as Lindelöf in terms of topological properties of  $X$ , where  $C_p(X)$  is the space of continuous real-valued functions on  $X$  under the topology of pointwise convergence. A number of people have obtained partial

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results for this “Lindelöf problem,” but no full characterization is known.

There has recently been much interest in the multifunction space of usc, usco, and cusco maps on a topological space  $X$  under various hyperspace topologies (see [10], [14], [15], [20], [24]). In particular, an interesting and classical problem that leads to the study of cusco maps is to characterize the closure of  $C(X)$  in the space  $CL(X \times \mathbb{R})$  of all nonempty closed subsets of  $X \times \mathbb{R}$  for various hyperspace topologies on  $CL(X \times \mathbb{R})$  (see [5], [14], [13], [16], [17], [18]). The cusco maps and minimal cusco maps are also important tools in convex analysis (see [7]).

Various topological properties of the space  $L_V(X)$  of cusco maps from  $X$  to the real line  $\mathbb{R}$  with the Vietoris topology were studied in [15]: metrizability, complete metrizability, and countability properties, but not Lindelöf. It was shown that for a normal space  $X$ , the metrizability, complete metrizability, second countability, separability, and countable chain condition of  $L_V(X)$  are all equivalent to  $X$  being compact and metrizable. In this paper, we consider the Lindelöf problem for the multifunction space  $L_V(X)$  of cusco maps on  $X$  under the Vietoris topology. It is not surprising that it is difficult to characterize when  $L_V(X)$  is a Lindelöf space in terms of properties of  $X$ . But we give several necessary properties of  $X$  and several sufficient properties of  $X$ , along with examples showing why these properties are not both necessary and sufficient for  $L_V(X)$  to be a Lindelöf space. This leaves the full characterization as an open problem. We also consider the “Lindelöf problem” for the same space  $L_F(X)$  except having the Fell topology.

## 2. PRELIMINARIES

We refer to Gerald Beer [6] and Ryszard Engelking [12] for basic notions. If  $X$  and  $Y$  are nonempty sets, a set-valued mapping or multifunction from  $X$  to  $Y$  is a mapping that assigns to each element of  $X$  a (possibly empty) subset of  $Y$ . If  $T$  is a set-valued mapping from  $X$  to  $Y$ , then its graph is  $\{(x, y) : y \in T(x)\}$ .

If  $F$  is a subset of  $X \times Y$  and  $x \in X$ , define  $F(x) = \{y \in Y : (x, y) \in F\}$ . We assign to each subset  $F$  of  $X \times Y$  a set-valued mapping which takes the value  $F(x)$  at each point  $x \in X$ . Then  $F$  is the graph of the set-valued mapping. In this paper, we identify mappings with their graphs.

Let  $X$  and  $Y$  be topological spaces, and let  $T$  be a set-valued mapping from  $X$  to  $Y$ . Then  $T$  is called *upper semicontinuous (usc)* if for each  $x \in X$  and any open set  $V$  containing  $T(x)$ , there exists a neighborhood  $U_x$  of  $x$  such that  $T(z) \subseteq V$  for all  $z \in U_x$ . Following Jens Peter Reus Christensen [8], we say that  $T$  is a *usco map* if  $T$  is a usc map such that  $T(x)$  is a nonempty compact set for all  $x \in X$ . Similarly, we say that  $T$  is *cusco* if it is usco and  $T(x)$  is connected for all  $x \in X$ . In the literature, the notation *cusco* [7] is also used for usco maps with convex values in a topological vector space. Since we are working only with multifunctions with values in  $\mathbb{R}$ , both of these notations coincide in our case.

To describe the hypertopologies that we are using in this paper, we need to introduce the following notation. Let  $(X, \tau)$  be a topological space and  $CL(X)$  be the hyperspace of all nonempty closed subsets of  $X$ . For  $U \subseteq X$ , define  $U^+ = \{A \in CL(X) : A \subseteq U\}$  and  $U^- = \{A \in CL(X) : A \cap U \neq \emptyset\}$ . If  $\mathcal{U}$  is a family of sets in  $X$ , define  $\mathcal{U}^- = \cap\{U^- : U \in \mathcal{U}\}$ .

A subbase for the Vietoris (Fell, resp.) topology on  $CL(X)$  (see [6]) is the family of sets of the form  $U^+$  with  $U \in \tau$  ( $U$  has compact complement in  $X$ , resp.) and of the form  $\mathcal{U}^-$  with  $\mathcal{U} \subseteq \tau$  finite.

In this paper,  $X$  always denotes a Hausdorff space. The sets of real numbers and natural numbers are denoted by  $\mathbb{R}$  and  $\mathbb{N}$ , respectively. For any subset  $A$  of a topological space  $X$ , the closure, boundary, and complement of  $A$  are denoted by  $\bar{A}$ ,  $bd(A)$ , and  $X \setminus A$ , respectively. However, in the case of  $X \times \mathbb{R}$ , for any subset  $A$  of  $X \times \mathbb{R}$ , we denote the complement of  $A$  in  $X \times \mathbb{R}$  by  $A^c$  also. Further, for any hyperspace  $H(X)$ , we denote by  $H_V(X)$  ( $H_F(X)$ , resp.),  $H(X)$  with the Vietoris (Fell, resp.) topology.

The results given in this section, although used in the next two sections, also have their independent importance.

The first result in this section states some basic facts on graphs of cusco maps which will be used in several results in this paper.

**Lemma 2.1** ([15, Lemma 3.1]). *For a Hausdorff space  $X$ , the following statements are equivalent.*

- (a)  $F \subset X \times \mathbb{R}$  is the graph of a cusco map.
- (b)  $F$  is a closed, locally bounded subset of  $X \times \mathbb{R}$  with  $F(x)$  nonempty and connected for each  $x \in X$ .

- (c) *There exist real-valued functions  $f$  and  $g$  on  $X$  with  $f \leq g$  and  $f$  and  $g$  lower and upper semicontinuous, respectively, such that  $F(x) = [f(x), g(x)]$  for each  $x \in X$ .*

**Theorem 2.2.** *For a Hausdorff space  $X$ , the following assertions hold.*

- (a)  *$CL_V(X)$  can be homeomorphically embedded into  $L_V(X)$ .*  
 (b)  *$CL_F(X)$  can be homeomorphically embedded into  $L_F(X)$ .*  
 (c) *If  $X$  is regular, then  $CL_V(X)$  can be embedded as a closed subspace of  $L_V(X)$ .*

*Proof:* Parts (a) and (c) follow from [15, Proposition 3.2 and Proposition 3.3], respectively.

Now, as in the proof of [15, Proposition 3.2], for each  $E \in CL(X)$ , define

$$F_E = E \times [0, 1] \cup X \times \{0\}.$$

Then  $F_E \in L(X)$ . Consider a map  $\phi : CL(X) \rightarrow L(X)$  defined by  $\phi(E) = F_E$  for each  $E \in CL(X)$ . We shall show that  $\phi$  is a homeomorphism from  $CL_F(X)$  to  $L_F(X)$ . Clearly,  $\phi$  is one-one.

First, let  $E \in CL(X)$  and  $K$  be a compact set in  $X \times \mathbb{R}$  such that  $F_E \in (K^c)^+$ . Without loss of generality, we can assume that  $K \subseteq X \times [0, 1]$ . Then it can be verified that the set

$$K_X = \{x \in X : \text{there exists some } t \text{ in } [0, 1] \text{ such that } (x, t) \in K\}$$

is a compact set in  $X$  with  $E \in (K_X^c)^+$  and  $\phi(X \setminus K_X)^+ \subseteq (K^c)^+$ . Therefore, by [15, Proposition 3.2], we obtain that  $\phi$  is continuous from  $CL_F(X)$  to  $L_F(X)$ .

Now, for any compact set  $C$  in  $X$ , let

$$K = C \times \{1\}.$$

Then  $K$  is a compact set in  $X \times \mathbb{R}$  and  $\phi((X \setminus C)^+) = (K^c)^+ \cap \phi(CL(X))$ . Hence, again by [15, Proposition 3.2], we finally obtain that  $\phi$  is a homeomorphism.  $\square$

**Remark 2.3.** Consider the subspace

$$CL^*(X \times \mathbb{R}) = \{F \in CL(X \times \mathbb{R}) : F(x) \neq \emptyset \text{ for all } x \in X\}$$

of  $CL(X \times \mathbb{R})$ . Then since for each  $x \in X$ ,  $((X \setminus \{x\}) \times \mathbb{R})^+$  is an open set in  $CL_V(X \times \mathbb{R})$  containing no member of  $CL^*(X \times \mathbb{R})$ ,  $CL_V^*(X \times \mathbb{R})$  is a closed subspace of  $CL_V(X \times \mathbb{R})$ .

The space  $L_V(X)$  is never closed in  $CL_V(X \times \mathbb{R})$  since the set

$$\mathcal{M} = \{A \cup (C \times \mathbb{R}) : A \in L(X), C \text{ is a closed subset of } X\}$$

is contained in the closure of  $L_V(X)$  in  $CL_V(X \times \mathbb{R})$ . However, for a compact space  $X$ ,  $L_V(X)$  is closed in the space  $K_V(X \times \mathbb{R})$  of nonempty compact subsets of  $X \times \mathbb{R}$  if and only if  $X$  is locally connected, as shown by the following theorem. In fact, the theorem gives an equivalent characterization for  $L_V(X)$  to be locally compact for a Hausdorff space  $X$ .

**Theorem 2.4.** *For a Hausdorff space  $X$ , the following assertions are equivalent.*

- (a)  $X$  is compact and locally connected.
- (b)  $L_V(X)$  is a closed subset of  $K_V(X \times \mathbb{R})$ .
- (c)  $L_V(X)$  is locally compact.

*Proof:* (a)  $\implies$  (b): Since  $X$  is compact, by [12, Problem 3.12.23(g)], every member  $A$  of  $L(X)$  is contained in  $X \times [-n, n]$  for some  $n \in \mathbb{N}$ . Therefore,  $L(X) \subseteq K(X \times \mathbb{R})$ . We shall show that  $L_V(X)$  is closed in  $K_V(X \times \mathbb{R})$ . Let  $F \in \overline{L(X)}$  in  $K_V(X \times \mathbb{R})$ . In a similar way as in Remark 2.3, we have that  $F(x) \neq \emptyset$  for each  $x \in X$ .

Now suppose for some  $x \in X$  that  $F(x)$  is not connected, that is, there exist real numbers  $r, s, t$  with  $r < s < t$  such that  $r, t \in F(x)$  but  $s \notin F(x)$ . If  $x$  is an isolated point of  $X$ , then

$$\begin{aligned} & (X \setminus \{x\} \times \mathbb{R} \cup (\{x\} \times ((-\infty, s) \cup (s, \infty))))^+ \cap \\ & (\{x\} \times (-\infty, s))^- \cap (\{x\} \times (s, \infty))^- \end{aligned}$$

is an open neighborhood of  $F$  containing no member of  $L(X)$ . So let  $x$  be a nonisolated point of  $X$ . Since  $F$  is closed in  $X \times \mathbb{R}$ , we can find some open neighborhood  $U$  of  $x$  such that  $s \notin F(z) \forall z \in U$ . Let  $U_0$  be a connected open neighborhood of  $x$  such that  $\overline{U_0} \subseteq U$ . Let  $W_0 = (X \setminus \overline{U_0} \times \mathbb{R}) \cup (U \times (-\infty, s) \cup (s, \infty))$ ,  $W_1 = U_0 \times (-\infty, s)$ , and  $W_2 = U_0 \times (s, \infty)$ . Then  $F \in W_0^+ \cap W_1^- \cap W_2^-$ . Since  $F \in \overline{L(X)}$  and  $x$  is a nonisolated point of  $X$ , by [15, Lemma 4.1], [16, Lemma 4.1], and [4, Proposition 7], we can find some  $f \in C(X) \cap W_0^+ \cap W_1^- \cap W_2^-$ . Since  $U_0$  is connected,  $f(U_0)$  is connected. But this is a contradiction as  $f(U_0) \subseteq (-\infty, s) \cup (s, \infty)$ , and  $f(U_0) \cap (-\infty, s) \neq \emptyset$  and  $f(U_0) \cap (s, \infty) \neq \emptyset$ . Therefore,  $F(x)$  is connected for each  $x \in X$ . Also since  $F \in K(X \times \mathbb{R})$ ,  $F$  is locally bounded and hence belongs to  $L(X)$ .

(b)  $\implies$  (c): Since  $X \times \{0\} \in L(X) \subseteq K(X \times \mathbb{R})$ ,  $X$  is compact. Let  $F \in L(X)$ . Again, since  $X$  is compact, we can find some  $n \in \mathbb{N}$  such that  $F \subseteq X \times (-n, n)$ . Now, for any neighborhood  $W$  of  $F$ ,  $\overline{W} \cap (X \times (-n, n))^+$  is a closed subset of the compact space  $K_V(X \times [-n, n])$ , and hence is compact. This proves that  $L_V(X)$  is locally compact.

(c)  $\implies$  (a): Let  $L_V(X)$  be locally compact. We shall show that  $X$  is compact. First, suppose, by way of contradiction, that  $X$  is not countably compact; that is, there exists an infinite closed discrete set  $M = \{x_n : n \in \mathbb{N}\}$  in  $X$ .

Let  $F_0 = X \times \{0\}$  denote the zero function. It can be verified that the set  $\mathcal{B} = \{W^+ : W \text{ is an open set containing } F_0\}$  forms a base for  $L_V(X)$  at  $F_0$ . Let  $W^+ \in \mathcal{B}$  be such that  $\overline{W^+}$  is compact in  $L_V(X)$ . For each  $n \in \mathbb{N}$ , let  $t_n \in \mathbb{R}$  such that  $(x_n, t_n) \in W$ , and let  $F_n = (X \times \{0\}) \cup (\{x_n\} \times [0, t_n])$ . Since  $\overline{W^+}$  is compact, the sequence  $(F_n)$  has a cluster point  $F$  in  $\overline{W^+}$ .

If for some  $x \in X$ ,  $F(x) \cap (\mathbb{R} \setminus \{0\}) \neq \emptyset$ , then since  $M$  is a closed discrete set in  $X$ , we can find an open set  $U_x$  such that  $x \in U_x$  and  $x_n \in U_x$  for at most one  $n \in \mathbb{N}$ , so that we have  $F \in (U_x \times \mathbb{R} \setminus \{0\})^-$  but  $F_n \in (U_x \times \mathbb{R} \setminus \{0\})^-$  for at most one  $n \in \mathbb{N}$ . This is a contradiction to the fact that  $F$  is a cluster point of  $(F_n)$ . Hence,  $F$  must be  $F_0$ . Now let  $W_0 = W \setminus \{(x_n, t_n) : n \in \mathbb{N}\}$ . Then  $W_0$  is an open subset of  $X \times \mathbb{R}$  such that  $F_0 \in W_0^+$ , but no  $F_n \in W_0^+$ , which is again a contradiction to  $F_0$  being a cluster point of  $(F_n)$ . But this shows that the sequence  $(F_n)$  has no cluster point in  $\overline{W^+}$ . This is not possible as  $\overline{W^+}$  is compact. Hence,  $X$  must be countably compact.

Now, again suppose, by way of contradiction, that  $X$  is not compact. Since  $X$  is countably compact, every upper semicontinuous function is bounded above, and hence, it can be verified that the set  $\mathcal{B}' = \{(X \times (-1/n, 1/n))^+ : n \in \mathbb{N}\}$  forms a base for  $L_V(X)$  at  $F_0$ . Since  $L_V(X)$  is locally compact, we can find an  $m \in \mathbb{N}$  such that  $(X \times (-1/m, 1/m))^+$  is compact. As  $X$  is not compact, we can find a net  $(x_i)$  in  $X$  which has no cluster point in  $X$ . Now, for each  $i$ , define  $F_i = (X \times \{0\}) \cup (\{x_i\} \times [0, r])$ , where  $0 < r < 1/m$ . Then, as in the above paragraph, we can show that the net  $(F_i)$  has no cluster point in  $(X \times (-1/m, 1/m))^+$ , a contradiction to the

compactness of  $\overline{(X \times (-1/m, 1/m))^+}$ . Hence, we obtain that  $X$  is compact.

It remains to show that  $X$  is locally connected. Again, by compactness and local compactness of  $X$ , we can find some  $m \in \mathbb{N}$  such that for every  $n \geq m$ ,  $\overline{(X \times (-1/n, 1/n))^+}$  is compact in  $L_V(X)$ . Suppose, by way of contradiction,  $X$  is not locally connected, that is, there exists some  $x \in X$  and some open neighborhood  $U_0$  of  $x$  such that every set containing  $x$  as its interior point and contained in  $U$  is not connected. Let  $U$  be an open neighborhood of  $x$  such that  $\overline{U} \subseteq U_0$ . Clearly,  $\overline{U}$  is not connected.

Define a set

$$S = \{A \subseteq \overline{U} : x \in A, A \text{ is a clopen subset in } \overline{U}\}.$$

Now, partially order  $S$  by the relation  $\geq$  defined by  $A_1 \geq A_2$  if and only if  $A_1 \subseteq A_2$ . Fix some  $k \in \mathbb{N}$  such that  $k > m$ . For each  $A \in S$ , consider  $f_A \in C(X)$  such that  $0 \leq f_A(y) \leq 1/k, \forall y \in X, f_A(A) = \{0\}$ , and  $f_A(\overline{U} \setminus A) = \{1/k\}$ .

Note that  $\bigcap_{A \in S} A$  is not an open set in  $\overline{U}$ . Because if it were so, then  $\bigcap_{A \in S} A \in S$ . But since  $\bigcap_{A \in S} A \cap U$  is an open set containing  $x$  contained in  $U_0$ , we can find some nonempty disjoint closed subsets  $A', B'$  such that  $A' \cup B' = \bigcap_{A \in S} A$  and  $x \in A'$ . Then  $A' \in S$ . This implies  $\bigcap_{A \in S} A = A'$ , which is not possible since  $B'$  is nonempty. Hence,  $\bigcap_{A \in S} A$  is not an open set in  $\overline{U}$ .

Therefore, there exists some  $y \in \bigcap_{A \in S} A$  such that  $V$  is not contained in  $\bigcap_{A \in S} A$  for every open neighborhood  $V$  of  $y$ .

Now, since  $K_V(X \times [0, 1/k])$  is compact, the net  $(f_A)_{A \in S}$  has a cluster point  $F \in K(X \times [0, 1/k])$ . We show that  $F(y)$  is not connected. First, we show that  $0, 1/k \in F(y)$ . If possible, let  $0 \notin F(y)$ . Then we can find some open neighborhood  $V$  of  $y$  such that  $0 \notin F(V)$ . Consider the open set  $W = (V \times (\mathbb{R} \setminus \{0\})) \cup ((X \setminus \overline{V_0}) \times \mathbb{R})$ , where  $V_0$  is open neighborhood of  $y$  such that  $\overline{V_0} \subseteq V$ . Then  $F \in W^+$ , but no  $f_A \in W^+$  since  $f_A(y) = 0$  for each  $A \in S$ . Hence,  $0 \in F(y)$ . Similarly, from the fact that for every open neighborhood  $V$  of  $y$  contained in  $U$ , there exists some



$A_V$  such that  $V$  is not contained in  $A_V$  and hence,  $\forall A \geq A_V$ ,  $1/k \in f_A(V)$ . Now suppose that  $1/(2k) \in F(y)$ . Consider any open neighborhood  $V$  of  $y$  contained in  $U$ . Then  $F \in (V \times (0, 1/k))^-$ , but since  $f_A(A) = \{0\}$  and  $f_A(\bar{U} \setminus A) = \{1/k\}$  for each  $A \in S$ , no  $f_A \in (V \times (0, 1/k))^-$ . Hence,  $F(y)$  is not connected. This is a contradiction to the compactness of  $(X \times (-1/m, 1/m))^+$ , and hence,  $X$  must be locally connected.  $\square$

**Definition 2.5.** A metric space  $(X, d)$  is called *boundedly compact* if every closed and bounded subset of  $(X, d)$  is compact.

**Corollary 2.6.** *The space  $L_V(X)$  admits a boundedly compact metric if and only if  $X$  is compact, locally connected, and metrizable.*

*Proof:* Let  $X$  be a compact, locally connected, and metrizable space. Then since  $X$  is compact and metrizable, by [15, Theorem 5.6],  $L_V(X)$  is separable and metrizable. Now by Theorem 2.4 and [26, Theorem 2] and ([12, Exercise 4.2.C]),  $L_V(X)$  admits a boundedly compact metric.

Conversely, if  $L_V(X)$  admits a boundedly compact metric, then again by [15, Theorem 5.6],  $X$  is compact and metrizable, and by Theorem 2.4,  $X$  is locally connected.  $\square$

**Proposition 2.7.** *If  $X$  and  $Y$  are any Hausdorff spaces, then the space  $L_V(X \oplus Y)$  ( $L_F(X \oplus Y)$ , resp.) is homeomorphic to the space  $L_V(X) \times L_V(Y)$  ( $L_F(X) \times L_F(Y)$ , resp.).*

*Proof:* Define a map  $\lambda : L(X \oplus Y) \rightarrow L(X) \times L(Y)$  by

$$\lambda(F) = (F \cap (X \times \mathbb{R}), F \cap (Y \times \mathbb{R})) \quad \forall F \in L(X \oplus Y).$$

Then, one can verify that  $\lambda$  is a homeomorphism from  $L_V(X \oplus Y)$  ( $L_F(X \oplus Y)$ , resp.) onto  $L_V(X) \times L_V(Y)$  ( $L_F(X) \times L_F(Y)$ , resp.).  $\square$

### 3. NECESSARY CONDITIONS

In this section, we study conditions that are necessary for the space  $L_V(X)$  to be Lindelöf.

**Theorem 3.1.** *If  $X$  is a regular space such that  $L_V(X)$  is Lindelöf, then  $X$  is compact.*

*Proof:* By Theorem 2.2, since  $X$  is regular,  $CL_V(X)$  is closed in  $L_V(X)$  and consequently, is normal. Therefore, by the Theorem in [27],  $X$  is compact.  $\square$

**Remark 3.2.** The normality of the space  $CL(X)$  with the Vietoris topology is equivalent to its compactness; so that  $CL_V(X)$  is normal if and only if  $X$  is compact. The first result in this direction was established for well-ordered spaces with the order topology by V. M. Ivanova in [21]. In 1970, the result was proved for general topological spaces under the assumption of continuum hypothesis by James Keesling in [22]. The assumption of continuum hypothesis was weakened to Martin's axioms in [23] and finally the general result was proved in 1975 by N. V. Veličko in [27]. A shorter and simpler proof to this result was given in [11] in 2004.

In a similar way, as in the case of  $L_V(X)$ , we can show that the space  $USCO_V(X)$  of all usco maps from  $X$  to  $\mathbb{R}$  is Lindelöf only if  $X$  is compact. However, the compactness of  $X$  is also sufficient for  $USCO_V(X)$  to be Lindelöf.

**Proposition 3.3.** *For a regular space  $X$ ,  $USCO_V(X)$  is Lindelöf if and only if  $X$  is compact.*

*Proof:* In a similar way, as in case of  $L_V(X)$ , it can be shown that  $CL_V(X)$  is a closed subspace of  $USCO_V(X)$  whenever  $X$  is a regular space. Hence, compactness of  $X$  is necessary for  $USCO_V(X)$  to be Lindelöf.

Conversely, for a compact space  $X$ ,  $USCO_V(X)$  is a closed subspace of the  $\sigma$ -compact space  $K_V(X \times \mathbb{R})$ . Therefore,  $USCO_V(X)$  is also  $\sigma$ -compact and consequently, Lindelöf.  $\square$

Note that compactness of  $X$  is not a sufficient condition for  $L_V(X)$  to be Lindelöf. For example, let  $W$  denote the set of all ordinal numbers less than or equal to the first uncountable ordinal number  $\omega_1$ . The set  $W$  is well-ordered by the natural order  $<$ . Consider on  $W$  the topology generated by the base  $\mathcal{B}$  consisting of the sets of the form  $(y, x] = \{z \in W : y < z \leq x\}$ , where  $y < x \leq \omega_1$  and the one-point set  $\{0\}$ . Then  $W$  is compact but  $L_V(W)$  is not Lindelöf, as will be shown by Theorem 3.5.

Further note that even if a space  $X$ , in addition to being compact, is hereditarily separable and perfectly normal, then also  $L_V(X)$

need not be Lindelöf. Consider the double arrow space  $X$ , given in the following example.

**Example 3.4.** Let  $X = ((0, 1] \times \{0\}) \cup ([0, 1) \times \{1\}) \subseteq \mathbb{R}^2$  be the double arrow space (called the two arrow space in [12]). A base for the topology on  $X$  consists of clopen subsets of  $X$  of the form  $G(a, b) = ((a, b] \times \{0\}) \cup ([a, b) \times \{1\})$ , where  $0 \leq a < b \leq 1$ . Then  $X$  is a compact, perfectly normal, hereditarily separable, non-metrizable space, but still  $L_V(X)$  is not Lindelöf, as shown by Theorem 3.5 below. In fact, the double arrow space  $X$  is a zero-dimensional space, and later in this paper, we shall show that if  $X$  is a zero-dimensional space, then  $L_V(X)$  is Lindelöf if and only if  $X$  is compact and metrizable.

Let  $C$  be a closed subset of  $X$ . The set

$$\mathcal{S}(C) = \{[A \cap C, B \cap C] : A, B \text{ are open in } X, A \cap B = \emptyset, C \subseteq A \cup B\}$$

is called the disconnection set for  $C$ . Note that if  $C$  is connected, then  $\mathcal{S}(C) = \{[C, \emptyset], [\emptyset, C]\}$ ; and if  $X$  is normal, then the converse is also true; that is, if  $X$  is normal, then a closed subset  $C$  of  $X$  is connected if and only if  $\mathcal{S}(C) = \{[C, \emptyset], [\emptyset, C]\}$ . In fact, for a normal space  $X$  and any closed subset  $C$  of  $X$ , a subset  $A$  of  $C$  is clopen in  $C$  if and only if  $[A, C \setminus A] \in \mathcal{S}(C)$ .

We say that  $C$  has the *countable disconnection property* if  $\mathcal{S}(C)$  is countable. A space  $X$  is said to have  *$G_\delta$  countable disconnection property* if every closed  $G_\delta$ -set in  $X$  has the countable disconnection property.

**Theorem 3.5.** *If  $X$  is a regular space such that  $L_V(X)$  is Lindelöf, then  $X$  has  $G_\delta$  countable disconnection property.*

*Proof:* First, note that since  $X$  is regular and  $L_V(X)$  is Lindelöf, by Theorem 3.1,  $X$  is compact. Obviously, the empty set has the countable disconnection property. So let  $C$  be a nonempty closed subset of  $X$  such that  $C = \bigcap_{n=1}^{\infty} U_n$ , where  $(U_n)$  is a decreasing sequence of open sets in  $X$ , and let  $p, q$  be rational numbers such that  $p < q$ . First, we show that

$$S = (C \times [p, q])^{c+} = \bigcup_{m, n \in \mathbb{N}} (U_m \times (p - 1/n, q + 1/n))^{c+}.$$

Let  $F \in L(X)$  such that  $F \cap C \times [p, q] = \emptyset$ . Then we can find some open set  $V$  containing  $C$  and some  $n \in \mathbb{N}$  such that  $C \times [p, q] \subseteq V \times (p - 1/n, q + 1/n)$  and  $F \cap V \times (p - 1/n, q + 1/n) = \emptyset$ . Now since  $C = \bigcap_{n \in \mathbb{N}} U_n$  and  $X$  is compact, we can find some  $U_m$  such that  $U_m \subseteq V$ . Hence,  $F \cap U_m \times (p - 1/n, q + 1/n) = \emptyset$ , that is,  $F \in U_m \times (p - 1/n, q + 1/n)^{c+}$ . Consequently,  $S$  is an  $F_\sigma$ -set in  $L_V(X)$  and hence is Lindelöf.

We show that the disconnection set  $\mathcal{S}(C)$  is countable. Define

$$\mathcal{S}_0(C) = \{[A, B] : A, B \text{ open in } X \text{ such that } A \cap B = \emptyset, C \subseteq A \cup B\}.$$

For each  $[E, F] \in \mathcal{S}(C)$ , let  $[A, B] \in \mathcal{S}_0(C)$  such that  $E = A \cap C$  and  $F = B \cap C$ . Define the set

$$M[E, F] = A \times \{p-1\} \cup B \times \{q+1\} \cup (X \setminus (A \cup B)) \times [p-1, q+1] \in S.$$

Now consider the family

$$\mathcal{U} = \{(A \times (-\infty, p) \cup B \times (q, \infty) \cup X \setminus C \times \mathbb{R})^+ : [A, B] \in \mathcal{S}_0(C)\}.$$

Then it can be verified that  $\mathcal{U}$  is an open cover of  $S$ . Since  $S$  is Lindelöf, there exists a countable family  $\mathcal{S}(p, q) \subseteq \mathcal{S}_0(U)$  such that

$$\mathcal{U}_0 = \{(A \times (-\infty, p) \cup B \times (q, \infty) \cup (X \setminus C) \times \mathbb{R})^+ : [A, B] \in \mathcal{S}(p, q)\}$$

is an open subcover of  $\mathcal{U}$  for  $S$ . This gives that if  $F \in S$ , then there exists some  $[A, B] \in \mathcal{S}(p, q)$  such that  $F \in (A \times (-\infty, p) \cup B \times (q, \infty) \cup (X \setminus C) \times \mathbb{R})^+$ ; that is, for each  $x \in C$ ,  $F(x) \subseteq (-\infty, p)$  if  $x \in A$  and  $F(x) \subseteq (q, \infty)$  if  $x \in B$ . In particular, for any  $[E_0, F_0] \in \mathcal{S}(C)$ , there exists some  $[A_0, B_0] \in \mathcal{S}(p, q)$  such that  $M[E_0, F_0](x) \subseteq (-\infty, p)$  for all  $x \in A_0 \cap C$  and  $M[E_0, F_0](x) \subseteq (q, \infty)$  for all  $x \in B_0 \cap C$ . This implies  $[E_0, F_0] = [A_0 \cap C, B_0 \cap C]$ . Since  $\mathcal{S}(p, q)$  is countable,  $\mathcal{S}(C)$  is also countable.  $\square$

**Corollary 3.6.** *Let  $X$  be a perfectly normal space such that  $L_V(X)$  is Lindelöf. Then  $X$  is compact and every nonempty closed subset of  $X$  has the countable disconnection property.*

**Corollary 3.7.** *Let  $X$  be a regular space such that  $L_V(X)$  is Lindelöf. Then  $X$  is compact and the set of all clopen subsets of  $X$  is countable.*

We would like to point out that if  $X$  is a space such that  $L_V(X)$  is Lindelöf, then  $X$  need not satisfy the countable chain condition, as shown by the following example, which defines the space

long segment  $V$  that does not satisfy the countable chain condition but  $L_V(V)$  is Lindelöf. Further, this example shows that if  $X$  is a space for which  $L_V(X)$  is Lindelöf, then it is not necessary that every closed subset of  $X$  has the countable disconnection property. Therefore we have that if  $X$  is any space such that  $L_V(X)$  is Lindelöf and if  $Y$  is a closed subset of  $X$ , then  $L_V(Y)$  need not be Lindelöf; the space  $W$  of all ordinals less than or equal to  $\omega_1$  is a closed subset of  $V$  such that  $W$  does not have  $G_\delta$  countable disconnection property, and hence,  $L_V(W)$  is not Lindelöf.

**Example 3.8.** Let  $W_0$  be the set of all countable ordinal numbers. In the set  $V_0 = W_0 \times [0, 1)$ , consider the linear order defined by letting  $(\alpha_1, t_1) < (\alpha_2, t_2)$  whenever  $\alpha_1 < \alpha_2$  or  $\alpha_1 = \alpha_2$  and  $t_1 < t_2$ ; the set  $V_0$  with the topology induced by the linear order  $<$  is called the *long line*. Adjoining the point  $\omega_1$  to  $V_0$  and assuming that  $x < \omega_1$  for all  $x \in V_0$ , we obtain a linearly ordered set  $V$ . The set  $V$  with the topology induced by the linear order  $<$  is called the *long segment* (see [12, Problem 3.12.19]). The long segment is a compact, locally connected space, and hence  $L_V(V)$  is Lindelöf (see Theorem 4.3). Note that  $V$  does not satisfy the countable chain condition. The necessary condition given in Theorem 3.5 implies a condition (see Theorem 3.9 below) that we call the weak boundary countable chain condition (wbccc) that is similar to, but weaker than the countable chain condition. One can check that  $V$  does satisfy the wbccc.

For an uncountable family  $\mathcal{U}$  of  $X$ , define a point  $x$  of  $X$  to be a *strong closure point* of  $\mathcal{U}$  provided that every neighborhood of  $x$  intersects all but countably many members of  $\mathcal{U}$ . Then define  $X$  to have the *weak boundary countable chain condition* provided that for every uncountable pairwise disjoint family  $\mathcal{U}$  of nonempty open subsets of  $X$ , every strong closure point of  $\mathcal{U}$  is contained in the closure of  $\cup\{bd(U) : U \in \mathcal{U}\}$  in  $X$ . Clearly having ccc implies having wbccc. But the converse is not true; every first countable space vacuously has the wbccc but, obviously, need not have ccc.

**Theorem 3.9.** *If  $X$  is a compact space having  $G_\delta$  countable disconnection property, then  $X$  has weak boundary countable chain condition. Consequently, if  $X$  is a regular space such that  $L_V(X)$  is Lindelöf, then  $X$  has weak boundary countable chain condition.*

*Proof:* Suppose that  $X$  is a compact space that does not have wbccc. Then there exists an uncountable pairwise disjoint family  $\mathcal{U}$  of nonempty open subsets of  $X$  and a strong closure point  $x$  of  $\mathcal{U}$  that is not contained in  $C$ , where  $C$  is the closure of  $\cup\{bd(U) : U \in \mathcal{U}\}$  in  $X$ .

By induction, define a sequence  $U_n$  of neighborhoods of  $x$  such that  $U_1 = X \setminus C$  and for each  $n > 1$ ,  $\overline{U_n} \subseteq U_{n-1}$ . Then define  $G = \cap\{U_n : n \in \mathbb{N}\}$  so that  $G = \cap\{\overline{U_n} : n \in \mathbb{N}\}$ , and hence,  $G$  is a closed  $G_\delta$ -set in  $X$ .

We want to show that  $G$  intersects uncountably many members of  $\mathcal{U}$ . For each  $n \in \mathbb{N}$ , let  $\mathcal{U}_n = \{U \in \mathcal{U} : U \cap U_n = \emptyset\}$ , which is countable; therefore,  $\cup\{\mathcal{U}_n : n \in \mathbb{N}\}$  is countable. Then define  $\mathcal{U}' = \mathcal{U} \setminus \cup\{\mathcal{U}_n : n \in \mathbb{N}\}$ , which is uncountable.

Now let  $U \in \mathcal{U}'$ . We want to show that  $G$  intersects  $U$ . For each  $n \in \mathbb{N}$ , we have  $U_n \cap U \neq \emptyset$  and  $U_n \cap bd(U) = \emptyset$ . Therefore,  $\overline{U_n} \cap U = \overline{U_n} \cap \overline{U}$  is compact. Thus,  $\{\overline{U_n} \cap U : n \in \mathbb{N}\}$  is a nested family of nonempty compact sets, and so  $\cap\{\overline{U_n} \cap U : n \in \mathbb{N}\} \neq \emptyset$ ; that is,  $U$  intersects  $\{\overline{U_n} : n \in \mathbb{N}\} = G$ .

Finally, since  $G$  is disjoint from  $C$ , for each  $U \in \mathcal{U}'$ ,  $U \cap G$  is a clopen subset of  $G$ ; and since  $\mathcal{U}$  is a pairwise disjoint family,  $\{U \cap G : U \in \mathcal{U}'\}$  is an uncountable family of clopen subsets of  $G$ . Consequently,  $X$  does not have  $G_\delta$  countable disconnection property. □

**Question 3.10.** If  $X$  is a compact space having  $G_\delta$  countable disconnection property, is  $L_V(X)$  a Lindelöf space?

#### 4. SUFFICIENT CONDITIONS

In this section, we study conditions that are sufficient for  $L(X)$  with the Vietoris and Fell topologies to be Lindelöf. As we shall see, in addition to the compactness of  $X$ , if we assume either metrizable or local connectedness of  $X$ , then  $L_V(X)$  will be Lindelöf.

**Theorem 4.1.** *If  $X$  is compact and metrizable, then  $L_V(X)$  is Lindelöf. Consequently, for a metrizable space  $X$ ,  $L_V(X)$  is Lindelöf if and only if  $X$  is compact.*

*Proof:* By [15, Theorem 5.6],  $L_V(X)$  is second countable and hence Lindelöf. □

The condition that  $X$  is compact and metrizable is not necessary for  $L_V(X)$  to be Lindelöf, as shown by Theorem 4.3 below, but is necessary for  $L_V(X)$  to be hereditarily Lindelöf.

**Theorem 4.2.** *The space  $L_V(X)$  is hereditarily Lindelöf if and only if  $X$  is compact and metrizable.*

*Proof:* By regularity of  $X$  and Lindelöfness of  $L_V(X)$ ,  $X$  is compact. Thus,  $C(X)$  with the fine topology is a subspace of  $L_V(X)$  and hence Lindelöf. Consequently,  $X$  is compact and metrizable (see [9]).  $\square$

**Theorem 4.3.** *If  $X$  is compact and locally connected, then  $L_V(X)$  is  $\sigma$ -compact and hence Lindelöf. Consequently, for a regular locally connected space  $X$ ,  $L_V(X)$  is Lindelöf if and only if  $X$  is compact.*

*Proof:* Since  $X$  is compact and locally connected, by Theorem 2.4,  $L_V(X)$  is a closed subset of the  $\sigma$ -compact set  $K_V(X \times \mathbb{R})$ . Thus,  $L_V(X)$  is also  $\sigma$ -compact and hence Lindelöf.  $\square$

In contrast to locally connected spaces, zero-dimensional spaces require metrizability of  $X$  for Lindelöfness of  $L_V(X)$ . The characterization of Lindelöfness of  $L_V(X)$  in the realm of zero-dimensional spaces follows from Theorem 4.1 and Corollary 3.7.

**Corollary 4.4.** *For a zero-dimensional space  $X$ ,  $L_V(X)$  is Lindelöf if and only if  $X$  is compact and metrizable.*

Since the product of a Lindelöf and a  $\sigma$ -compact space is Lindelöf, from Proposition 2.7 and theorems 4.1 and 4.3, we can infer the following result.

**Theorem 4.5.** *If  $X$  is compact and can be expressed as a sum of a locally connected and a metrizable space, then  $L_V(X)$  is Lindelöf.*

In the next result, we investigate the Lindelöfness of  $L_V(Y)$ , when  $Y$  is a continuous image of a space  $X$  such that  $L_V(X)$  is Lindelöf. But, before stating the theorem, we would like to give the following notations and lemmas.

Let  $\phi : X \rightarrow Y$  be a continuous surjection. Then  $\phi \times id : X \times \mathbb{R} \rightarrow Y \times \mathbb{R}$  is a naturally defined continuous function. Define  $\phi^\circ : L(Y) \rightarrow L(X)$  by  $\phi^\circ(F) = (\phi \times id)^{-1}(F)$  for all  $F \in L(Y)$ .

**Lemma 4.6.** *For every  $F \in L(Y)$ ,  $\phi^\circ(F) \in L(X)$ .*

*Proof:* Since  $\phi \times id$  is continuous,  $\phi^\circ(F)$  is closed in  $X \times \mathbb{R}$ . Moreover, for each  $x \in X$ ,  $\phi^\circ(F)(x) = F(\phi(x))$  is a bounded interval. Also, since  $F$  is locally bounded, there exist an open neighborhood  $U_x$  of  $\phi(x)$  and a positive  $b \in \mathbb{R}$  such that  $F(y) \subseteq [-b, b]$  for all  $y \in U_x$ . Then we have  $\phi^\circ(F)(z) \subseteq [-b, b]$  for all  $z$  in the open neighborhood  $\phi^{-1}(U_x)$  of  $x$ , thus showing that  $\phi^\circ(F)$  is locally bounded. This finishes the proof that  $\phi^\circ(F) \in L(X)$ .  $\square$

For each basic open set  $V = W^+ \cap W^-$  in  $L_V(Y)$ , define the basic open set  $V^\circ = (W^\circ)^+ \cap (W^\circ)^-$  in  $L_V(X)$  by taking  $W^\circ = (\phi \times id)^{-1}(W)$  and  $W^\circ = \{(\phi \times id)^{-1}(W') : W' \in \mathcal{W}\}$ .

The proof of the following lemma is simple and hence omitted.

**Lemma 4.7.** *For a basic open set  $V$  in  $L_V(Y)$ , the following are true.*

- (1)  $\phi^\circ(V) = V^\circ \cap \phi^\circ(L(Y))$ .
- (2) For each  $F \in L(Y)$ ,  $F \in V$  if and only if  $\phi^\circ(F) \in \phi^\circ(V)$ .

**Lemma 4.8.** *If  $X$  is compact and  $F \in \overline{\phi^\circ(L(Y))}$  in  $L_V(X)$ , then  $(\phi \times id)(F) \in L(Y)$ .*

*Proof:* First,  $(\phi \times id)(F)(y)$  is nonempty for all  $y \in Y$  because  $\phi$  is a surjection. Also, since  $X$  is compact,  $F$  is compact and thus,  $(\phi \times id)(F)$  is closed in  $Y \times \mathbb{R}$  by the continuity of  $\phi \times id$ .

To show that  $(\phi \times id)(F)$  is locally bounded, suppose, by way of contradiction, that  $(\phi \times id)(F)$  is not locally bounded at some  $y \in Y$ . Then for each neighborhood  $V$  of  $y$  and for each natural number  $n$ , there exist  $y_{V,n} \in V$  and  $t_{V,n} \in [n, \infty)$  such that  $(y_{V,n}, t_{V,n}) \in (\phi \times id)(F)$ . Let  $x_{V,n} \in X$  be such that  $(x_{V,n}, t_{V,n}) \in F$  and  $(y_{V,n}, t_{V,n}) = (\phi \times id)(x_{V,n}, t_{V,n})$ . Since  $X$  is compact, the net  $(x_{V,n})$  has a cluster point  $x$  in  $X$ . So for every neighborhood  $U$  of  $x$  and for every  $n \in \mathbb{N}$ , there exist a  $V$  and a natural number  $m > n$  such that  $x_{V,m} \in U$ . This contradicts  $F$  being locally bounded at  $x$ , which shows that  $(\phi \times id)(F)$  is indeed locally bounded.

To show that  $(\phi \times id)(F)(y)$  is connected for every  $y$  in  $Y$ , suppose, by way of contradiction, that there exists a  $y$  in  $Y$  such that  $(\phi \times id)(F)(y) \subseteq (-\infty, t) \cup (t, \infty)$  for some  $t \in \mathbb{R}$  where  $(\phi \times id)(F)(y) \cap (-\infty, t) \neq \emptyset$  and  $(\phi \times id)(F)(y) \cap (t, \infty) \neq \emptyset$ . Define  $A = \{x \in \phi^{-1}(y) : F(x) \subseteq (-\infty, t)\}$  and  $B = \{x \in \phi^{-1}(y) : F(x) \subseteq (t, \infty)\}$ . Now  $A$  and  $B$  are disjoint nonempty subsets of  $X$  with  $A \cup B = \phi^{-1}(y)$ . To see that  $A$  is closed in  $X$ , let  $(a_i)$  be a



net in  $A$  that converges to some  $a$  in  $X$ . Since  $\phi^{-1}(y)$  is closed in  $X$ ,  $a \in \phi^{-1}(y)$ . We want to show that  $F(a) \subseteq (-\infty, t)$ , so suppose not. Then  $F(a) \subseteq (t, \infty)$ . For each  $i$ , let  $t_i = \max F(a_i)$ , which is less than  $t$ . Since  $F$  is compact and  $(a_i)$  converges to  $a$ , the net  $((a_i, t_i))$  has a cluster point  $(a, t_0)$  in  $F$  for some  $t_0$  that must be less than or equal to  $t$ . But this contradicts  $F(a) \subseteq (t, \infty)$ . Therefore,  $F(a) \subseteq (-\infty, t)$  so that  $a \in A$ . This shows that  $A$  is closed in  $X$  and a similar argument shows that  $B$  is also closed in  $X$ .

Finally, define  $W = ((A \times [t, \infty)) \cup (V \times (-\infty, t]))^c$ . Now since  $F \in W^+ \cap \overline{\phi^\circ(L(Y))}$ , there exists  $G \in L(Y)$  with  $\phi^\circ(G) \in W^+$ , and hence,  $(\phi \times id)^{-1}(G) \subseteq W$ . Because  $\phi^{-1}(y) = A \cup B$ , it follows that  $G(y) \subseteq (-\infty, t) \cup (t, \infty)$ . Also, since  $A$  and  $B$  are nonempty,  $G(y) \cap (-\infty, t) \neq \emptyset$  and  $G(y) \cap (t, \infty) \neq \emptyset$ , which contradicts  $G(y)$  being connected. This shows that  $(\phi \times id)(F)(y)$  is connected for all  $y \in Y$  and finishes the proof that  $(\phi \times id)(F) \in L(Y)$ .  $\square$

**Lemma 4.9.** *Let  $X$  be compact, let  $\mathcal{B}$  be a family of basic open sets in  $L_V(Y)$ , and let  $\mathcal{B}^\circ = \{B^\circ : B \in \mathcal{B}\}$ . Then the following assertions are equivalent.*

- (1)  $\mathcal{B}$  covers  $L(Y)$ .
- (2)  $\mathcal{B}^\circ$  covers  $\overline{\phi^\circ(L(Y))}$ .
- (3)  $\mathcal{B}^\circ$  covers  $\overline{\phi^\circ(L(Y))}$  (here, closure of  $\phi^\circ(L(Y))$  is taken in  $L_V(X)$ ).

*Proof:* That (3)  $\implies$  (2) is obvious, and (2)  $\implies$  (1) follows from Lemma 4.7.

(1)  $\implies$  (3): Let  $F \in \overline{\phi^\circ(L(Y))}$ . Then, by Lemma 4.8,  $A = (\phi \times id)(F) \in L(Y)$  and hence,  $A \in B$  for some  $B \in \mathcal{B}$ . Observe that  $F \subseteq \phi^\circ(A)$ , and if for some open set  $W$  of  $Y \times \mathbb{R}$ , some  $(y, t) \in A \cap W$ , then there exists some  $x \in X$  such that  $\phi(x) = y$  and  $(x, t) \in F \cap (\phi \times id)^{-1}(W)$ . Therefore, we obtain that  $F \in B^\circ$  and consequently,  $\mathcal{B}^\circ$  covers  $\overline{\phi^\circ(L(Y))}$ .  $\square$

**Theorem 4.10.** *If  $Y$  is the continuous image of a regular space  $X$  and  $L_V(X)$  is Lindelöf, then  $L_V(Y)$  is Lindelöf.*

*Proof:* Let  $\mathcal{B}$  be a family of basic open sets in  $L_V(Y)$  which covers  $L(Y)$ . Then, by Lemma 4.9,  $\mathcal{B}^\circ$  covers  $\overline{\phi^\circ(L(Y))}$ , which is a closed subset of the Lindelöf space  $L_V(X)$ . Therefore, we can find a countable subfamily  $\mathcal{G}^\circ$  of  $\mathcal{B}^\circ$  that also covers  $\overline{\phi^\circ(L(Y))}$ . Again,

by Lemma 4.9, the corresponding subfamily  $\mathcal{G}$  of  $\mathcal{B}$  covers  $L(Y)$ , showing that  $L_V(Y)$  is Lindelöf.  $\square$

Finally, from Theorem 4.5 and Theorem 4.10, we obtain the following sufficient condition for Lindelöfness of  $L_V(X)$ .

**Corollary 4.11.** *If a compact space  $X$  can be expressed as a union of two closed subspaces  $Y$  and  $Z$  (not necessarily disjoint) such that, with respect to their relative subspace topologies inherited from  $X$ , the subspace  $Y$  is locally connected and the subspace  $Z$  is metrizable, then  $L_V(X)$  is Lindelöf.*

**Question 4.12.** Let  $\mathcal{C}$  be the smallest class of topological spaces that contains the compact metrizable spaces and the compact locally connected spaces and is closed under finite sums and continuous images. If  $L_V(X)$  is a Lindelöf space, is  $X \in \mathcal{C}$ ?

**Question 4.13.** If  $X$  is a compact metrizable space and  $Y$  is a compact locally connected space, is  $L_V(X \times Y)$  a Lindelöf space?

Note that if the answer to Question 4.13 is no, then the answer to Question 3.10 will obviously be no. If the answer to Question 4.13 is yes, then this probably gives a negative answer to Question 4.12, so that in this case one should weaken Question 4.12 by having  $\mathcal{C}$  closed under finite sums, finite products, and continuous images.

The remainder of the section is devoted to the study of Lindelöf property of  $L(X)$  with the Fell topology. We start by giving an equivalent characterization for second countability of  $L_F(X)$ .

**Proposition 4.14.** *The space  $L_F(X)$  is second countable if and only if  $X$  is locally compact and second countable.*

*Proof:* By [19, Theorem 3.8],  $CL_F(X)$  is second countable if and only if  $X$  is locally compact and second countable. Further, since local compactness and second countability of  $X$  implies local compactness and second countability of  $X \times \mathbb{R}$ ,  $CL_F(X \times \mathbb{R})$  is second countable if and only if  $X$  is locally compact and second countable. Hence, by Theorem 2.2,  $L_F(X)$  is second countable if and only if  $X$  is locally compact and second countable.  $\square$

**Corollary 4.15.** *If  $X$  is locally compact and second countable, then  $L_F(X)$  is Lindelöf.*

Further, since the Fell topology is weaker than the Vietoris topology, any condition sufficient for  $L_V(X)$  to be Lindelöf is also sufficient for  $L_F(X)$  to be Lindelöf. However, we can find a space  $X$  for which  $L_F(X)$  is Lindelöf but neither  $L_V(X)$  is second countable nor  $L_V(X)$  is Lindelöf. As an example, the long line  $V_0$ , defined in Example 3.8, is countably compact, locally compact, and locally connected but neither compact nor second countable. Therefore, neither  $L_V(V_0)$  is Lindelöf nor  $L_F(V_0)$  is second countable. But Theorem 4.17 below shows that  $L_F(V_0)$  is Lindelöf.

In addition, observe that if  $X$  is compact, then for each  $n \in \mathbb{N}$ , the Fell topology and the Vietoris topology coincide on  $L^n(X) = L(X) \cap (X \times [-n, n])^+$ ; and hence, for any compact space  $X$ ,  $L_F(X)$  is Lindelöf if and only if  $L_V(X)$  is Lindelöf. So we obtain the following result analogous to Theorem 4.10.

**Theorem 4.16.** *If  $Y$  is the continuous image of a compact space  $X$  and  $L_F(X)$  is Lindelöf, then  $L_F(Y)$  is Lindelöf.*

We end this section, as well as this paper, with a result that gives another sufficient condition for  $L_F(X)$  to be Lindelöf.

**Theorem 4.17.** *For a locally compact space  $X$ , if  $X$  is a sum of a second countable space and a countably compact, locally connected space, then  $L_F(X)$  is Lindelöf.*

*Proof:* First, we show that if  $X$  is a locally compact, countably compact, and locally connected space, then  $L_F(X)$  is  $\sigma$ -compact. So, let  $X$  be countably compact, locally compact, and locally connected. Then, by [12, Problem 3.12.23(g)], each  $F \in L(X)$  is contained in  $X \times [-n, n]$  for some  $n \in \mathbb{N}$  and hence,  $L(X) = \cup\{L^n(X) : n \in \mathbb{N}\}$ . Now, by using local compactness of  $X$  and similar arguments, as in the proof of (a)  $\implies$  (b) in Theorem 2.4, we can prove that  $L_F^n(X)$  is a closed subset of the compact space  $2_F^{X \times [-n, n]}$ , and hence, is compact, too. Therefore,  $L_F(X)$  is  $\sigma$ -compact.

Now, let  $X$  be a sum of a locally compact, second countable space  $Y$  and a countably compact, locally compact, locally connected space  $Z$ . Then, by Proposition 2.7,  $L_F(X) = L_F(Y \oplus Z)$  is homeomorphic to  $L_F(Y) \times L_F(Z)$ , and hence is Lindelöf.  $\square$

## REFERENCES

- [1] A. V. Arhangel'skii, *On relationships between topological properties of  $X$  and  $C_p(X)$* , in *General Topology and its Relations to Modern Analysis and Algebra*, V. Ed. J. Novák. Sigma Series in Pure Mathematics, 3. Berlin: Heldermann, 1983. 24–36
- [2] ———, *Topological Function Spaces*. Translated from the Russian by R. A. M. Hoksbergen. Mathematics and its Applications (Soviet Series), 78. Dordrecht: Kluwer Academic Publishers Group, 1992.
- [3] ———,  *$C_p$ -theory*, in *Recent Progress in General Topology*. Ed. Miroslav Hušek and Jan van Mill. Amsterdam: North-Holland, 1992. 1–56
- [4] Giuliano Artico, Ľubica Holá, Umberto Marconi, and Roberto Moresco, *Approximation by continuous functions in the Fell topology*. Preprint.
- [5] Gerald Beer, *On a theorem of Cellina for set valued functions*, Rocky Mountain J. Math. **18** (1988), no. 1, 37–47.
- [6] ———, *Topologies on Closed and Closed Convex Sets*. Mathematics and its Applications, 268. Dordrecht: Kluwer Academic Publishers Group, 1993.
- [7] J. M. Borwein, *Minimal CUSCOS and subgradients of Lipschitz functions*, in *Fixed Point Theory and Applications*. Ed. M. A. Théra and J.-B. Bailon. Pitman Research Notes in Mathematics Series, 252. Essex, England: Longman Scientific and Technical, 1991. 57–81
- [8] Jens Peter Reus Christensen, *Theorems of Namioka and R. E. Johnson type for upper semicontinuous and compact valued set-valued mappings*, Proc. Amer. Math. Soc. **86** (1982), no. 4, 649–655.
- [9] G. Di Maio, L. Holá, D. Holý, and R. A. McCoy, *Topologies on the space of continuous functions*, Topology Appl. **86** (1998), no. 2, 105–122.
- [10] Giuseppe Di Maio, Enrico Meccariello, and Somashekhar Naimpally, *Graph topologies on closed multifunctions*, Appl. Gen. Topol. **4** (2003), no. 2, 445–465.
- [11] ———, *Normal Vietoris implies compactness: A short proof*, Czechoslovak Math. J. **54(129)** (2004), no. 1, 181–182.
- [12] Ryszard Engelking, *General Topology*. Translated from the Polish by the author. 2nd ed. Sigma Series in Pure Mathematics, 6. Berlin: Heldermann Verlag, 1989.
- [13] Ľubica Holá, *On relations approximated by continuous functions*, Acta Univ. Carolin. Math. Phys. **28** (1987), no. 2, 67–72.
- [14] ———, *Hausdorff metric on the space of upper semicontinuous multifunctions*, Rocky Mountain J. Math. **22** (1992), no. 2, 601–610.
- [15] Ľubica Holá, Tanvi Jain, and R. A. McCoy, *Topological properties of the multifunction space ofusco maps*. To appear in *Mathematica Slovaca*.
- [16] Ľ. Holá and R. A. McCoy, *Relations approximated by continuous functions*, Proc. Amer. Math. Soc. **133** (2005), no. 7, 2173–2182 (electronic).

- [17] ———, *Relations approximated by continuous functions in the Vietoris topology*, *Fund. Math.* **195** (2007), no. 3, 205–219.
- [18] Ľ. Holá, R. A. McCoy, and J. Pelant, *Approximations of relations by continuous functions*, *Topology Appl.* **154** (2007), no. 11, 2241–2247.
- [19] Ľubica Holá, Jan Pelant, and László Zsilinszky, *Developable hyperspaces are metrizable*, *Appl. Gen. Topol.* **4** (2003), no. 2, 351–360.
- [20] D. Holý and P. Vadovič, *Hausdorff graph topology, proximal graph topology and the uniform topology for densely continuous forms and minimal USCO maps*, *Acta Math. Hungar.* **116** (2007), no. 1-2, 133–144.
- [21] V. M. Ivanova, *On the theory of spaces of subsets* (Russian), *Dokl. Akad. Nauk SSSR (N.S.)* **101** (1955), 601–603.
- [22] James Keesling, *On the equivalence of normality and compactness in hyperspaces*, *Pacific J. Math.* **33** (1970), 657–667.
- [23] V. I. Malyhin and B. É. Šapirovič, *Martin's axiom, and properties of topological spaces* (Russian), *Dokl. Akad. Nauk SSSR* **213** (1973), 532–535.
- [24] Robert A. McCoy, *Comparison of hyperspace and function space topologies*, in *Recent Progress in Function Spaces*. Ed. G. Di Maio and L. Holá. *Quaderni di Matematica*, 3. Caserta, Italy: Dipartimento di Matematica, Seconda Università degli Studi di Napoli, 1998. 241–258
- [25] Robert A. McCoy and Ibula Ntantu, *Topological Properties of Spaces of Continuous Functions*. *Lecture Notes in Mathematics*, 1315. Berlin: Springer-Verlag, 1988.
- [26] H. E. Vaughan, *On locally compact metrisable spaces*, *Bull. Amer. Math. Soc.* **43** (1937), no. 8, 532–535.
- [27] N. V. Veličko, *The space of closed subsets* (Russian), *Sibirsk. Mat. Ž.* **16** (1975), no. 3, 627–629, 646. English translation, *Siberian Math. J.* **16** (1975), no. 3, 484–486.

(Jain) DEPARTMENT OF MATHEMATICS; INDIAN INSTITUTE OF TECHNOLOGY  
 DELHI; NEW DELHI 110016, INDIA  
*E-mail address:* tanvij1705@rediffmail.com

(McCoy) DEPARTMENT OF MATHEMATICS; VIRGINIA TECH; BLACKSBURG,  
 VA 24060-0123, USA  
*E-mail address:* mccoy@math.vt.edu