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ON EXTREMELY AMENABLE GROUPS OF HOMEOMORPHISMS

VLADIMIR USPENSKIJ

Abstract. A topological group G is *extremely amenable* if every compact G-space has a G-fixed point. Let X be compact and $G \subset \operatorname{Homeo}(X)$. We prove that the following are equivalent: (1) G is extremely amenable; (2) every minimal closed G-invariant subset of $\operatorname{Exp} R$ is a singleton, where R is the closure of the set of all graphs of $g \in G$ in the space $\operatorname{Exp}(X^2)$ (Exp stands for the space of closed subsets); (3) for each $n = 1, 2, \ldots$ there is a closed G-invariant subset Y_n of $(\operatorname{Exp} X)^n$ such that $\cup_{n=1}^{\infty} Y_n$ contains arbitrarily fine covers of X and for every $n \geq 1$ every minimal closed G-invariant subset of $\operatorname{Exp} Y_n$ is a singleton. This yields an alternative proof of Pestov's theorem that the group of all order-preserving self-homeomorphisms of the Cantor middle-third set (or of the interval $[0, 1]$) is extremely amenable.

1. Introduction

With every[1] topological group G one can associate the *greatest ambit* $\mathcal{S}(G)$ and the *universal minimal compact G-space* $\mathcal{M}(G)$. To define these objects, recall some definitions. A *G-space* is a topological space X with a continuous action of G, that is, a map

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[1]All spaces are assumed to be Tikhonov, and all *maps* are assumed to be continuous.

$G \times X \to X$ satisfying $g(hx) = (gh)x$ and $1x = x$ ($g, h \in G$, $x \in X$). A map $f : X \to Y$ between two G-spaces is *G-equivariant*, or a *G-map* for short, if $f(gx) = gf(x)$ for every $g \in G$ and $x \in X$.

A *semigroup* is a set with an associative multiplication. A semigroup X is *right topological* if it is a topological space and for every $y \in X$ the self-map $x \mapsto xy$ of X is continuous. (Sometimes the term *left topological* is used for the same thing.) A subset $I \subset X$ is a *left ideal* if $XI \subset I$. If G is a topological group, a *right topological semigroup compactification* of G is a right topological compact semigroup X together with a continuous semigroup morphism $f : G \to X$ with a dense range such that the map $(g, x) \mapsto f(g)x$ from $G \times X$ to X is jointly continuous (and hence X is a G-space). The *greatest ambit* $\mathcal{S}(G)$ for G is a right topological semigroup compactification which is universal in the usual sense: for any right topological semigroup compactification X of G there is a unique morphism $\mathcal{S}(G) \to X$ of right topological semigroups such that the obvious diagram commutes. Considered as a G-space, $\mathcal{S}(G)$ is characterized by the following property: there is a distinguished point $e \in \mathcal{S}(G)$ such that for every compact G-space Y and every $a \in Y$ there exists a unique G-map $f : \mathcal{S}(G) \to Y$ such that $f(e) = a$.

We can take for $\mathcal{S}(G)$ the compactification of G corresponding to the C^*-algebra $\mathrm{RUCB}(G)$ of all bounded right uniformly continuous functions on G, that is, the maximal ideal space of that algebra. (A complex function f on G is *right uniformly continuous* if

$$\forall \epsilon > 0\, \exists V \in \mathcal{N}(G)\, \forall x, y \in G\, (xy^{-1} \in V \Rightarrow |f(y) - f(x)| < \epsilon),$$

where $\mathcal{N}(G)$ is the filter of neighbourhoods of unity.) The G-space structure on $\mathcal{S}(G)$ comes from the natural continuous action of G by automorphisms on $\mathrm{RUCB}(G)$ defined by $gf(h) = f(g^{-1}h)$ ($g, h \in G$, $f \in \mathrm{RUCB}(G)$). We shall identify G with a subspace of $\mathcal{S}(G)$. Closed G-subspaces of $\mathcal{S}(G)$ are the same as closed left ideals of $\mathcal{S}(G)$.

A G-space X is *minimal* if it has no proper G-invariant closed subsets or, equivalently, if the orbit Gx is dense in X for every $x \in X$. The *universal minimal compact G-space* $\mathcal{M}(G)$ is characterized by the following property: $\mathcal{M}(G)$ is a minimal compact G-space, and for every compact minimal G-space X there exists a G-map of $\mathcal{M}(G)$ onto X. Since Zorn's lemma implies that every compact G-space has a minimal compact G-subspace, it follows that for every compact G-space X, minimal or not, there exist a G-map of $\mathcal{M}(G)$ to X. The space $\mathcal{M}(G)$ is unique up to a G-space isomorphism and is isomorphic to any minimal closed left ideal of $\mathcal{S}(G)$, see e.g. [1], [9, Section 4.1], [11, Appendix], [10, Theorem 3.5].

A topological group G is *extremely amenable* if $\mathcal{M}(G)$ is a singleton or, equivalently, if G has the *fixed point on compacta property*: every compact G-space X has a G-fixed point, that is, a point $p \in X$ such that $gp = p$ for every $g \in G$. Examples of extremely amenable groups include $\mathrm{Homeo}_+[0,1]$ = the group of all orientation-preserving self-homeomorphisms of $[0,1]$; $U_s(H)$ = the unitary group of a Hilbert space H, with the topology inherited from the product H^H; $\mathrm{Iso}\,(U)$ = the group of isometries of the Urysohn universal metric space U. See Pestov's book [9] for the proof. Note that a locally compact group $\neq \{1\}$ cannot be extremely amenable, since every locally compact group admits a free action on a compact space [12], [9, Theorem 3.3.2].

We refer the reader to Pestov's book [9] for various intrinsic characterizations of extremely amenable groups. These characterizations reveal a close connection between Ramsey theory and the notion of extreme amenability. The aim of the present paper is to give another characterization of extremely amenable groups, based on a different approach. For a compact space X let $H(X)$ be the group of all self-homeomorphisms of X, equipped with the compact-open topology. Let G be a topological subgroup of $H(X)$. There is an obvious necessary condition for G to be extremely amenable: every minimal closed G-subset of X must be a singleton. However, this condition is not sufficient. For example, let X be the Hilbert cube, and let $G \subset H(X)$ be the stabilizer of a given point $p \in X$. Then the only minimal closed G-subset of X is the singleton $\{p\}$, but G is not extremely amenable [11], since G acts without fixed points on the compact space Φ_p of all maximal chains of closed subsets of X starting at p. The space Φ_p is a subspace of the compact G-space $\mathrm{Exp}\,\mathrm{Exp}\,X$, where for a compact space K we denote by $\mathrm{Exp}\,K$ the compact space of all closed non-empty subsets of K, equipped with the Vietoris topology[2]. It was indeed necessary to use the second exponent in this example, the first exponent

[2]If F is closed in K, the sets $\{A \in \mathrm{Exp}\,K : A \subset F\}$ and $\{A \in \mathrm{Exp}\,K : A$ meets $F\}$ are closed in $\mathrm{Exp}\,K$, and the Vietoris topology is generated by the closed sets of this form. If K is a G-space, then so is $\mathrm{Exp}\,K$, in an obvious way.

would not work. One can ask whether in general for every group $G \subset H(X)$ which is not extremely amenable there exists a compact G-space X' derived from X by applying a small number of simple functors, like powers, probability measures, exponents, etc., such that X' contains a closed G-subspace (which can be taken minimal) on which G acts without fixed points. We answer this question in the affirmative.

Consider the action of G on $\mathrm{Exp}\,(X^2)$ defined by the composition of relations: if $g \in G$, $F \subset X^2$, and $\Gamma_g \subset X^2$ is the graph of g, then $gF = \Gamma_g \circ F = \{(x, gy) : (x, y) \in F\}$. This amounts to considering X^2 as the product of two different G-spaces: the first copy of X has the trivial G-structure, and the second copy is the given G-space X. If G is not extremely amenable, then there is a closed minimal G-subspace Y of $\mathrm{Exp}\,\mathrm{Exp}\,(X^2)$ that is not a singleton (and hence fixed point free). This follows from:

Theorem 1.1. *Let X be compact, G a subgroup of $H(X)$. Denote by R the closure of the set $\{\Gamma_g : g \in G\}$ of the graphs of all $g \in G$ in the space $\mathrm{Exp}\,(X^2)$. Then G is extremely amenable if and only if every minimal closed G-subset of $\mathrm{Exp}\,R$ is a singleton.*

Here X^2 is the product of the trivial G-space and the given G-space X, as in the paragraph preceding Theorem 1.1, and R is considered as a G-subspace of $\mathrm{Exp}\,(X^2)$.

For example, let $X = I = [0, 1]$ be the closed unit interval. Consider the group $G = H_+([0, 1])$ of all orientation-preserving self-homeomorphisms of I. The space R in this case consists of all curves Γ in the square I^2 that connect the lower left and upper right corners and "never go down": if $(x, y) \in \Gamma$, $(x', y') \in \Gamma$ and $x < x'$, then $y \le y'$ (see the picture in [8, Example 2.5.4]). It can be verified that the only minimal compact G-subsets of $\mathrm{Exp}\,R$ are singletons (they are of the form {a closed union of G-orbits in R}). The proof depends on the following lemma:

Lemma 1.2. *Let Δ^n be the n-simplex of all n-tuples $(x_1, \dots, x_n) \in I^n$ such that $0 \le x_1 \le \cdots \le x_n \le 1$. Equip Δ^n with the natural action of the group $G = H_+([0, 1])$. Then every minimal closed G-subset of $\mathrm{Exp}\,\Delta^n$ is a singleton (= {a union of some faces of Δ^n}).*

The idea to consider the action of $G = H_+([0,1])$ on Δ^n is borrowed from [2], where it is shown that the geometric realization of any simplicial set can be equipped with a natural action of G. We shall not prove Lemma 1.2, since this lemma follows from Pestov's theorem that G is extremely amenable, and I am not aware of a short independent proof of the lemma. The essence of the lemma is that every subset of Δ^n can be either pushed (by an element of G) into the ϵ-neighbourhood of the boundary of the simplex or else can be pushed to approximate the entire simplex within ϵ. Some Ramsey-type argument seems to be necessary for this. Actually Lemma 1.2 may be viewed as a topological equivalent of the finite Ramsey theorem [9, Theorem 1.5.2], since Pestov showed that this theorem has an equivalent reformulation in terms of the notion of a "finitely oscillation stable" dynamical system [9, Section 1.5], and extremely amenable groups are characterized in the same terms [9, Theorem 2.1.11].

An important example of an extremely amenable group is the Polish group $\mathrm{Aut}\,(\mathbb{Q})$ of all automorphisms of the ordered set $\mathbb{Q}$ of rationals [6], [9, Theorem 2.3.1]. This group is considered with the topology inherited from $(\mathbb{Q}_d)^{\mathbb{Q}}$, where $\mathbb{Q}_d$ is the set of rationals with the discrete topology. Let $K \subset [0,1]$ be the usual middle-third Cantor set. The topological group $\mathrm{Aut}\,(\mathbb{Q})$ is isomorphic to the topological group $G = H_<(K) \subset H(K)$ of all order-preserving self-homeomorphisms of K. To see this, note that pairs of the endpoints of "deleted intervals" (= components of $[0,1] \setminus K$) form a set which is order-isomorphic to $\mathbb{Q}$, whence a homomorphism $G \to \mathrm{Aut}\,(\mathbb{Q})$ which is easily verified to be a topological isomorphism. One can prove that the group $G \simeq \mathrm{Aut}\,(\mathbb{Q})$ is extremely amenable with the aid of Theorem 1.1. The proof is essentially the same as in the case of the group $G = H_+([0,1])$. The space R considered in Theorem 1.1 again is the space of "curves", this time in K^2, that go from $(0,0)$ to $(1,1)$ and "look like graphs", with the exception that they may contain vertical and horizontal parts. The evident analogue of Lemma 1.2 holds for "Cantor simplices" of the form $\{(x_1, \dots, x_n) \in K^n : 0 \le x_1 \le \dots \le x_n \le 1\}$.

Theorem 1.1 may help to answer the following:

Question 1.3. *Let P be pseudoarc, $G = H(P)$, and let G_0 be the stabilizer of a given point $x \in P$. Is G_0 extremely amenable?*

As explained in [11], this question is motivated by the observation that the argument involving maximal chains, which shows that the stabilizer $G_0 \subset H(X)$ of a point $p \in X$ is not extremely amenable if X is either a Hilbert cube or a compact manifold of dimension > 1, does not work for the pseudoarc. A positive answer to Question 1.3 would imply that the pseudoarc P can be identified with $\mathcal{M}(G)$ for $G = H(P)$. The problem whether this is the case was raised in [11] and appears as Problem 6.7.20 in [9].

The *suspension* ΣX of a space X is the quotient of $X \times I$ obtained by collapsing the "bottom" $X \times \{0\}$ and the "top" $X \times \{1\}$ to points. Let $q : \Sigma X \to I$ be the natural projection. The inverse image under q of the maximal chain $\{[0, x] : x \in I\}$ of closed subsets of I is a maximal chain of closed subsets of ΣX.

Question 1.4. *Let $Q = I^\omega$ be the Hilbert cube, and C be the maximal chain of subcontinua of ΣQ considered above. If $G = H(\Sigma Q)$ and $G_0 \subset G$ is the stabilizer of C, is G_0 extremely amenable?*

This question is motivated by the search for a good candidate for the space $\mathcal{M}(G)$, where $G = H(Q)$. The space Φ_c of all maximal chains of subcontinua of Q, proved to be minimal by Y. Gutman [5], may be such a candidate [9, Problem 6.4.13]. Recall that for the group $G = H(K)$, where $K = 2^\omega$ is the Cantor set, $\mathcal{M}(G)$ can be identified with the space $\Phi \subset \operatorname{Exp}\operatorname{Exp} K$ of all maximal chains of closed subsets of K [4], [9, Example 6.7.18].

There is another characterization (Theorem 1.5) of extremely amenable groups in the spirit of Theorem 1.1 which, in combination with Lemma 1.2, readily implies Pestov's results that $H_+([0, 1])$ and $\operatorname{Aut}(\mathbb{Q})$ are extremely amenable. Let X be compact, $Y_n \subset (\operatorname{Exp} X)^n$ for $n = 1, 2, \dots$. We say that $\cup_{n=1}^\infty Y_n$ *contains arbitrarily fine covers* if for every open cover α of X there are $n \ge 1$ and $(F_1, \dots, F_n) \in Y_n$ such that $\cup_{i=1}^n F_i = X$ and the cover $\{F_i\}_{i=1}^n$ of X refines α.

Theorem 1.5. *Let X be compact, G a subgroup of $H(X)$. Let Y_n be a closed G-invariant subset of $(\operatorname{Exp} X)^n$ $(n = 1, 2, \dots)$ such that $\cup_{n=1}^\infty Y_n$ contains arbitrarily fine covers of X. Then G is extremely amenable if and only if for every $n \ge 1$ every minimal closed G-invariant subset of $\operatorname{Exp} Y_n$ is a singleton.*

Observe that Pestov's theorem asserting that $G = H_+([0,1])$ is extremely amenable follows from Theorem 1.5 and Lemma 1.2: it suffices to take for Y_{n+1} the collection of all sequences

$$([0,x_1],[x_1,x_2],\dots,[x_n,1]),$$

where $0 \le x_1 \le \dots \le x_n \le 1$. The G-space Y_{n+1} is isomorphic to the n-simplex Δ^n considered in Lemma 1.2. The argument for $\mathrm{Aut}(\mathbb{Q}) \simeq H_<(K)$ is similar.

The proof of Theorems 1.1 and 1.5 depends on the notion of a representative family of compact G-spaces. We introduce this notion in Section 2 and observe that a topological group G is extremely amenable if (and only if) there exists a representative family $\{X_\alpha\}$ such that any minimal closed G-subset of any X_α is a singleton (Theorem 2.2). In Section 3 we prove that the single space $\mathrm{Exp}\, R$ considered in Theorem 1.1 constitutes a representative family (Theorem 3.1). The conjunction of Theorems 2.2 and 3.1 proves Theorem 1.1. In Section 4 we prove that under the conditions of Theorem 1.5 the sequence $\{\mathrm{Exp}\, Y_n\}$ is representative (Theorem 4.2). The conjunction of Theorems 2.2 and 4.2 proves Theorem 1.5.

2. Representative families of G-spaces

Let G be a topological group, X a compact G-space. For $g \in G$ the *g-translation* of X is the map $x \mapsto gx$, $x \in X$. The *enveloping semigroup* (or the *Ellis semigroup*) $E(X)$ of the dynamical system (G, X) is the closure of the set of all g-translations, $g \in G$, in the compact space X^X. This is a right topological semigroup compactification of G, as defined in Section 1. The natural map $G \to E(X)$ extends to a G-map $\mathcal{S}(G) \to E(X)$ which is a morphism of right topological semigroups.

Definition 2.1. A family $\{X_\alpha : \alpha \in A\}$ of compact G-spaces is *representative* if the family of natural maps $\mathcal{S}(G) \to E(X_\alpha)$, $\alpha \in A$, separates points of $\mathcal{S}(G)$ (and hence yields an embedding of $\mathcal{S}(G)$ into $\prod_{\alpha \in A} E(X_\alpha)$).

Theorem 2.2. *Let G be a topological group, $\{X_\alpha\}$ a representative family of compact G-spaces. Then G is extremely amenable if (and only if) every minimal closed G-subset of every X_α is a singleton.*

This is a special case of a more general theorem:

Theorem 2.3. *If $\{X_\alpha\}$ is a representative family of compact G-spaces, the universal minimal compact G-space $\mathcal{M}(G)$ is isomorphic (as a G-space) to a G-subspace of a product $\prod Y_\beta$, where each Y_β is a minimal compact G-space isomorphic to a G-subspace of some X_α.*

Proof. By definition of a representative family, the greatest ambit $\mathcal{S}(G)$ can be embedded (as a G-space) into the product $\prod E(X_\alpha)$ and hence also into the product $\prod X_\alpha^{X_\alpha}$. Consider $\mathcal{M}(G)$ as a subspace of $\mathcal{S}(G)$ and take for the Y_β 's the projections of $\mathcal{M}(G)$ to the factors X_α. □

We now give a sufficient condition for a family of compact G-spaces to be representative. Let us say that two subsets A, B of G are *far from each other with respect to the right uniformity* if one of the following equivalent conditions holds: (1) the neutral element 1_G of G is not in the closure of the set BA^{-1}; (2) for some neighbourhood U of 1_G the sets A and UB are disjoint; (3) there exists a right uniformly continuous function $f : G \to [0,1]$ such that $f = 0$ on A and $f = 1$ on B; (4) A and B have disjoint closures in $\mathcal{S}(G)$.

Proposition 2.4. *Let $\mathcal{F}$ be a family of compact G-spaces. Suppose that the following holds:*

(*) if $A, B \subset G$ are far from each other with respect to the right uniformity, then there exists $X \in \mathcal{F}$ and $p \in X$ such that the sets Ap and Bp have disjoint closures in X.

Then $\mathcal{F}$ is representative.

Proof. Consider the natural map $G \to \prod\{E(X) : X \in \mathcal{F}\}$. It defines a compactification bG of G. We must prove that this compactification is equivalent to $\mathcal{S}(G)$.

Let A, B be any two subsets of G with disjoint closures in $\mathcal{S}(G)$. Then A and B are far from each other with respect to the right uniformity. According to the condition (*), there exists $X \in \mathcal{F}$ and $p \in X$ such that the sets Ap and Bp have disjoint closures in X. It follows that the images of A and B in $E(X)$ have disjoint closures, and *a fortiori* the images of A and B in bG have disjoint closures. It follows that $\mathcal{S}(G)$ and bG are equivalent compactifications of G [3, Theorem 3.5.5]. □

3. Proof of Theorem 1.1

Recall the setting of Theorem 1.1: X is compact, G is a topological subgroup of $H(X)$. For $g \in G$ let $\Gamma_g = \{(x, gx) : x \in X\} \subset X^2$ be the graph of g, and let R be the closure of the set $\{\Gamma_g : g \in G\}$ in the compact space $\mathrm{Exp}\,(X^2)$. We consider the action of G on $\mathrm{Exp}\,(X^2)$ defined by $gF = \{(x, gy) : (x, y) \in F\}$ ($g \in G$, $F \in \mathrm{Exp}\,(X^2)$), and consider R as a G-subspace of $\mathrm{Exp}\,(X^2)$.

Theorem 3.1. *Let X be a compact space, $G \subset H(X)$. Let $R \subset \mathrm{Exp}\,(X^2)$ be the compact G-space defined above. The family consisting of the single compact G-space* $\mathrm{Exp}\,R$ *is representative.*

In other words, $\mathcal{S}(G)$ is isomorphic to the enveloping semigroup of $\mathrm{Exp}\,R$.

Proof. Let $A, B \subset G$ be far from each other (that is, 1_G is not in the closure of BA^{-1}). In virtue of proposition 2.4, it suffices to find $p \in Y = \mathrm{Exp}\,R$ such that Ap and Bp have disjoint closures in Y.

Let p be the closure of the set $\{\Gamma_g : g \in A^{-1}\}$ in the space $\mathrm{Exp}\,(X^2)$. Then p is a closed subset of R and hence $p \in Y$. We claim that p has the required property: Ap and Bp have disjoint closures in Y or, which is the same, in $\mathrm{Exp}\,\mathrm{Exp}\,(X^2)$.

There exist a continuous pseudometric d on X and $\delta > 0$ such that

$$\forall f \in A\ \forall g \in B\ \exists x \in X\ \ (d(gf^{-1}(x), x) \geq \delta).$$

Let $\Delta \subset X^2$ be the diagonal. Let $C \subset X^2$ be the closed set defined by

$$C = \{(x, y) \in X^2 : d(x, y) \geq \delta\}.$$

Let $K \subset \mathrm{Exp}\,X^2$ be the closed set defined by

$$K = \{F \subset X^2 : F \text{ meets } C\}.$$

Consider the closed sets $L_1, L_2 \subset \mathrm{Exp}\,\mathrm{Exp}\,(X^2)$ defined by

$$L_1 = \{q \subset \mathrm{Exp}\,(X^2) : q \text{ is closed and } \Delta \in q\}$$

and

$$L_2 = \{q \subset \mathrm{Exp}\,(X^2) : q \text{ is closed and } q \subset K\}.$$

Since $\Delta \notin K$, the sets L_1 and L_2 are disjoint. It suffices to verify that $Ap \subset L_1$ and $Bp \subset L_2$.

The first inclusion is immediate: if $g \in A$, then for $h = g^{-1}$ we have $\Delta = g\Gamma_h \in gp$, hence $gp \in L_1$. Thus $Ap \subset L_1$. We now prove that $Bp \subset L_2$. Let $g \in B$. If $f \in A$ and $h = f^{-1}$, there exists $x \in X$ such that $d(gh(x), x) \geq \delta$, which means that Γ_{gh} meets C. Hence $g\Gamma_h = \Gamma_{gh} \in K$. It follows that the closed set $g^{-1}K$ contains the set $\{\Gamma_h : h \in A^{-1}\}$ and hence also its closure p. In other words, $gp \subset K$ and hence $gp \in L_2$. □

As noted in Section 1, Theorem 1.1 follows from Theorems 2.2 and 3.1.

Combining Theorems 2.3 and 3.1, we obtain the following generalization of Theorem 1.1:

Theorem 3.2. *Let X be a compact space, G a subgroup of $H(X)$. Let R be the same as in Theorems 1.1 and 3.1. Let $\mathcal{F}$ be the family of all minimal closed G-subspaces of* $\operatorname{Exp} R$. *Then $\mathcal{M}(G)$ is isomorphic to a subspace of a product of members of $\mathcal{F}$ (some factors may be repeated).*

4. Proof of Theorem 1.5

Theorem 3.1 implies that for any subgroup $G \subset H(X)$ the one-point family $\{\operatorname{Exp}\operatorname{Exp}(X^2)\}$ is representative (recall that we consider the trivial action on the first factor X). I do not know whether X^2 can be replaced here by X. On the other hand, the following holds:

Theorem 4.1. *Let X be a compact space, G a subgroup of $H(X)$. The sequence*
$\{\operatorname{Exp}((\operatorname{Exp} X)^n)\}_{n=1}^{\infty}$ *of compact G-spaces is representative.*

This is a special case of a more general theorem:

Theorem 4.2. *Let X be a compact space, G a subgroup of $H(X)$. Let Y_n be a closed G-invariant subset of* $(\operatorname{Exp} X)^n$ *($n = 1, 2, \dots$) such that $\cup_{n=1}^{\infty} Y_n$ contains arbitrarily fine covers of X. Then the sequence* $\{\operatorname{Exp} Y_n\}_{n=1}^{\infty}$ *of compact G-spaces is representative.*

Proof. Let $A, B \subset G$ be two sets that are far from each other with respect to the right uniformity. In virtue of proposition 2.4, it suffices to find n and a point $p \in \operatorname{Exp} Y_n$ such that Ap and Bp have disjoint closures in $\operatorname{Exp} Y_n$ or, which is the same, in $Z_n = \operatorname{Exp}((\operatorname{Exp} X)^n)$.

There exist a continuous pseudometric d on X and $\delta > 0$ such that A and B are $(d, 2\delta)$-far from each other, in the sense that

$$\forall f \in A\ \forall g \in B\ \exists x \in X\ (d(f(x), g(x)) > 2\delta).$$

The assumption that $\cup_{n=1}^{\infty} Y_n$ contains arbitrarily fine covers implies that we can find $n \geq 1$ and closed sets $C_1, \dots, C_n \subset X$ of d-diameter $\leq \delta$ such that $(C_1, \dots, C_n) \in Y_n$ and $\cup_{i=1}^{n} C_i = X$. For each $g \in G$ let $F_g = (g^{-1}(C_1), \dots, g^{-1}(C_n)) \in (\operatorname{Exp} X)^n$. Since Y_n is G-invariant, we have $F_g \in Y_n$. Let p be the closure of the set $\{F_g : g \in A\}$ in the space $(\operatorname{Exp} X)^n$. Then $p \in \operatorname{Exp} Y_n$. We claim that p has the required property: Ap and Bp have disjoint closures in Z_n.

Let $D_i = \{x \in X : d(x, C_i) \geq \delta\}$, $i = 1, \dots, n$. Consider the closed sets $K_1, K_2 \subset (\operatorname{Exp} X)^n$ defined by

$$K_1 = \{(F_1, \dots, F_n) \in (\operatorname{Exp} X)^n : F_i \subset C_i,\ i = 1, \dots, n\}$$

and

$$K_2 = \{(F_1, \dots, F_n) \in (\operatorname{Exp} X)^n : F_i \text{ meets } D_i \text{ for some } i = 1, \dots, n\}.$$

Consider the closed sets $L_1, L_2 \subset Z_n$ defined by

$$L_1 = \{q \subset (\operatorname{Exp} X)^n : q \text{ is closed and } q \text{ meets } K_1\}$$

and

$$L_2 = \{q \subset (\operatorname{Exp} X)^n : q \text{ is closed and } q \subset K_2\}.$$

Clearly K_1 and K_2 are disjoint, hence L_1 and L_2 are disjoint as well. It suffices to verify that $Ap \subset L_1$ and $Bp \subset L_2$.

The first inclusion is immediate: if $g \in A$, then $F_g \in p$ and $gF_g = (C_1, \dots, C_n) \in K_1 \cap gp$, hence gp meets K_1 and $gp \in L_1$. We now prove that $Bp \subset L_2$. Let $h \in B$. If $g \in A$, we can find $x \in X$ such that $d(g(x), h(x)) > 2\delta$ and an index i, $1 \leq i \leq n$, such that $g(x) \in C_i$. Since $\operatorname{diam} C_i \leq \delta$, we have $h(x) \in D_i$ and therefore $h(x) \in hg^{-1}(C_i) \cap D_i \neq \emptyset$. It follows that $hF_g = (hg^{-1}(C_1), \dots, hg^{-1}(C_n)) \in K_2$. This holds for every $g \in A$, and thus we have shown that the closed set $h^{-1}K_2 \subset (\operatorname{Exp} X)^n$ contains the set $\{F_g : g \in A\}$ and hence also its closure p. In other words, $hp \subset K_2$ and hence $hp \in L_2$. $\square$

Theorem 1.5 follows from Theorems 4.2 and 2.2.

Combining Theorems 2.3 and 4.2, we obtain the following generalization of Theorem 1.5:

Theorem 4.3. *Let X be a compact space, G a subgroup of $H(X)$. Let Y_n be a closed G-invariant subset of $(\mathrm{Exp}\, X)^n$ $(n = 1, 2, \dots)$ such that $\cup_{n=1}^{\infty} Y_n$ contains arbitrarily fine covers of X. Let $\mathcal{F}$ be the family of all (up to an isomorphism) minimal closed G-subspaces of $\mathrm{Exp}\, Y_n$, $n = 1, 2, \dots$. Then $\mathcal{M}(G)$ is isomorphic to a subspace of a product of members of $\mathcal{F}$ (some factors may be repeated).*

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