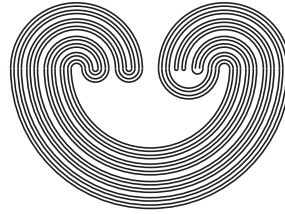

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ON TWO PROBLEMS CONCERNING TOPOLOGICAL CENTERS

by

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ON TWO PROBLEMS CONCERNING TOPOLOGICAL CENTERS

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ABSTRACT. Let Γ be an infinite discrete group and $\beta\Gamma$ its Čech-Stone compactification. Using the well known fact that a free ultrafilter on an infinite set is nonmeasurable, we show that for each element p of the remainder $\beta\Gamma \setminus \Gamma$, left multiplication $L_p : \beta\Gamma \rightarrow \beta\Gamma$ is not Borel measurable. Next assume that Γ is abelian. Let $\mathcal{D} \subset \ell^\infty(\Gamma)$ denote the subalgebra of distal functions on Γ and let $D = \Gamma^{\mathcal{D}} = |\mathcal{D}|$ denote the corresponding universal distal (right topological group) compactification of Γ . Our second result is that the topological center of D (i.e. the set of $p \in D$ for which $L_p : D \rightarrow D$ is a continuous map) is the same as the algebraic center and that for $\Gamma = \mathbb{Z}$, this center coincides with the canonical image of Γ in D .

1. INTRODUCTION

This short note is a direct outcome of the topology conference held in Castellón in the summer of 2007. I was presented during the conference with two problems relating to the topological center of certain right topological semigroups. (A compact semigroup A such that for each $p \in A$ the corresponding right multiplication $R_p : q \mapsto qp$ is continuous is called a *right topological semigroup*. The collection of elements $p \in A$ for which the corresponding left multiplication $L_p : q \mapsto pq$ is continuous is called the *topological center* of A .) The first was a question of Michael Megrelishvili:

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Given an infinite discrete group Γ , which are the elements of $\beta\Gamma$ for which $L_p : \beta\Gamma \rightarrow \beta\Gamma$ is a Baire class 1 map? (It is known that the topological center of $\beta\Gamma$ is exactly Γ itself, considered as a subset of $\beta\Gamma$, see e.g. [5].) The second problem is due to Mahmoud Filali: If $D = D(\Gamma)$ is the universal distal Ellis group of Γ , identify the topological center of D .

I present here a complete answer to Megrelishvili's problem, based on the well known result that a free ultrafilter on an infinite set is nonmeasurable, and an answer to Filali's problem in the case $\Gamma = \mathbb{Z}$, the group of integers.

The interested reader is referred to [3, chapter 1] and the bibliography list thereof, for more information on the abstract theory of topological dynamics, and to [5] for information concerning $\beta\Gamma$.

I thank both Megrelishvili and Filali for addressing to me these nice problems. I also thank the organizers of the Castellón meeting for the formidable effort they put into the details of the conference and for their warm hospitality.

2. ON THE CENTER OF $\beta\Gamma$

We consider the Čech-Stone compactification $\beta\Gamma$ as the collection of ultrafilters on Γ . The topology on $\beta\Gamma$ is given by the collection of basic clopen sets $\{V_A : A \subset \Gamma\}$, where $V_A = \{p \in \beta\Gamma, A \in p\}$. We also recall that the semigroup product on $\beta\Gamma$ is defined by

$$A \in pq \iff \{\gamma \in \Gamma : \gamma^{-1}A \in q\} \in p.$$

For more details see e.g. [2].

Theorem 2.1. *Let Γ be an infinite countable discrete group and $\beta\Gamma$ its Čech-Stone compactification. For each element p of the remainder $\beta\Gamma \setminus \Gamma$, left multiplication $L_p : \beta\Gamma \rightarrow \beta\Gamma$ is not Borel measurable.*

Proof. Let $\mathcal{P}(\Gamma)$ denote the collection of all subsets of Γ . Let $\Omega = \{0, 1\}^\Gamma$ and let $\chi : \mathcal{P}(\Gamma) \rightarrow \Omega$ denote the canonical map $\chi(A) = \mathbf{1}_A$ for $A \subset \Gamma$. We regard Ω as a compact space and let \mathcal{B} denote its Borel σ -algebra. Let $\mu = (\frac{1}{2}(\delta_0 + \delta_1))^\Gamma$ denote the product probability measure on Ω and let \mathcal{B}_μ denote the completion of \mathcal{B} with respect to μ .

A well known and easy fact, which for completeness we will reproduce below (Lemma 2.3), is that a free ultrafilter on an infinite set is nonmeasurable: Viewing an element $p \in \beta\Gamma \setminus \Gamma$ as an ultrafilter on Γ , the collection $\{\chi(A) : A \in p\} \subset \Omega$, is not μ -measurable; i.e. not an element of \mathcal{B}_μ . In particular it is not a Borel subset of Ω . (In fact, it is not even Baire measurable, [6] and [7].)

The compact space Ω becomes a dynamical system when we let Γ act on it by permuting the coordinates:

$$\gamma\omega(\gamma') = \omega(\gamma^{-1}\gamma').$$

Of course the measure μ is Γ -invariant, but we will not need this fact. The action of Γ on Ω extends to an action of $\beta\Gamma$ in the natural way and we write $p\omega$ for the image of $\omega \in \Omega$ under $p \in \beta\Gamma$. (In fact via this “action” $\beta\Gamma$ is identified with the enveloping semigroup of the system (Ω, Γ) , see [3, chapter 1].)

For $A \subset \Gamma$ and $p \in \beta\Gamma$ set

$$p \star A = \{\gamma \in \Gamma : \gamma A^{-1} \in p\}$$

and check that $\gamma \mathbf{1}_A = \mathbf{1}_{\gamma A}$ and $p\chi(A) = p\mathbf{1}_A = \mathbf{1}_{p \star A} = \chi(p \star A)$. Moreover if $q \in \beta\Gamma$ then

$$pq \star A = p \star (q \star A).$$

For convenience I sometimes identify a subset $A \subset \Gamma$ with the corresponding element $\mathbf{1}_A = \chi(A)$ in Ω .

Let $\pi_e : \Omega \rightarrow \{0, 1\}$ denote the projection on the e -component of Ω . Here e is the neutral element of Γ . Let $\omega_0 = \mathbf{1}_D$ be some fixed element of Ω whose Γ orbit is dense in Ω . Let $\psi : \gamma \mapsto \gamma\omega_0$ be the orbit map and let $\hat{\psi}$ denote its unique extension to $\beta\Gamma$. Thus $\hat{\psi}(q) = q\omega_0$ for $q \in \beta\Gamma$.

Consider the map $L_p : \beta\Gamma \rightarrow \beta\Gamma$ and write $\phi : \beta\Gamma \rightarrow \{0, 1\}$ for the map $\phi = \pi_e \circ \hat{\psi} \circ L_p$. (Thus $\phi(q) = (pq\omega_0)(e)$.) We define $J : \Omega \rightarrow \Omega$ by $(J(\omega))(\gamma) = \omega(\gamma^{-1})$.

We have

$$\mathcal{Q} := \phi^{-1}(1) = \{q \in \beta\Gamma : pq\omega_0(e) = 1\}.$$

Now $pq\omega_0(e) = 1$ iff $e \in \chi^{-1}(pq\omega_0)$ hence

$$\begin{aligned} \mathcal{Q} &= \{q \in \beta\Gamma : pq\omega_0(e) = 1\} \\ &= \{q \in \beta\Gamma : e \in p \star q\omega_0\} \\ &= \{q \in \beta\Gamma : e \in p \star q \star D\} \\ &= \{q \in \beta\Gamma : e \in \{\gamma \in \Gamma : \gamma(q \star D)^{-1} \in p\}\} \\ &= \{q \in \beta\Gamma : (q \star D)^{-1} \in p\} \\ &= \{q \in \beta\Gamma : J(q\omega_0) \in p\}. \end{aligned}$$

Thus $(J \circ \hat{\psi})(\mathcal{Q}) = p$ and since also $(\hat{\psi}^{-1} \circ J^{-1})(p) = \mathcal{Q}$ we conclude that $\mathcal{Q} = \phi^{-1}(1)$ is not Borel measurable in $\beta\Gamma$. Finally, since also $\mathcal{Q} = L_p^{-1}(\{q \in \beta\Gamma : (q\omega_0)(e) = 1\})$ we see that L_p is not Borel measurable. \square

In the next two lemmas we follow [7] (see also [1]). Let I be an infinite set, $\Omega = \{0, 1\}^I$. As above, we identify subsets A of I with their characteristic functions $\mathbf{1}_A \in \Omega = \{0, 1\}^I$ and, accordingly, filters on I with subsets of Ω . Let $\phi : \Omega \rightarrow \Omega$ denote the ‘‘flip’’ function defined by $\phi(\omega)_i = 1 - \omega_i$, $i \in I$. We consider the measure space $(\Omega, \Sigma_\lambda, \lambda)$, where $\Omega = \{0, 1\}^I$, λ is the Bernoulli measure $\lambda = (\frac{1}{2}(\delta_0 + \delta_1))^I$, and Σ_λ denotes the completion of the Borel σ -algebra with respect to λ . As usual we use the notation λ_* and λ^* for the induced inner and outer measures.

The assertions of the next lemma are easily verified.

- Lemma 2.2.**
1. *The involution ϕ is measurable and it preserves λ .*
 2. *For $A \subset I$ we have $\phi(\mathbf{1}_A) = \mathbf{1}_{A^c}$.*
 3. *If \mathcal{F} is a filter on I then $\phi\mathcal{F} \cap \mathcal{F} = \emptyset$.*
 4. *If \mathcal{F} is a free filter on I (i.e. $\bigcap \mathcal{F} = \emptyset$) then, considered as a collection of subsets of $\{0, 1\}^I$ it is a ‘‘tail event’’, that is, for every finite subset $F \subset I$, $\mathcal{F} = \{0, 1\}^F \times \mathcal{F}'$, with $\mathcal{F}' \subset \{0, 1\}^{I \setminus F}$.*
 5. *A filter \mathcal{F} on I is an ultrafilter iff $\phi(\mathcal{F}) \cup \mathcal{F} = \Omega$.*

Lemma 2.3. *Let \mathcal{F} be a free filter on I . Then*

1. $\lambda_*(\mathcal{F}) = 0$.
2. $\lambda^*(\mathcal{F}) \in \{0, 1\}$.
3. $\lambda^*(\mathcal{F}) = 1$ if \mathcal{F} is an ultrafilter.

4. A free filter \mathcal{F} is measurable iff $\lambda^*(\mathcal{F}) = 0$, and nonmeasurable iff $\lambda^*(\mathcal{F}) = 1$. In particular, every free ultrafilter is nonmeasurable.

Proof. If \mathcal{F} is a free filter on I and $\mathcal{G} \subset \mathcal{F}$ is a measurable tail event then it has measure either 0 or 1. Thus $\lambda^*(\mathcal{F}) \in \{0, 1\}$. This proves part 2. We also have $\lambda_*(\mathcal{F}) \in \{0, 1\}$ and since $\phi(\mathcal{F}) \cap \mathcal{F} = \emptyset$ it follows that

$$1 = \lambda(\Omega) \geq \lambda_*(\phi\mathcal{F}) + \lambda_*(\mathcal{F}) = 2\lambda_*(\mathcal{F}).$$

We conclude that $\lambda_*(\mathcal{F}) = 0$, proving part 1. If \mathcal{F} is an ultrafilter then $\mathcal{F} \cup \phi\mathcal{F} = \{0, 1\}^I$ and we conclude that

$$1 = \lambda(\Omega) \leq \lambda^*(\phi\mathcal{F}) + \lambda^*(\mathcal{F}) = 2\lambda^*(\mathcal{F}),$$

whence $\lambda^*(\mathcal{F}) = 1$. This proves part 3. Part 4 is now clear. \square

3. ON THE CENTER OF $\Gamma^{\mathcal{D}}$, THE UNIVERSAL DISTAL ELLIS GROUP OF Γ

Let Γ be a discrete abelian group. Let \mathcal{D} denote the closed Γ -invariant subalgebra of (complex valued) distal functions in $\ell^\infty(\Gamma)$. Let $D = \Gamma^{\mathcal{D}} = |\mathcal{D}|$ denote the corresponding Gelfand space. It is well known that D is the largest right topological group compactification of Γ .

Theorem 3.1. *Let Γ be an infinite discrete abelian group. The topological center of $D = \Gamma^{\mathcal{D}}$ is the same as the algebraic center and, when $\Gamma = \mathbb{Z}$, it also coincides with the canonical image of Γ in D .*

Proof. In order to simplify our notation we will identify elements of Γ with their images in D . The coincidence of the topological and algebraic centers of D is easy: Suppose first that $p \in D$ is in the algebraic center of this group. Then, as right multiplication is always continuous, we have for any convergent net $q_\alpha \rightarrow q$ in D

$$pq = qp = \lim q_\alpha p = \lim pq_\alpha,$$

i.e. $L_p : D \rightarrow D$ is continuous.

Conversely, assume that p is in the topological center; i.e. $L_p : D \rightarrow D$ is continuous. We note that if q is an element of D then $\gamma q = q\gamma$ for every $\gamma \in G$. In fact choosing a convergent net $\Gamma \ni \gamma_\alpha \rightarrow q$, by the commutativity of Γ ,

$$\gamma q = \gamma \lim \gamma_\alpha = \lim \gamma \gamma_\alpha = \lim \gamma_\alpha \gamma = q\gamma.$$

Now, with this in mind, we have

$$pq = p \lim \gamma_\alpha = \lim p\gamma_\alpha = \lim \gamma_\alpha p = qp,$$

so that p is indeed an element of the center.

Now to the more delicate task of showing that this center coincides with Γ . Let $p \in D$ be a central element. If $p \notin \Gamma$ then there exists a *metric* minimal distal dynamical system (Y, Γ) and a point $y_0 \in Y$ such that

$$(3.1) \quad py_0 \notin \Gamma y_0.$$

By assumption the map $L_p : D \rightarrow D$ is continuous (in fact a homeomorphism) and as we have seen it also commutes with every element of Γ . In other words, L_p is an automorphism of the system (D, Γ) . Now the dynamical system (D, Γ) is the universal distal system and therefore, it admits a unique homomorphism of dynamical systems $\hat{\phi} : (D, \Gamma) \rightarrow (E(Y, \Gamma), \Gamma)$ onto the enveloping semigroup $E = E(Y, \Gamma)$ (which by a theorem of Ellis is in fact a group) such that $\hat{\phi}(e_D) = e_E$. Now the map $\phi : p \mapsto \hat{\phi}(p)y_0$ (which we write simply as $p \mapsto py_0$) is a homomorphism $\phi : (D, \Gamma) \rightarrow (Y, \Gamma)$ with $\phi(e) = y_0$. If $y_\alpha \rightarrow y$ is a convergent net in Y then there are $q_\alpha \in D$ with $y_\alpha = q_\alpha y_0$. With no loss of generality we have $q_\alpha \rightarrow q$ in D , so that in particular $y = \lim y_\alpha = \lim q_\alpha y_0 = qy_0$. Now we see that

$$\begin{aligned} py &= pqy_0 = (p \lim q_\alpha)y_0 \\ &= (\lim pq_\alpha)y_0 = \lim p(q_\alpha y_0) \\ &= \lim py_\alpha. \end{aligned}$$

Thus p acts continuously on Y . Since also $p\gamma = \gamma p$ for every $\gamma \in \Gamma$ we conclude that p is an automorphism of the system (Y, Γ) .

Note that this argument shows that p acts as an automorphism of every factor of (D, Γ) . Therefore, our proof will be complete when we find a minimal distal dynamical system (X, Γ) extending (Y, Γ) , say $\pi : (X, \Gamma) \rightarrow (Y, \Gamma)$, where p is not an automorphism.

At this stage, in order to be able to use a method of construction developed by Glasner and Weiss in [4], we specialize to the case $\Gamma = \mathbb{Z}$. In particular the system (Y, Γ) which was singled out in the above discussion has now the form (Y, T) where $T : Y \rightarrow Y$ is a self homeomorphism of Y determined by the element $1 \in \mathbb{Z}$. Of course we can assume that Y is non-periodic (i.e. infinite).

The following construction is a special case of a general setup designed in [4] for providing minimal extensions of a given non-periodic minimal \mathbb{Z} -system (Y, T) . We refer the reader to [4] for more details.

Set $X = Y \times K$ where K denotes the circle group $K = S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Let Θ be the family of continuous maps $\theta : Y \rightarrow K$. For each $\theta \in \Theta$ let $G_\theta : X \rightarrow X$ be the map $G_\theta(y, z) = (y, z\theta(y))$ and $S_\theta = G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta$. Thus

$$(3.2) \quad S_\theta : X \rightarrow X, \quad S_\theta(y, z) = (Ty, z\theta(y)\theta(Ty)^{-1}).$$

Form the collection

$$\mathcal{S}(T) = \{G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta : \theta \in \Theta\}.$$

Theorem 1 of [4] ensures that in the set $\overline{\mathcal{S}(T)}$ (closure with respect to the uniform convergence topology in $\text{Homeo}(X)$) there is a dense G_δ subset \mathcal{R} such that for every $R \in \mathcal{R}$ the system (X, R) is minimal, distal, and the projection map $\pi : X \rightarrow Y$ is a homomorphism of dynamical systems ($\pi R(y, z) = T\pi(y, z) = Ty$). Note that every $R \in \overline{\mathcal{S}(T)}$ has the form

$$R = T_\phi : X \rightarrow X, \quad \text{where } T_\phi(y, z) = (Ty, z\phi(y)),$$

for some continuous map $\phi : Y \rightarrow K$. We will often use the fact that for $n \in \mathbb{N}$ the n -th iteration of T_ϕ has the form

$$(3.3) \quad T_\phi^n(y, z) = (T^n y, z\phi_n(y)), \quad \text{where } \phi_n(y) = \phi(T^{n-1}y) \cdots \phi(Ty)\phi(y).$$

Note that when ϕ has the very special form $\phi(y) = \theta(Ty)^{-1}\theta(y)$ for some continuous $\theta : Y \rightarrow K$, the equation (3.3) collapses:

$$(3.4) \quad \phi_n(y) = \theta(T^n y)^{-1}\theta(y), \quad \text{hence } S_\theta^n(y, z) = (T^n y, z\theta(T^n y)^{-1}\theta(y)).$$

We temporarily fix an element $R = T_\phi \in \mathcal{R}$. As observed above, the element $p \in D$ defines an automorphism of the system (X, T_ϕ) ; moreover we have for every $x = (y, z) \in X$:

$$\pi(px) = p\pi(x) = py.$$

This last observation implies that $p : X \rightarrow X$ has the form $p(y, z) = (py, \omega(y, z))$ for some continuous map $\omega : Y \times K \rightarrow K$.

Lemma 3.2. *The function ω has the form $\omega(y, z) = z\psi(y)$ for some continuous map $\psi : Y \rightarrow K$, whence*

$$p(y, z) = (py, z\psi(y)).$$

Proof. There exists a net $\{n_\nu\}_{\nu \in I}$ in \mathbb{Z} such that $p = \lim n_\nu$ in D . Thus, for every $(y, z) \in X$

$$p(y, z) = \lim T_\phi^{n_\nu}(y, z) = \lim(T_\phi^{n_\nu}y, z\phi_{n_\nu}(y)) = (py, z\psi(y)),$$

where the *point-wise* limit $\psi(y) := \lim \phi_{n_\nu}(y)$ is necessarily a continuous function. \square

The commutation relation $pT_\phi = T_\phi p$ now reads:

$$\begin{aligned} pT_\phi(y, z) &= p(Ty, z\phi(y)) = (pTy, z\phi(y)\psi(Ty)) \\ &= T_\phi p(y, z) = T_\phi(py, z\psi(y)) \\ &= (Tpy, z\psi(y)\phi(py)). \end{aligned}$$

In turn this implies:

$$(3.5) \quad \phi(y)\psi(Ty) = \psi(y)\phi(py).$$

Similarly the commutation relations $pT_\phi^n = T_\phi^n p$ yield:

$$(3.6) \quad \phi_n(y)\psi(T^n y) = \psi(y)\phi_n(py).$$

Next consider any sequence $n_i \nearrow \infty$ such that

- $\lim T^{n_i} y_0 = y_0$,
- $\lim \phi_{n_i}(y_0) = z'$, and
- $\lim \phi_{n_i}(py_0) = z''$.

Applying (3.6) and taking the limit as $i \rightarrow \infty$ we get $\psi(y_0)z'' = z'\psi(y_0)$, whence necessarily also $z' = z''$.

The proof of Theorem 3.1 will be complete when we next show that for a residual subset \mathcal{R}_1 of $\overline{\mathcal{S}(T)}$, we have $\lim \phi_{n_i}(y_0) = z' \neq z'' = \lim \phi_{n_i}(py_0)$, whenever $R = T_\phi \in \mathcal{R}_1$. Then for any element $T_\phi \in \mathcal{R} \cap \mathcal{R}_1$, (X, T_ϕ) will serve as a minimal distal system where p is not an automorphism.

Proposition 3.3. *For a given sequence $n_i \nearrow \infty$ with $\lim T^{n_i} y_0 = y_0$, the set*

$$\mathcal{R}_1 = \{T_\phi \in \overline{\mathcal{S}(T)} : \forall i \exists j > i, |\phi_{n_j}(y_0) - \phi_{n_j}(py_0)| > 1\}$$

is a residual subset of $\overline{\mathcal{S}(T)}$.

Proof. For $i \in \mathbb{N}$ and $\eta > 0$ set

$$E_{i,\eta} = \{T_\phi \in \overline{\mathcal{S}(T)} : \exists j > i, |\phi_{n_j}(y_0) - \phi_{n_j}(py_0)| > 1 + \eta\}.$$

Clearly $E_{i,\eta}$ is an open subset of $\overline{\mathcal{S}(T)}$ and for $i < k$ we have $E_{k,\eta} \subset E_{i,\eta}$.

Lemma 3.4. *Given i and $\eta > 0$, for every $\theta_0 \in \Theta$ there exists an $i_0 > i$ such that*

$$G_{\theta_0}^{-1} E_{i_0,\eta} G_{\theta_0} \subset E_{i_0,\eta/2}.$$

Proof. Fix $\theta_0 \in \Theta$. For sufficiently large i_0 , for all $j > i_0$ the distances $d(T^{n_j} y_0, y_0)$ and $d(T^{n_j} py_0, py_0)$ are so small that

$$|\theta(T^{n_j} y_0)^{-1} \theta(y_0) \phi_{n_j}(y_0) - \theta(T^{n_j} py_0)^{-1} \theta(y_0) \phi_{n_j}(py_0)| > 1 + \eta/2$$

holds whenever

$$|\phi_{n_j}(y_0) - \phi_{n_j}(py_0)| > 1 + \eta. \quad \square$$

We will show that $E_{i,\eta}$ is also dense in $\overline{\mathcal{S}(T)}$. For this it suffices to show that $G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta \in \overline{E_{i,\eta}}$ for every $\theta \in \Theta$, i.e. $T \times \text{id} \in G_\theta \overline{E_{i,\eta}} G_\theta^{-1}$.

Now for a fixed θ_0 there is by Lemma 3.4, an $i_0 > i$ with $G_{\theta_0}^{-1} E_{i_0,2\eta} G_{\theta_0} \subset E_{i_0,\eta}$, hence it suffices to show that $T \times \text{id} \in \overline{E_{i_0,2\eta}}$, since then

$$T \times \text{id} \in \overline{E_{i_0,2\eta}} \subset G_{\theta_0} \overline{E_{i_0,\eta}} G_{\theta_0}^{-1} \subset G_{\theta_0} \overline{E_{i,\eta}} G_{\theta_0}^{-1}.$$

Finally the next lemma will prove this last assertion and therefore also the density of $E_{i,\eta}$ for every i and $0 < \eta < 1$.

Lemma 3.5. *Given $i \in \mathbb{N}$, $0 < \eta < 1$ and $\varepsilon > 0$ there exists $\theta \in \Theta$ such that*

1. $d(T \times \text{id}, G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta) < \varepsilon$.
2. $G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta \in E_{i,\eta}$.

Proof. Let $I = [0, 1]$ and set $h(0) = h(1/3) = h(2/3) = 1$, $h(1) = -1$ and extend this function in an arbitrary way to a continuous $h : I \rightarrow S^1$. Choose $\delta > 0$ such that $|t-s| < \delta$ implies $|h(t)^{-1} h(s) - 1| < \varepsilon$. Let $m \in \mathbb{N}$ be such that $2/m < \delta$. Let U_1 and U_2 be open neighborhoods of y_0 and py_0 , respectively, in Y such that for $s = 1, 2$, the sets $U_s, TU_s, \dots, T^{m-1}U_s$ are mutually disjoint. (Here we use the facts that Y is infinite and that $py_0 \notin \{T^j y_0 : j \in \mathbb{Z}\}$ (3.1).) Choose $k > i$ so that $T^{n_k} y_0 \in U_1$ and $T^{n_k} py_0 \in U_2$.

Let $K_s \subset U_s$, $s = 1, 2$, be Cantor sets such that $y_0, T^{n_k}y_0 \in K_1$ and $py_0, T^{n_k}py_0 \in K_2$.

Next define:

$$g(y_0) = 0, \quad g(T^{n_k}y_0) = 1/3, \quad g(py_0) = 2/3, \quad g(T^{n_k}py_0) = 1$$

and extend this function in an arbitrary way to a continuous function $g : K_1 \cup K_2 \rightarrow S^1$. We now extend g to the set $\cup_{j=0}^{m-1} T^j(K_1 \cup K_2)$ by setting $g(y) = g(T^{-j}y)$ for $y \in T^j(K_1 \cup K_2)$. Extend g continuously over all of Y in an arbitrary way.

Set

$$\tilde{g}(y) = \frac{1}{m} \sum_{j=0}^{m-1} g(T^j y).$$

Clearly $\tilde{g} \upharpoonright (K_1 \cup K_2) = g \upharpoonright (K_1 \cup K_2)$, so that

$$\tilde{g}(y_0) = 0, \quad \tilde{g}(T^{n_k}y_0) = 1/3, \quad \tilde{g}(py_0) = 2/3, \quad \tilde{g}(T^{n_k}py_0) = 1.$$

Finally define $\theta : Y \rightarrow S^1$ by $\theta(y) = h(\tilde{g}(y))$. Note that

$$(3.7) \quad \theta(y_0) = \theta(T^{n_k}y_0) = \theta(py_0) = 1, \quad \text{and} \quad \theta(T^{n_k}py_0) = -1.$$

Now

$$\begin{aligned} G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta(y, z) &= (Ty, z\theta(Ty)^{-1}\theta(y)) \\ &= (Ty, zh(\tilde{g}(Ty))^{-1}h(\tilde{g}(y))). \end{aligned}$$

But

$$|\tilde{g}(Ty) - \tilde{g}(y)| < 2/m < \delta,$$

hence $|h(\tilde{g}(Ty))^{-1}h(\tilde{g}(y)) - 1| < \varepsilon$ and therefore also

$$d(T \times \text{id}, G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta) < \varepsilon.$$

This proves part (1) of the lemma and we now proceed to prove part (2). We have to show that $G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta \in E_i$. But this map has the form

$$S_\theta : X \rightarrow X, \quad S_\theta(y, z) = G_\theta^{-1} \circ (T \times \text{id}) \circ G_\theta = (Ty, z\theta(Ty)^{-1}\theta(y)),$$

so that, by (3.4), we have to show that there exists $j > i$ with

$$|\theta(T^{n_j}y_0)^{-1}\theta(y_0) - \theta(T^{n_j}py_0)^{-1}\theta(py_0)| > 1 + \eta.$$

Since, by the choice of θ (3.7), we have

$$|\theta(T^{n_k}y_0)^{-1}\theta(y_0) - \theta(T^{n_k}py_0)^{-1}\theta(py_0)| = |1 - (-1)| = 2 > 1 + \eta,$$

this completes the proof of the lemma. \square

To conclude the proof of Proposition 3.3 observe that, for instance, the dense G_δ set $\bigcap_{i=1}^{\infty} E_{i,1/2}$ is contained in \mathcal{R}_1 . \square

This also concludes the proof of Theorem 3.1. \square

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