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ON THE PROPERNESS OF SOME ALGEBRAIC EQUATIONS APPEARING IN FUCHSIAN GROUPS

RAQUEL DÍAZ AND AKIRA USHIJIMA

ABSTRACT. In this paper we give necessary and sufficient conditions on three orientation-preserving hyperbolic isometries T_1, T_2, T_3 so that for any point $x \in \mathbb{H}^2$ the orthogonal bisectors of the three segments with endpoints $x, T_i(x)$ intersect. This is equivalent to proving the properness of certain algebraic sets. As a corollary, we give a new proof for the existence and density of generic fundamental polygons for cocompact Fuchsian groups.

1. INTRODUCTION

A Kleinian group is a discrete subgroup of orientation-preserving isometries of the hyperbolic space $\mathbb{H}^3$. One can understand this group by taking a fundamental region for the action of Γ on $\mathbb{H}^3$. A nice kind of fundamental regions is the *Dirichlet fundamental polyhedron* centred at y, denoted by $\mathcal{P}_0(y)$, where y is any point in $\mathbb{H}^3$ (see Subsection 2.3 for precise definitions). The polyhedron $\mathcal{P}_0(y)$ and its images under the elements of Γ tessellate $\mathbb{H}^3$. We look for points y such that both the polyhedron and the corresponding tessellation are as *generic* as possible, in the sense (roughly) that: at each vertex of $\mathcal{P}_0(y)$ there are three edges of $\mathcal{P}_0(y)$ meeting; at

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each vertex of the tessellation there are four polyhedra meeting; at each edge, there are three polyhedra meeting. If, for a point $y \in \mathbb{H}^3$ we do not have the above situation, then it is reasonable to believe that we can move the point y a little bit to obtain a generic situation.

This problem is treated by Jorgensen and Marden in their paper entitled "Generic fundamental polyhedra for Kleinian groups" ([3]). The strategy for the proof is as follows. First they claim that certain sets of "bad points" (namely, those y for which there is a non-generic situation) are algebraic sets, i.e., solution sets of polynomials. An algebraic set is either equal to the whole ambient space or it is closed and has empty interior; in the latter case, it is called *proper*. Assume that each of the above algebraic sets is proper; then, because there is countable number of them, and the Baire theorem, the union of all of them still has empty interior, and its complement is dense. Since any point in the complement gives a generic polyhedron, this shows the existence and density of generic fundamental polyhedra. Then, the main effort is to show that the above algebraic sets are proper, and they do it by finding a point in the complementary of any of them.

Unfortunately, there is an error in their proof, coming from the fact that the set of bad points is not exactly an algebraic set, but is strictly contained in an algebraic set (it is actually a *semialgebraic* set, since it needs some inequalities). Then, their arguments are not enough to guaranty that these algebraic sets are proper.

Our initial motivation was (and still is) to fill the gap in their proof. We try to do that by, first, carefully establishing the relationship between the sets "bad points" (of geometric nature) and certain algebraic sets, and then proving the properness of those algebraic sets. So far we have not got the complete proof of this fact.

In this paper we provide a complete proof for the analogous problem in dimension 2, following the above strategy of Jorgensen and Marden. Precisely, to any three isometries T_1, T_2, T_3 of the hyperbolic plane, we assign a certain algebraic set and we give necessary and sufficient conditions on the T_i so that this set is proper. This is the main result of the paper, and we provide two independent proofs for it, one of them using the projective model of hyperbolic plane and the other using the upper half-plane model.

The reason why we give these two proofs is our initial motivation to prove the analogous results in higher dimensions. In fact, some parts of both proofs can already be generalized to dimension 3 or higher.

We remark that, in the above result, the three isometries are not necessarily elements of a Fuchsian group, and actually there is a strange condition found there (Theorem 4.3(2)(a), Theorem 3.8(ii)) that never occurs in Fuchsian groups. When we apply the above results to Fuchsian groups, we obtain a new proof for the existence and density of generic fundamental polygons (for dimension 2, this result is known, see [1], Theorem 9.4.5).

The paper is organized as follows. In Section 2 we recall the two mentioned models of hyperbolic plane. Section 3 contains the whole proof using the projective model. Section 4 does the same using the upper half-plane model. The algebraic set considered in Section 3 is denoted by $\mathcal{A}_{T_1T_2T_3}$, and the main result characterizing its properness is Theorem 3.8. The algebraic set considered in Section 4 is denoted by $F_{T_1T_2T_3} = 0$, and the main result characterizing its properness is Theorem 4.3, which is proved in Subsection 4.3. The results for Fuchsian groups are proved in Corollaries 3.10, 4.4, and 3.11.

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2. Background

In this paper we will be using two models for hyperbolic plane: the projective model, denoted by D^2, and the Poincaré upper half-plane model, denoted by H^2. Nevertheless, we will use the notation $\mathbb{H}^2$ to refer to hyperbolic plane when there is no need to specify one concrete model. We briefly recall here their definitions and main properties.

2.1. **Projective model.** For details in this model, see [2], [4], [5]. In this model, the hyperbolic plane, denoted by D^2, is the open unit disc. We see this model as a subset of the real projective plane $\mathbb{P}^2$, under the usual embedding $(x,y) \mapsto (1:x:y) \in \mathbb{P}^2$. The boundary of D^2 is the unit circle, $\mathbb{S}^1$, and points on $\mathbb{S}^1$ are called *points at infinity* of the hyperbolic plane.

The *hyperbolic lines* in this model are the intersection of projective lines with D^2. If L is a hyperbolic line, we will denote by $\tilde{L}$ the projective line containing L. Two hyperbolic lines L_1, L_2 either intersect or they are *parallel* (when $\tilde{L}_1 \cap \tilde{L}_2$ is a point at infinity) or *ultraparallel* (when $\tilde{L}_1 \cap \tilde{L}_2$ is a point in $\mathbb{P}^2 - (D^2 \cup \mathbb{S}^1)$).

The hyperbolic distance of two points $A, B \in D^2$ is given by the cross-ratio $d_D(A,B) = \frac{1}{2}\log[U,V,A,B]$, where U, V are the points at infinity of the line containing A, B, and here the cross-ratio is the usual one for four aligned points in a projective space. We remark that this formula gives a *signed* distance: if we orientate the hyperbolic line containing A, B from V to U, then $d_D(A,B)$ is positive for all the points B in the ray from A to U.

Orthogonality in this model corresponds to orthogonality with respect to $\mathbb{S}^1$, namely, two hyperbolic lines L_1, L_2 are orthogonal if and only if $\tilde{L}_1$ contains the pole of $\tilde{L}_2$ with respect to $\mathbb{S}^1$.

Hyperbolic isometries are the projective transformations that preserve D^2. We can represent a hyperbolic isometry uniquely by a 3×3 matrix M such that $M^t F M = F$ and the entry $M[1,1]$ of M is positive, where $F = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The isometry preserves orientation if and only if $\det M > 0$. Along the paper we will always represent a hyperbolic isometry by a matrix satisfying the above properties. We denote by $\mathrm{Isom}^+(\mathrm{D}^2)$ the group of orientation-preserving hyperbolic isometries. There are three kinds of them (not trivial): *hyperbolic*, *parabolic* or *elliptic*, depending on whether the isometry fixes two points at infinity, or exactly one point at infinity or exactly one point on D^2.

2.2. **Upper half-plane model.** Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. An important subset of $\widehat{\mathbb{C}}$ is $\widehat{H^2} = H^2 \cup \partial H^2$, where $H^2 = \{z \in \mathbb{C} \,|\, \Im z > 0\}$ and $\partial H^2 = \{z \in \mathbb{C} \,|\, \Im z = 0\} \cup \{\infty\}$. The set H^2 turns out to be a model of the two-dimensional hyperbolic space, and it is well-known that the orientation-preserving isometry group is

$$\mathrm{PSL}_2(\mathbb{R}) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \middle| \, a, b, c, d \in \mathbb{R}, a\,d - b\,c = 1 \right\} \Big/ \{\pm Id\}.$$

An element $T \in \mathrm{PSL}_2(\mathbb{R})$ acts on H^2 as the restriction of the Möbius transformation $T(z) = \frac{a\,z+b}{c\,z+d}$. Since $a, b, c, d \in \mathbb{R}$, this action preserves ∂H^2. The different types of elements in $\mathrm{PSL}_2(\mathbb{R})$ are characterized by their traces: an element T different to the identity is hyperbolic, parabolic or elliptic if and only if $(\mathrm{tr}T)^2 > 4$, $(\mathrm{tr}T)^2 = 4$, or $(\mathrm{tr}T)^2 < 4$, respectively. They are also characterized by the number of fixed points in $\widehat{H}^2$: hyperbolic elements fix two points in ∂H^2, parabolic elements fix one point in ∂H^2 and elliptic elements fix one point in H^2.

By a *circle of* $\widehat{\mathbb{C}}$ we will understand either an Euclidean circle in $\mathbb{C}$ or the union of an Euclidean line in $\mathbb{C}$ and the point ∞. The intersection of a circle of $\widehat{\mathbb{C}}$ with H^2 has always a meaning in hyperbolic geometry. Indeed, if the circle of $\widehat{\mathbb{C}}$ is fully contained in H^2, then it is also a hyperbolic circle; if it is contained in $\widehat{H}^2$ but tangent to ∂H^2 at a point p then its intersection with H^2 is called a *horosphere with centre* p; if the circle of $\widehat{\mathbb{C}}$ intersects ∂H^2 orthogonally, then its intersection with H^2 is a hyperbolic geodesic; finally, if the circle of $\widehat{\mathbb{C}}$ intersects ∂H^2 with any other angle, then its intersection with H^2 is an equidistant curve to the geodesic with the same endpoints.

Recall that the cross-ratio of four points $z_1, z_2, z_3, z_4 \in \mathbb{C}$ (which is the cross-ratio of these points in the complex projective line) is the quotient

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

The main tool that we will use to prove the results using the upper half-plane model is the well known fact: four different points in $\widehat{\mathbb{C}}$ are contained in a circle of $\widehat{\mathbb{C}}$ if and only if the cross-ratio of these points is real. This idea appears also in [1].

2.3. **Bisecting lines, Fuchsian groups and Dirichlet domains.**

Let T be an orientation-preserving isometry of $\mathbb{H}^2$ and p a point in $\mathbb{H}^2$ that is not fixed by T. Then, there is a hyperbolic bisecting line of the segment with endpoints $p, T(p)$, which we denote by $B(p, T)$. (We remark that the definition of $B(p, T)$ that we give here corresponds to $B(p; T^{-1})$ in [3].)

A *Fuchsian* group is a discrete subgroup of isometries of hyperbolic plane. Let Γ be a Fuchsian group and y a point in $\mathbb{H}^2$. The *Dirichlet fundamental polygon* centred at y is the set

$$\mathcal{P}_0(y) = \{x \in \mathbb{H}^2 \mid d(x,y) \leq d(x,T(y)), \quad \text{for any } T \in \Gamma\}$$

(where $d(\ ,\ )$ denotes the hyperbolic distance). If we denote by $B(y,T)^-$ the set of points $x \in \mathbb{H}^2$ such that $d(x,y) \leq d(x,T(y))$, then we have that $\mathcal{P}_0(y) = \bigcap_{T\in\Gamma} B(y,T)^-$, and so the vertices of $\mathcal{P}_0(y)$ are points of intersection of several bisecting lines $B(y,T_i)$. The family of polygons $\{T(\mathcal{P}_0(y)) \mid T \in \Gamma\}$ tessellates $\mathbb{H}^2$.

For simplicity, we restrict ourselves to the case that Γ is cocompact, so that the Dirichlet fundamental domain is a compact polygon. Then, we say that the Dirichlet fundamental polygon is *generic* if around any vertex of the polygon there are exactly three polygons in the above tessellation.

It is well known that if three hyperbolic isometries T_1, T_2, T_3 are elements of a Fuchsian group then they have some restrictions. For instance, if T_1, T_2 are hyperbolic and have a common fixed point, then they must have the same axis. Also, if T_1, T_2 have the same fixed points (in $\mathbb{H}^2$ or at infinity), then they must be powers of some other isometry. See Chapter 5 of [1] for example.

3. Main results using in the projective model.

The strategy in this model is as follows.

1.- First, given $T \in \mathrm{Isom}^+(\mathrm{D}^2)$, we consider the map that assigns to each point $x \in D^2$ its bisecting line with respect to T. This map extends to a projective map $\Psi_T \colon \mathbb{P}^2 \dashrightarrow (\mathbb{P}^2)^*$ (the arrow $\dashrightarrow$ indicates that the map may not be defined in the whole $\mathbb{P}^2$). We define this map and prove some properties in Propositions 3.1 and 3.3.

2.- Next, given $T_1, T_2 \in \mathrm{Isom}^+(\mathrm{D}^2)$, we consider the map $\Phi_{T_1T_2}$ that assigns to each point $x \in D^2$ the intersection point of its bisecting lines with respect to T_1, T_2. In SubSection 3.2, we define precisely the map $\Phi_{T_1T_2}$ and study carefully its properties, depending on T_1, T_2.

3.- Now, given $T_1, T_2, T_3 \in \mathrm{Isom}^+(\mathrm{D}^2)$, we have the maps $\Phi_{T_1T_2}$, $\Phi_{T_1T_3}$, $\Phi_{T_2T_3}$. If these maps are defined on a point x, then it is clear that two of these maps are equal at x if and only if the three of them are equal at x. Such a point x is "bad" in the sense that the three intersecting lines intersect. Then, imposing the condition that $\Phi_{T_1T_2} = \Phi_{T_1T_3}$ we obtain an algebraic subset $\mathcal{A}_{T_1T_2T_3}$ of $\mathbb{P}^2$. This set is proper if and only if the maps $\Phi_{T_1T_2}, \Phi_{T_1T_3}$ are different, and the main theorem (Theorem 3.8) gives necessary and sufficient conditions for it.

3.1. The maps Ψ_T and ψ_T.

Proposition 3.1. *Let T be an orientation-preserving hyperbolic isometry. Then there exists a projective map Ψ_T from $\mathbb{P}^2$ to its dual, $(\mathbb{P}^2)^*$, such that for any $p \in D^2$ with $T(p) \neq p$, we have $\Psi_T(p) = \tilde{B}(p, T)$. Moreover, if M is the matrix representing T, then a matrix representing Ψ_T is $F(Id - M)$, where Id is the identity matrix.*

Proof. We construct Ψ_T as follows. Take a point $p \in \mathbb{P}^2$ such that $T(p) \neq p$, so that they determine a line $\tilde{L}$. Let q be the pole of $\tilde{L}$ with respect to $\mathbb{S}^1$. On the other hand, denote by $\sqrt{T}$ the unique hyperbolic isometry such that $\sqrt{T} \circ \sqrt{T} = T$ (this isometry exists because T preserves orientation; moreover $\sqrt{T}$ also preserves orientation). Then we take $\Psi_T(p)$ to be the projective line going through q and containing the point $\sqrt{T}(p)$.

When p is a point in D^2, then the hyperbolic lines orthogonal to $\tilde{L} \cap D^2$ are those lines N such that $\tilde{N}$ goes through q. Then, by the construction, $\Psi_T(p)$ is orthogonal to the the line containing $p, T(p)$. Also, since $\sqrt{T}$ is a hyperbolic isometry, we have $d_D(p, \sqrt{T}(p)) = d_D(\sqrt{T}(p), T(p))$, and therefore $\sqrt{T}(p)$ is in the bisecting line of $p, T(p)$. Hence, $\Psi_T(p) = \tilde{B}(p, T)$.

To see that Ψ_T is a projective map, we compute its analytic expression. For this, let $(p_0 : p_1 : p_2)$ be the homogeneous coordinates of p, and let A be the matrix expression of $\sqrt{T}$, (then, $A^2 = M$). Let L_p and $L_{T(p)}$ be the polar lines of the points $p, T(p)$, respectively, with respect to $\mathbb{S}^1$. Their respective equations are

$$(x_0, x_1, x_2)F \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} = 0 \quad \text{and} \quad (x_0, x_1, x_2)FA^2 \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} = 0.$$

A line goes through the point q if and only if it is a line in the sheaf generated by L_p and $L_{T(p)}$. Since $\Psi_T(p)$ is the line on this sheaf containing $\sqrt{T}(p)$, then we look for $(\lambda : \mu)$ such that

$$(p_0, p_1, p_2)A^t\Big(\lambda F \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} + \mu F A^2 \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}\Big) = 0. \tag{3.1}$$

Notice that $A^tF = (FA)^t$, and $A^tFA^2 = (A^tFA)A = FA$, and therefore,the previous expression is equivalent to

$$(\lambda + \mu)\Big((p_0, p_1, p_2)FA \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}\Big) = 0.$$

Hence we can take $\lambda = -\mu = 1$, and then the line $\Psi_T(p)$ is

$$(x_0, x_1, x_2)F(Id - A^2) \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} = 0.$$

Therefore, a matrix representing Ψ_T is $F(Id - M)$. □

In Proposition 3.3 we will see that the projective map Ψ_T is never bijective. Namely, its image is contained in a line of $(\mathbb{P}^2)^*$ and, therefore, its domain is $\mathbb{P}^2$ minus one point. First, we study some geometric properties of Ψ_T.

Notation. Let T be an element in $\mathrm{Isom}^+\mathrm{D}^2$ different to the identity. We call *centre* of T, denoted by p_T, to the following point:

(i) if T is of hyperbolic type with axis $\mathrm{Ax}(T)$, then p_T is the pole of the line $\mathrm{Ax}(T)$ with respect to $\mathbb{S}^1$; notice that p_T is also a fixed point of T.

(ii) if T is elliptic or parabolic, then p_T is the (unique) fixed point of T in $\mathbb{P}^2$.

Denote by $\mathcal{F}_T$ the sheaf of lines going through p_T. Given two different points x, y, we denote by $\overline{x\,y}$ the line containing x, y. It will always be clear by the context if the line is hyperbolic or projective. In the next lemma we also use this notation for the segment with endpoints x, y.

Lemma 3.2. *Let T be an element in $\mathrm{Isom}^+(\mathrm{D}^2)$ and q any point in D^2 with $q \neq T(q)$. Then the line $\tilde{B}(q, T)$ contains the point p_T.*

Proof. If T is elliptic, then the result is clear, since $B(q, T(q))$ consists on the points in D^2 equidistant to q and $T(q)$, and the centre p_T of T is one of such points.

If T is hyperbolic and q is on $\mathrm{Ax}(T)$, then $T(q)$ is also in $\mathrm{Ax}(T)$ and hence $\tilde{B}(q, T(q))$ goes through $p_{_T}$. Now, take a point q not in $\mathrm{Ax}(T)$; then q and $T(q)$ lie on the same equidistant curve to $\mathrm{Ax}(T)$, so that if $m, n \in \mathrm{Ax}(T)$ are the orthogonal projections of $q, T(q)$ onto $\mathrm{Ax}(T)$, then the quadrilateral with vertices $q, m, n, T(q)$ has right angles at m, n and the sides $\overline{q\,m}$ and $\overline{T(q)\,n}$ have equal length. Now, consider the bisecting line B of the segment $\overline{m\,n}$. Since $m, n \in \mathrm{Ax}(T)$, then $p_{_T} \in \tilde{B}$. On the other hand, the reflection on this line preserves the quadrilateral. This implies that B is also the bisecting line of the segment $\overline{q\,T(q)}$.

If T is parabolic, there is a line B with $p_{_T} \in \tilde{B}$ and such that the reflection on B maps the line $\overline{p_{_T} q}$ onto $\overline{p_{_T} T(q)}$. Since $p_{_T} \in \tilde{B}$, this reflection also preserves the horospheres centred at $p_{_T}$. Now, q and $T(q)$ lie on the same of such horospheres; hence the reflection on B maps q on $T(q)$, and therefore B is the bisecting line of these two points. □

Proposition 3.3. *Let T be an element in* $\mathrm{Isom}^+\mathrm{D}^2$ *different to the identity. Then:*

(a) *The domain of the map Ψ_T is $\mathbb{P}^2 - \{p_{_T}\}$ and its image is the sheaf $\mathcal{F}_T$.*
(b) *The map $\psi_{_T}$ defined as $\psi_{_T}(\overline{p_{_T}\,q}) = \Psi_T(q)$ is a projective transformation of $\mathcal{F}_T$ whose fixed points are the lines in $\mathcal{F}_T$ tangent to $\mathbb{S}^1$; moreover, $\psi_{_T}$ is an involution if and only if T is elliptic of rotation angle π.*

Proof. (a) Notice that, if $p_1, p_2 \in \mathbb{P}^2$ are aligned with $p_{_T}$, then also $\sqrt{T}(p_1)$, $\sqrt{T}(p_2)$ are aligned with $\sqrt{T}(p_{_T}) = p_{_T}$. If, in addition, p_1, p_2 are points in D^2, then by Lemma 3.2, both $\tilde{B}(p_1, T)$ and $\tilde{B}(p_2, T)$ contain $p_{_T}$. Therefore, $\Psi_T(p_1) = \Psi_T(p_2)$. This argument also shows that the image of Ψ_T is contained in the sheaf $\mathcal{F}_T$, and it is easy to see that this image contains at least two lines in this sheaf.

This is enough to show that Ψ_T is not injective and therefore, being a projective map, its image is precisely the sheaf $\mathcal{F}_T$, and its domain is $\mathbb{P}^2$ minus one point. This point must be $p_{_T}$, since this is the only point where the geometric construction of Ψ_T fails.

(b) We can see the map ψ_T as the composition of the projective maps $\pi\colon \mathcal{F}_T \to \tilde{L}$ and $(\Psi_T)_{|_{\tilde{L}}}\colon \tilde{L} \to \mathcal{F}_T$, where $\tilde{L}$ is any line not containing p_T and π maps each line in $\mathcal{F}_T$ to its intersection with $\tilde{L}$. Therefore, ψ_T is a projective transformation. If T is hyperbolic, we can consider as $\tilde{L}$ the line containing the axis of T. Then, it is clear that the two tangent lines through p_T to $\mathbb{S}^1$ are fixed points of ψ_T and that ψ_T is not an involution. If T is parabolic, then, the tangent through p_T to $\mathbb{S}^1$ is the only fixed point of ψ_T, and hence ψ_T is not an involution. Finally, if T is elliptic of rotation angle α, then ψ_T rotates the lines through p_T with angle $\alpha/2$. Therefore, it does not have any fixed points, and it is an involution if and only if $\alpha = \pi$. □

3.2. **Intersection of bisecting lines.** Given two hyperbolic isometries, to any point x in D^2 we assign the intersection of the two bisecting lines $B(x, T_1), B(x, T_2)$, if any. We extend this assignment to get an algebraic map of $\mathbb{P}^2$.

Definition of $\Phi_{T_1T_2}$ and $\tilde{\Phi}_{T_1T_2}$. Given $T_1, T_2 \in \mathrm{Isom}^+(\mathrm{D}^2)$, we consider the map

$$\Phi_{T_1T_2}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad \Phi_{T_1T_2}(x) = \Psi_{T_1}(x) \cap \Psi_{T_2}(x),$$

whenever $\Psi_{T_i}(x), i = 1, 2$ are defined and $\Psi_{T_1}(x) \neq \Psi_{T_2}(x)$. Notice that, for $x \in D^2$, $\Phi_{T_1T_2}(x) = \tilde{B}(x, T_1) \cap \tilde{B}(x, T_2)$, whenever these two lines exist and are different.

Let M_1, M_2 be the matrices representing T_1, T_2. Then, using Proposition 3.1, we see that $\Phi_{T_1T_2}(x_0 : x_1 : x_2)$ is the cross product of the vectors $F(Id - M_1)\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$ and $F(Id - M_2)\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$. Thus,

$$\Phi_{T_1T_2}(x_0 : x_1 : x_2) = (Q_0(x_0, x_1, x_2) : Q_1(x_0, x_1, x_2) : Q_2(x_0, x_1, x_2)),$$

where Q_0, Q_1, Q_2 are homogeneous polynomials of degree two. If T_1, T_2 are distinct and different to the identity, then $\Psi_{T_1} \neq \Psi_{T_2}$, and hence the map $\Phi_{T_1T_2}$ is defined on at least one point, which means that the three polynomials are not identically zero at the same time. The codomain of $\Phi_{T_1T_2}$ (complement of the domain) is the set $Q_0 = Q_1 = Q_2 = 0$. It may happen that Q_0, Q_1, Q_2 have a common factor R. Dividing by R, we obtain a map $\tilde{\Phi}_{T_1T_2}$ that extends continuously $\Phi_{T_1T_2}$ to the points of $R = 0$, which is a proper algebraic set. If R is also a quadratic polynomial, then $\tilde{\Phi}_{T_1T_2}$ will be a constant map, while if R is a linear polynomial, then $\tilde{\Phi}_{T_1T_2}$ is a projective map. All the cases will appear (Lemmas 3.5, 3.6, 3.7). Thus, the codomain of $\Phi_{T_1T_2}$ is the codomain of $\tilde{\Phi}_{T_1T_2}$ union the subset $R = 0$.

Definition. Let $T_1, T_2, T_3 \in \mathrm{Isom}^+(\mathrm{D}^2)$ distinct and different to the identity. We denote by $\mathcal{A}_{T_1T_2T_3}$ the subset

$$\begin{aligned}\mathcal{A}_{T_1T_2T_3} &= \{x \in \mathbb{P}^2 \mid \Phi_{T_1T_2}(x) = \Phi_{T_1T_3}(x)\} \cup \mathrm{codom}(\Phi_{T_1T_2}) \\ &\cup \ \mathrm{codom}(\Phi_{T_1T_3}),\end{aligned}$$

where we denote by $\mathrm{codom}(\Phi_{T_iT_j})$ the codomain of $\Phi_{T_iT_j}$.

Lemma 3.4. *The set $\mathcal{A}_{T_1T_2T_3}$ is an algebraic subset of $\mathbb{P}^2$.*

Proof. The map $\Phi_{T_1T_2}$ is the quadratic map $\Phi_{T_1T_2}(x_0 : x_1 : x_2) = (Q_0 : Q_1 : Q_2)$, where Q_0, Q_1, Q_2 are homogeneous quadratic polynomials. Similarly, $\Phi_{T_1T_3}(x_0 : x_1 : x_2) = (Q'_0 : Q'_1 : Q'_2)$. Now, consider the algebraic subset $\mathcal{S} \subset \mathbb{P}^2$ given by the equations $Q_0Q'_1 - Q_1Q'_0 = Q_0Q'_2 - Q_2Q'_0 = Q_1Q'_2 - Q_2Q'_1 = 0$. A point $x = (x_0 : x_1 : x_2)$ is in $\mathcal{S}$ if and only if either $Q_0(x) = Q_1(x) = Q_2(x) = 0$, or $Q'_0(x) = Q'_1(x) = Q'_2(x) = 0$, or $\Phi_{T_1T_2}(x) = \Phi_{T_1T_3}(x)$. Thus, $\mathcal{S} = \mathcal{A}_{T_1T_2T_3}$, and so that $\mathcal{A}_{T_1T_2T_3}$ is an algebraic set. □

We will first study in detail how are the maps $\Phi_{T_1T_2}$ and $\tilde{\Phi}_{T_1T_2}$ for the different cases of T_1, T_2 (Lemmas 3.5, 3.6, 3.7). Next, we will study necessary and sufficient conditions on three hyperbolic isometries T_1, T_2, T_3 so that the maps $\Phi_{T_1T_2}$ and $\Phi_{T_1T_3}$ be equal (Theorem 3.8). From these conditions we readily find conditions for the properness of the set $\mathcal{A}_{T_1T_2T_3}$ (Corollary 3.9).

For the study of the map $\Phi_{T_1T_2}$, we distinguish the cases: (i) the centre p_{T_1} of T_1 is equal to the centre p_{T_2} of T_2; (ii) both centres are different and the line containing them is not tangent to $\mathbb{S}^1$; and (iii) both centres are different and the line containing them is tangent to $\mathbb{S}^1$.

Lemma 3.5. *Let T_1, T_2 different elements of* $\mathrm{Isom}^+(\mathrm{D}^2)$ *and different to the identity and suppose that $p_{T_1} = p_{T_2} = p$. Then $\tilde{\Phi}_{T_1T_2}$ is the map constantly equal to p.*

Proof. If $p_{T_1} = p_{T_2} = p$, then all the lines in the image of Ψ_{T_1} and Ψ_{T_2} contain p. Thus, for x with $\Psi_{T_1}(x) \neq \Psi_{T_2}(x)$, we have that $\Phi_{T_1T_2}(x) = p$. Since $T_1 \neq T_2$, the set $\{x \mid \Psi_{T_1}(x) \neq \Psi_{T_2}(x)\}$ is dense, and therefore, $\tilde{\Phi}_{T_1T_2}$ is the constant map with value p. Alternatively, one can check the previous result directly by computation. If we do so, we see that $\Phi_{T_1T_2}$ is a quadratic map $\Phi_{T_1T_2} = (Q_0 : Q_1 : Q_2)$ with $Q_i = p_iQ'$, where Q' is also a quadratic polynomial and $p = (p_0 : p_1 : p_2)$. Moreover, we can see that the quadric $Q' = 0$ is: the union of the two tangent lines to $\mathbb{S}^1$ through p, if the T_i are hyperbolic or parabolic (if parabolic, then both lines are equal); or imaginary with the only real point p if the T_i are elliptic. This quadric is the codomain of $\Phi_{T_1T_2}$. □

Lemma 3.6. *Let $T_1, T_2 \in \mathrm{Isom}^+(\mathrm{D}^2)$ be different to the identity, such that $p_{T_1} \neq p_{T_2}$ and the line L containing p_{T_1}, p_{T_2} is not tangent to $\mathbb{S}^1$. Then the domain of $\Phi_{T_1T_2}$ is $\mathbb{P}^2 - \{p_{T_1}, p_{T_2}, q\}$, where q is a point not in L. Moreover, $\Phi_{T_1T_2}$ does not extend continuously to $\{p_{T_1}, p_{T_2}, q\}$, so in this case $\Phi_{T_1T_2} = \tilde{\Phi}_{T_1T_2}$.*

Proof. A point $x \neq p_{T_1}, p_{T_2}$ is not in the domain of $\Phi_{T_1T_2}$ if and only if $\psi_{T_1}(\overline{p_{T_1}x}) = L = \psi_{T_2}(\overline{p_{T_2}x})$. Since L is not tangent to $\mathbb{S}^1$, by Proposition 3.3, L is not a fixed point of ψ_{T_1} nor of ψ_{T_2}. Therefore $\psi_{T_1}^{-1}(L) \cap \psi_{T_2}^{-1}(L)$ is a point q not in L, and hence the domain of $\Phi_{T_1T_2}$ is $\mathbb{P}^2 - \{p_{T_1}, p_{T_2}, q\}$.

By the same reason as before, $\psi_{T_1}(L) \cap \psi_{T_2}(L)$ is a point q', which is not in L. Now, we have: if $x \in \overline{p_{T_1}q}$, $x \neq p_{T_1}, q$, then $\Phi_{T_1T_2}(x) = p_{T_2}$; if $x \in \overline{p_{T_2}q}$, $x \neq p_{T_2}, q$, then $\Phi_{T_1T_2}(x) = p_{T_1}$; if $x \in L = \overline{p_{T_1}p_{T_2}}$, $x \neq p_{T_1}, p_{T_2}$, then $\Phi_{T_1T_2}(x) = q'$. This shows that $\Phi_{T_1T_2}$ can not be extended continuously to none of p_{T_1}, p_{T_2}, q. (We remark that the map $\Phi_{T_1T_2}$ is, up to projective transformation, an example of a *Cremona* transformation, which is a birrational transformation of $\mathbb{P}^2$ that is injective in $\mathbb{P}^2$ minus the three lines $\overline{p_{T_1}p_{T_2}}$, $\overline{p_{T_1}q}$, $\overline{qp_{T_2}}$). □

Lemma 3.7. *Let $T_1, T_2 \in \mathrm{Isom}^+(\mathrm{D}^2)$ be different to the identity, such that $p_{T_1} \neq p_{T_2}$ and the line L containing p_{T_1}, p_{T_2} is tangent to $\mathbb{S}^1$. Then the map $\Phi_{T_1T_2}$ extends continuously to a projective transformation $\tilde{\Phi}_{T_1T_2}$ of $\mathbb{P}^2$. We have two cases:*

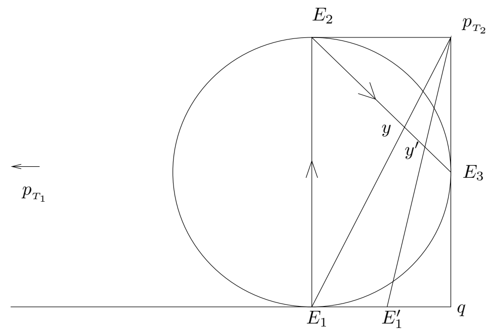

FIGURE 1

CASE 1. *T_1, T_2 are isometries of hyperbolic type whose axes share one point at infinity. Let $\overrightarrow{E_1E_2}$ be the oriented axis of T_1 and $\overrightarrow{E_2E_3}$ the oriented axis of T_2; denote by q the pole of the line $\overline{E_1E_3}$; finally, let $r, s \in \mathbb{R}$ be the signed translation distance of T_1, T_2, respectively (i.e., $r > 0$ if the sense of the translation agrees with the orientation of $\overrightarrow{E_1E_2}$, and similarly for s). Then*

1.1. *If $r = -s$, then the fixed points of $\tilde{\Phi}_{T_1T_2}$ are q and all the points of the line $\overline{p_{T_1}p_{T_2}}$.*
1.2. *If $r \neq -s$, then $\tilde{\Phi}_{T_1T_2}$ has exactly three fixed points, p_{T_1}, p_{T_2}, q; moreover, we have that $[p_{T_2}, q, x_1, \tilde{\Phi}_{T_1T_2}(x_1)] = e^r$, $[q, p_{T_1}, x_2, \tilde{\Phi}_{T_1T_2}(x_2)] = e^s$ and $[p_{T_1}, p_{T_2}, x_3, \tilde{\Phi}_{T_1T_2}(x_3)] = e^{-r-s}$, for any $x_1 \in \overline{p_{T_2}q}$, $x_2 \in \overline{q\,p_{T_1}}$, $x_3 \in \overline{p_{T_1}p_{T_2}}$.*

CASE 2. *T_1 is hyperbolic with oriented axis $\overrightarrow{E_1E_2}$ and T_2 is parabolic with fixed point E_2. Then: p_{T_1} and E_2 are the only fixed points of $\tilde{\Phi}_{T_1T_2}$ and $\overline{p_{T_1}E_1}, \overline{p_{T_1}E_2}$ are the two invariant lines.*

Proof. CASE 1. We conjugate T_1, T_2 by an element of $\mathrm{Isom}^+(\mathrm{D}^2)$ so that the axis of T_1 has endpoints $E_1 = (1 : 0 : -1), E_2 = (1 : 0 : 1)$ and the axis of T_2 has endpoints E_2 and $E_3 = (1 : 1 : 0)$ (see Figure 1). Then, $p_{T_1} = (0 : 1 : 0)$, $p_{T_2} = (1 : 1 : 1)$, and $q = (1 : 1 : -1)$, and the matrices M_1, M_2 representing T_1, T_2 are, respectively,

$$\begin{pmatrix} \cosh r & 0 & \sinh r \\ 0 & 1 & 0 \\ \sinh r & 0 & \cosh r \end{pmatrix},$$

$$\begin{pmatrix} 2\cosh s - 1 & 1 - \cosh s - \sinh s & 1 - \cosh s + \sinh s \\ -1 + \cosh s - \sinh s & 1 & 1 - \cosh s + \sinh s \\ -1 + \cosh s + \sinh s & 1 - \cosh s - \sinh s & 1 \end{pmatrix}.$$

Note that $r > 0$ means that E_2 is the attracting fixed point of T_1 and $s > 0$ means that E_3 is the attracting fixed point of T_3.

Now, the cross product of the vectors $F(Id - M_1)\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$ and $F(Id - M_2)\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$ are quadratic polynomials with the common factor $x_0 - x_2$. Dividing by this factor, we obtain the projective map $\tilde{\Phi}_{T_1T_2}$ represented by the matrix

$$M = \begin{pmatrix} A(s)\sinh r & 0 & A(s)B(r) \\ C(r,s) & (\cosh s + \sinh s - 1)A(s) & A(s)B(r) \\ A(s)B(r) & 0 & A(s)\sinh r \end{pmatrix},$$

where $A(s) = \cosh s - \sinh s - 1$, $B(r) = \cosh r - 1$ and $C(r,s) = (-\cosh s - \sinh s + 1)(\cosh r - 1) + 2\sinh r(\cosh s - 1)$.

The determinant of M is equal to

$$\begin{aligned} \det M &= (\cosh s - \sinh s - 1)^3(\cosh s + \sinh s - 1) \\ &\quad (\cosh r + \sinh r - 1)(-\cosh r + \sinh r + 1), \end{aligned}$$

which is equal to zero if and only if $r = 0$ or $s = 0$. Since $T_1, T_2 \neq 0$, then $r, s \neq 0$, and hence $\tilde{\Phi}_{T_1T_2}$ is a projective transformation. Now we study the properties of this transformation. Notice that the case 1.1 is a particular case of 1.2, since the cross-ratio $[p_{T_1}, p_{T_2}, x_3, \tilde{\Phi}_{T_1T_2}(x_3)] = 1$ means that $\tilde{\Phi}_{T_1T_2}(x_3) = x_3$. By direct computation, we check that the points p_{T_1}, p_{T_2}, q are fixed by $\tilde{\Phi}_{T_1T_2}$. Therefore the lines $\overline{q\,p_{T_1}}, \overline{q\,p_{T_2}}$ are invariant and the cross-ratio $[q, p_{T_1}, x, \tilde{\Phi}_{T_1T_2}(x)]$ is a fixed value for any $x \in \overline{q\,p_{T_1}}$. We compute this value for $x = E_1$. Let $y = \overline{p_{T_2}E_1} \cap \overline{E_2E_3}$, and $y' = \sqrt{T_2}(y)$, that is, $y' \in \overline{E_2E_3}$ and $d_D(y,y') = \frac{1}{2}\log[E_3,E_2,y,y'] = s/2$. Then

$$\tilde{\Phi}_{T_1T_2}(E_1) = \Phi_{T_1T_2}(E_1) = \Psi_{T_1}(E_1) \cap \Psi_{T_2}(E_1) = \overline{E_1 p_{T_1}} \cap \overline{p_{T_2} y'}.$$

Hence, projecting from p_{T_2} onto $\overline{E_2E_3}$, we have

$$[q, p_{T_1}, E_1, \tilde{\Phi}_{T_1T_2}(E_1)] = [E_3, E_2, y, y'] = e^s.$$

In similar way (and taking care of the orientations) we have that $[p_{T_2}, q, E_3, \tilde{\Phi}_{T_1T_2}(E_3)] = e^r$. The previous information completely determines the projective map $\tilde{\Phi}_{T_1T_2}$. In particular, it is easy to check that $[p_{T_1}, p_{T_2}, x_3, \tilde{\Phi}_{T_1T_2}(x_3)] = e^{-r-s}$, for any $x_3 \in \overline{p_{T_1}p_{T_2}}$.

Case 2. We conjugate T_1, T_2 by an element of $\mathrm{Isom}^+(\mathrm{D}^2)$ so that the axis of T_1 has endpoints $E_1 = (0,-1), E_2 = (0,1)$ and the fixed point of T_2 is E_2. The matrix representing T_2 is now:

$$M_2 = \begin{pmatrix} 1+\frac{t^2}{2} & -t & -\frac{t^2}{2} \\ -t & 1 & t \\ \frac{t^2}{2} & -t & 1-\frac{t^2}{2} \end{pmatrix},$$

with $t \in \mathbb{R}$. Now, the cross product of the vectors $F(Id-M_1)\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$ and $F(Id-M_2)\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$ are quadratic polynomials with the common factor $-t(x_0-x_2)$. Dividing by this factor, we obtain the projective map $\tilde{\Phi}_{T_1T_2}$ represented by the matrix

$$M = \begin{pmatrix} -\sinh r & 0 & -\cosh r + 1 \\ -\frac{t}{2}D(r) & D(r) & \frac{t}{2}D(r) \\ -\cosh r + 1 & 0 & -\sinh r \end{pmatrix},$$

with $D(r) = \cosh r - \sinh r - 1$. The determinant of M is equal to

$$-(\cosh r - \sinh r - 1)^2(\cosh r + \sinh r - 1),$$

which is never zero, since $T_1 \neq Id$. Therefore, $\tilde{\Phi}_{T_1T_2}$ is a projective transformation. Its fixed points and invariant lines can be computed directly using M. □

Using the previous lemmas, we now prove the following theorem.

Theorem 3.8. *Let* T_1, T_2, T_3 *be distinct elements of* $\mathrm{Isom}^+(\mathrm{D}^2)$ *and different to the identity. Then,* $\Phi_{T_1T_2} = \Phi_{T_1T_3}$ *if and only if one of the following happens:*

(i) *The three of* T_1, T_2, T_3 *are either hyperbolic with the same axis, or parabolic with the same fixed point or elliptic with the same fixed point.*

(ii) T_1, T_2, T_3 *are of hyperbolic type with oriented axes* A_1, A_2, A_3 *and translation distances* r_1, r_2, r_3*, respectively, such that* A_1, A_2, A_3 *form an ideal triangle cyclically oriented and* $r_1 + r_2 + r_3 = 0$.

Proof. Notice that $\Phi_{T_1T_2} = \Phi_{T_1T_3}$ (i.e., these maps are equal in the intersection of their domains) if and only if $\tilde{\Phi}_{T_1T_2} = \tilde{\Phi}_{T_1T_3}$. We will use the latter condition.

Consider first the case such that $p_{T_1} = p_{T_2} = p$. By Lemma 3.5, $\tilde{\Phi}_{T_1T_2}$ is constantly equal to p. On the other hand, the previous lemmas also imply that $\tilde{\Phi}_{T_1T_3}$ is constantly equal to p if and only if $p_{T_3} = p$. Therefore, T_1, T_2, T_3 are as in (i).

Secondly, consider the case such that the three points $p_{T_1}, p_{T_2}, p_{T_3}$ are different and that $\overline{p_{T_1} p_{T_2}}$ is not tangent to $\mathbb{S}^1$. By Lemma 3.6, the domain of $\tilde{\Phi}_{T_1T_2}$ is $\mathbb{P}^2 - \{p_{T_1}, p_{T_2}, q\}$, with $q \notin \overline{p_{T_1} p_{T_2}}$. Suppose that $\tilde{\Phi}_{T_1T_2} = \tilde{\Phi}_{T_1T_3}$. Since these maps do not extend continuously to their codomains, then their domains must be equal, and so necessarily $p_{T_3} = q$. Also, in the proof of that lemma, we saw that $\psi_{T_1}(\overline{p_{T_1}\, q}) = \overline{p_{T_1} p_{T_2}}$ and $\psi_{T_2}(\overline{p_{T_2}\, q}) = \overline{p_{T_1} p_{T_2}}$. Applying the same to the maps $\Phi_{T_1T_3}$ and $\Phi_{T_2T_3}$, we have that $\psi_{T_i}(\overline{p_{T_i} p_{T_j}}) = \overline{p_{T_i} p_{T_k}}$ for any permutation (i, j, k) of $(1, 2, 3)$. Then, the projective transformation ψ_{T_i} interchanges two points, and hence it is an involution. By Proposition 3.3, T_1, T_2, T_3 are elliptic elements of rotation angle π. Then the points $p_{T_1}, p_{T_2}, p_{T_3}$ are the vertices of a hyperbolic triangle with the three angles equal to $\pi/2$, which is impossible, and therefore $\tilde{\Phi}_{T_1T_2}, \tilde{\Phi}_{T_1T_3}$ are never equal in this case.

Finally, consider the case where $p_{T_1}, p_{T_2}, p_{T_3}$ are different and the three lines $\overline{p_{T_i} p_{T_j}}$ are tangent to $\mathbb{S}^1$. If T_2 is parabolic, then T_1, T_3 are hyperbolic, and $\tilde{\Phi}_{T_1T_2}$ and $\tilde{\Phi}_{T_2T_3}$ have exactly two fixed points (by Lemma 3.7). But $\tilde{\Phi}_{T_1T_3}$ has three fixed points, so $\tilde{\Phi}_{T_1T_2} \neq \tilde{\Phi}_{T_1T_3}$.

Hence, suppose that T_1, T_2, T_3 are hyperbolic, so that T_1, T_2 and T_1, T_3 are as in Case 1 of Lemma 3.7. Then $\tilde{\Phi}_{T_1T_2}$ and $\tilde{\Phi}_{T_1T_3}$ are the projective transformations described in that lemma. Looking carefully to the properties of these maps, we conclude that the only possibility for these maps to be equal is the condition described in (ii). □

Corollary 3.9. *Let T_1, T_2, T_3 be distinct elements of* $\mathrm{Isom}^+(\mathrm{D}^2)$ *and different to the identity, and not satisfying conditions (i) or (ii) in Theorem 3.8. Then the algebraic set $\mathcal{A}_{T_1T_2T_3}$ is proper. As a consequence, $\mathcal{A}_{T_1T_2T_3}$ is a closed subset of $\mathbb{P}^2$ with empty interior. Its complement,*

$$\mathcal{A}^c_{T_1T_2T_3} = \{x \in \mathbb{P}^2 \mid \Phi_{T_1T_2}(x) \neq \Phi_{T_1T_3}(x)\},$$

is an open dense subset of $\mathbb{P}^2$.

Proof. The hypothesis and Theorem 3.8 imply that $\Phi_{T_1T_2} \neq \Phi_{T_1T_3}$, that is, there is a point x in the domain of both maps such that $\Phi_{T_1T_2}(x) \neq \Phi_{T_1T_3}(x)$. Therefore, $x \in \mathbb{P}^2 - \mathcal{A}_{T_1T_2T_3}$, so that the subset $\mathcal{A}_{T_1T_2T_3}$ is proper. □

3.3. **Application to Fuchsian groups.** From the above results, we obtain consequences for Fuchsian groups.

Corollary 3.10. *Let Γ be a Fuchsian group without elliptic elements and let T_1, T_2, T_3 be distinct elements of a Γ and different to the identity. Then the set*

$$\mathcal{G}^D_{T_1T_2T_3} = \{x \in D^2 \mid B(x, T_1) \cap B(x, T_2) \cap B(x, T_3) = \emptyset\}$$

contains an open dense subset of D^2.

Proof. Suppose that T_1, T_2, T_3 are parabolic with the same centre (which, in this case, is equal to the common fixed point) or hyperbolic with the same centre, that we denote by p. Note that $p \notin D^2$. By Proposition 3.8, we have that $\tilde{\Phi}_{T_1T_2} = \tilde{\Phi}_{T_1T_3}$ is the map constantly equal to p. Then, for any $x \in D^2$, we have that $\tilde{B}(x, T_1) \cap \tilde{B}(x, T_2) \cap \tilde{B}(x, T_3) = \{p\}$, and so $B(x, T_1) \cap B(x, T_2) \cap B(x, T_3) = \emptyset$. Hence, in this case $\mathcal{G}_{T_1T_2T_3} = D^2$.

Suppose that T_1, T_2, T_3 is not as in the previous case. Since they are elements of a Fuchsian group without elliptic elements, then T_1, T_2, T_3 do not satisfy the conditions (i) or (ii) in Theorem 3.8. By Corollary 3.9, the subset $\mathcal{A}_{T_1T_2T_3}$ is closed with empty interior.

Now,

$$\begin{aligned} \mathcal{G}^D_{T_1T_2T_3} &\supset \{x \in D^2 \mid \tilde{B}(x,T_1) \cap \tilde{B}(x,T_2) \cap \tilde{B}(x,T_3) = \emptyset\} \\ &= D^2 \cap \{x \in \mathbb{P}^2 \mid \Phi_{T_1T_2}(x) \neq \Phi_{T_1T_3}(x)\} \\ &= D^2 \cap \mathcal{A}^c_{T_1T_2T_3}. \end{aligned}$$

Therefore, $\mathcal{G}^D_{T_1T_2T_3}$ contains a dense open subset of D^2. □

Corollary 3.11. *Let* Γ *be a finitely generated cocompact Fuchsian group without elliptic elements. Then, the set* $\{x \in D^2 \mid \mathcal{P}_0(x)$ *is generic*$\}$ *is dense.*

Proof. Though the proof of this corollary follows exactly the same argument as Theorem 4.6 in ([3]), we include it here for the sake of this paper to be self-contained. The Dirichlet fundamental polygon $\mathcal{P}_0(x)$ centred at a point x is not generic if and only if there is a vertex v of this polygon incident to, at least, four polygons in the tessellation. This is equivalent to say that there are (at least) three elements $T_1, T_2, T_3 \in \Gamma$ such that $v \in B(x,T_1) \cap B(x,T_2) \cap B(x,T_3)$ and therefore $x \notin \mathcal{G}^D_{T_1T_2T_3}$. As a consequence, all the points in $\cap \mathcal{G}^D_{T_1T_2T_3}$ are centres of generic fundamental polygons, where the intersection is taken over all the triples $T_1, T_2, T_3 \in \Gamma$. By Corollary 3.10, these sets contain open and dense subsets, and, since Γ is finitely generated, there are countably many of those triples. Since D^2 is a Baire space, we have that the previous intersection is dense. □

4. Proof using the upper half-plane model.

4.1. **The algebraic subset $F_{T_1T_2T_3} = 0$.** Let $T_1, T_2, T_3 \in \mathrm{PSL}_2(\mathbb{R})$ and define the set of "good points" for T_1, T_2, T_3 as the set

$$\mathcal{G}^H_{T_1T_2T_3} = \{x \in H^2 \mid B(x,T_1) \cap B(x,T_2) \cap B(x,T_3) = \emptyset\}.$$

Now we have:

$$\begin{aligned} H^2 - \mathcal{G}^H_{T_1T_2T_3} &= \{z \in H^2 \mid B(z,T_1) \cap B(z,T_2) \cap B(z,T_3) \neq \emptyset\} \\ &= \{z \in H^2 \mid z, T_1(z), T_2(z), T_3(z) \text{ are contained in a hyperbolic circle }\} \\ &\subset \{z \in H^2 \mid z, T_1(z), T_2(z), T_3(z) \text{ are contained in a circle of } \widehat{\mathbb{C}}\} \qquad (4.1) \\ &= H^2 \cap \{z \in \mathbb{C} \mid z, T_1(z), T_2(z), T_3(z) \text{ are contained in a circle of } \widehat{\mathbb{C}}\}. \end{aligned}$$

We denote by $\mathcal{B}^H_{T_1T_2T_3}$ the set of "bad points" $\{z \in \mathbb{C} \mid z, T_1(z), T_2(z), T_3(z)$ are contained in a circle of $\widehat{\mathbb{C}}\}$. We show next that this set is the solution set of the polynomial $F_{T_1T_2T_3}$ obtained from imposing the condition that the cross-ratio $[z, T_1(z), T_2(z), T_3(z)]$ is real. Precisely, let T_i be $T_i(z) = \frac{a_iz+b_i}{c_iz+d_i}$. The above cross-ratio is a quotient N/D of fractions; consider

$$Q_1(z) = N\cdot\overline{D} = \big(z - T_2(z)\big)\big(T_1(z) - T_3(z)\big)\overline{\big(z - T_3(z)\big)\big(T_1(z) - T_2(z)\big)}$$

Still Q_1 is a quotient N_1/D_1 of polynomials, where

$$D_1 = (c_1z + d_1)(c_2z + d_2)(c_3z + d_3)(c_1\bar{z} + d_1)(c_2\bar{z} + d_2)(c_3\bar{z} + d_3)$$

Consider $Q_2 = N_1 \cdot \overline{D_1}$. Now Q_2 is a polynomial in $z, \bar{z}$ with real coefficients. Calling $x = \Re(z), y = \Im(z)$, then $Q_2 = A(x, y) + iB(x, y)$, with A, B polynomials with real coefficients. We define $F_{T_1T_2T_3}(x, y)$ to be $B(x, y)$. Notice that, since $Q_2(\bar{z}) = \overline{Q_2(z)}$, we have that $F_{T_1T_2T_3}(x, -y) = -F_{T_1T_2T_3}(x, y)$; also, since T_i have real coefficients, then it is $F_{T_1T_2T_3}(x, 0) = 0$.

Then we have the following proposition.

Proposition 4.1. *Let T_1, T_2, T_3 be distinct elements of* $\mathrm{PSL}_2(\mathbb{R})$ *and different to the identity. Then the solution set of the polynomial $F_{T_1T_2T_3}(x, y)$ is the set $\mathcal{B}^H_{T_1T_2T_3}$.*

Proof. Let $z = x + iy \in \mathbb{C}$. Suppose first that $z, T_1(z), T_2(z), T_3(z)$ are all different and $z \neq -d_i/c_i$ for $i = 1, 2, 3$. Then, the fact that these four points are contained in a circle of $\widehat{\mathbb{C}}$ is equivalent to $[z, T_1(z), T_2(z), T_3(z)]$ being real, and this is equivalent to $F_{T_1T_2T_3}(x, y) = 0$.

If z is such that some of the points $z, T_1(z), T_2(z), T_3(z)$ are equal, then $z \in \mathcal{B}^H_{T_1T_2T_3}$ and also $F_{T_1T_2T_3}(x, y) = 0$, since $Q_1(z) = 0$.

Finally, suppose that $z = -d_i/c_i \in \mathbb{R}$ for some $i = 1, 2, 3$. In this case, since T_i have real coefficients, $T_i(z)$ is also real and hence the four points are on a circle of $\widehat{\mathbb{C}}$, hence $z \in \mathcal{B}^H_{T_1T_2T_3}$. On the other hand, $Q_2(z) = 0$, and therefore $F_{T_1T_2T_3}(x, y) = 0$. □

We remark that, if the T_i do not have real coefficients, then the solution set of $F_{T_1T_2T_3}$ is $\mathcal{B}^H_{T_1T_2T_3} \cup \{p_1, p_2, p_3, \bar{p}_1, \bar{p}_2, \bar{p}_3\}$, where $p_i = -d_i/c_i$.

Lemma 4.2. *Let T_1, T_2, T_3 be distinct elements of* $\mathrm{PSL}_2(\mathbb{R})$ *and different to the identity. Then $F_{T_1T_2T_3}(x,y)$ is not proper if and only if $H^2 \subset \mathcal{B}^H_{T_1T_2T_3}$.*

Proof. If $F_{T_1T_2T_3}(x,y)$ is not proper, then its solution set is the whole $\mathbb{C}$ and hence $H^2 \subset \mathcal{B}^H_{T_1T_2T_3}$.

Conversely, suppose $H^2 \subset \mathcal{B}^H_{T_1T_2T_3}$. Then, if $z = x + iy$ with $y < 0$, then $x - iy \in H^2$ and $F_{T_1T_2T_3}(x,y) = -F_{T_1T_2T_3}(x,-y) = 0$. Because we also have that $F_{T_1T_2T_3}(x,0) = 0$, then any point in $\mathbb{R}^2$ is solution of $F_{T_1T_2T_3} = 0$, and hence this polynomial is not proper. □

Using the previous lemma, the properness of the polynomial $F_{T_1T_2T_3}$ is now equivalent to finding a point $z \in H^2$ so that the four points $z, T_1(z), T_2(z), T_3(z)$ are not on a circle of $\widehat{\mathbb{C}}$. In the next theorem we characterize when the polynomial $F_{T_1T_2T_3}$ is proper. We prove it in Section 4.3.

Theorem 4.3. *Let T_1, T_2, T_3 be distinct and non-trivial elements in* $\mathrm{PSL}_2(\mathbb{R})$*. Then the following two conditions are equivalent:*

(1) *For any z in H^2, the four points $z, T_1(z), T_2(z), T_3(z) \in H^2$ are on a circle of $\widehat{\mathbb{C}}$ (i.e., $F_{T_1T_2T_3}(x,y)$ is not proper)*

(2) *The three elements T_1, T_2, T_3 satisfy one of the following conditions:*

   (a) *All of T_1, T_2, T_3 are of the same type (i.e., elliptic, parabolic or hyperbolic) with the same fixed point set.*

   (b) *Up to conjugation by an element of* $\mathrm{PSL}_2(\mathbb{R})$*, $T_1(z) = a\,z$, $T_2(z) = b\,z + 1 - b$, $T_3(z) = \dfrac{a\,z}{(a-b)\,z + b}$, where $a, b \in \mathbb{R} - \{0,1\}$. In this case T_1 fixes 0 and ∞, T_2 fixes 1 and ∞, and T_3 fixes 0 and 1. Moreover, if we orientate the three axes from 0 to ∞, from ∞ to 1 and from 1 to 0, respectively, and if r_1, r_2, r_3 are the signed translation distances, then $r_1 + r_2 + r_3 = 0$.*

4.2. **Applications to Fuchsian groups.** We now prove Corollary 4.4, which is the corresponding version to Corollary 3.10 in the upper half-plane model. From this, the corresponding version to Corollary 3.11 is proved in exactly the same way.

Corollary 4.4. *Let Γ be a Fuchsian group without elliptic elements and let T_1, T_2, T_3 be distinct elements of a Γ and different to the identity. Then the set $\mathcal{G}^H_{T_1T_2T_3}$ contains an open dense subset of H^2.*

Proof. We first notice that case (2) in Theorem 4.3 never happens in a Fuchsian group.

Suppose T_1, T_2, T_3 are parabolic elements with the same fixed point p. Then, all bisecting lines $B(z, T_i)$ for any $z \in H^2$ are parallel hyperbolic lines, with p as an endpoint. Then, for any $z \in H^2$, $B(z, T_1) \cap B(z, T_2) \cap B(z, T_3) = \emptyset$, and we have that $\mathcal{G}^H_{T_1T_2T_3} = H^2$. If T_1, T_2, T_3 are hyperbolic with the same fixed points, then any two different bisecting lines are ultraparallel. Hence we again have that $\mathcal{G}^H_{T_1T_2T_3} = H^2$. Finally, if T_1, T_2, T_3 are not as before, then by Theorem 4.3 we have $\{F_{T_1T_2T_3} = 0\}$ is a proper algebraic subset of $\mathbb{C}$, then it is closed and has empty interior. By (4.1), we have that $\mathcal{G}^H_{T_1T_2T_3}$ contains an open dense subset of H^2. □

4.3. **Properness of the set $F_{T_1T_2T_3} = 0$: proof of Theorem4.3.** We start the proof by classifying all the possibilities for T_1, T_2, T_3 with respect to their fixed points. For $T \in \mathrm{PSL}_2(\mathbb{R})$, let Fix(T) be the set of its fixed points in $\widehat{H^2}$. Let $F = \{T_1, T_2, T_3\}$. Then the situation can be separated into the following two cases:

- (i) There is a point in $\cup_{i=1}^{3}\mathrm{Fix(T_i)}$ that is fixed by exactly one element in F.
- (i') Not (i), i.e., any point in $\cup_{i=1}^{3}\mathrm{Fix(T_i)}$ is fixed by at least two elements in F.

Let us assume that F satisfies Case (i'). We have several possibilities. Suppose first that there is an elliptic element in F; in this case, (i') forces that $\mathrm{Fix(T_1)} = \mathrm{Fix(T_2)} = \mathrm{Fix(T_3)}$. Then we have the subcase:

- (ii) The set F consists of elliptic elements with the common fixed point.

Next, suppose there is no elliptic element in F and there is a parabolic element, say T_1. Its fixed point, p, must be fixed by another element, say T_2 of F. If T_2 is parabolic, then (i') forces that T_3 is also parabolic with the same fixed point. If T_2 is hyperbolic with fixed points p, q, then T_3 is either parabolic with fixed point q or hyperbolic with fixed points p, q. Then we have the subcases:

(iii) The set F consists of parabolic elements with the same fixed point.

(iv) The set F consists of a parabolic element with fixed point p a hyperbolic element with fixed points p, q and a third element which is either parabolic with fixed point q or hyperbolic with fixed points p, q.

Finally, suppose that F consists only of hyperbolic elements. Then (i') forces the following two subcases:

(v) The set F consists of hyperbolic elements satisfying $\mathrm{Fix(T_1)} = \mathrm{Fix(T_2)} = \mathrm{Fix(T_3)}$.

(vi) The set F consists of hyperbolic elements, $\cup_{i=1}^{3}\mathrm{Fix(T_i)}$ consists of three points, and each point in the set is fixed by exactly two elements in F.

We have thus classified all the possibilities for T_1, T_2, T_3 into Cases (i) to (vi). Since these cases are disjoint, to prove the theorem it is sufficient to show:

- Condition (1) never holds in the cases (i) and (iv).
- Condition (1) always holds in the cases (ii), (iii) and (v), which is exactly (2)(a).
- In the case (vi), condition (1) is equivalent to condition (2)(b).

Given three distinct and non-trivial elements $T_1, T_2, T_3 \in \mathrm{PSL}_2(\mathbb{R})$, consider the function

$$f(z) = [z, T_1(z), T_2(z), T_3(z)] = \frac{\big(z - T_2(z)\big)\big(T_1(z) - T_3(z)\big)}{\big(z - T_3(z)\big)\big(T_1(z) - T_2(z)\big)}.$$

It is a holomorphic function out of the set $C = \{z \in \widehat{\mathbb{C}} \mid T_3(z) = z\} \cup \{z \in \widehat{\mathbb{C}} \mid T_1(z) = T_2(z)\}$, which is a finite set.

Suppose that T_1, T_2, T_3 satisfy condition (1) of the theorem; then the holomorphic function f takes real values on $H^2 - C$. By the open mapping theorem we have that f must be a constant real function.

Now we consider the cases (i) to (vi) and for each one we study the necessary and sufficient conditions so that condition (1) holds.

Case (i). By the assumption in this case, there is a point z_1 that is fixed by one of the T_i but not fixed by any of the other two. Suppose, first, that $z_1 \in \mathrm{Fix(T_2)}$ but $z_1 \notin \mathrm{Fix(T_1)} \cup \mathrm{Fix(T_3)}$.

As explained above, condition (1) holds if and only if f is a constant real function. The value of this constant is $f(z_1)$, and we have

$$\begin{aligned} f(z_1) &= \frac{\big(z_1 - T_2(z_1)\big)\big(T_1(z_1) - T_3(z_1)\big)}{\big(z_1 - T_3(z_1)\big)\big(T_1(z_1) - T_2(z_1)\big)} \\ &= \frac{0 \cdot \big(T_1(z_1) - T_3(z_1)\big)}{\big(z_1 - T_3(z_1)\big)\big(T_1(z_1) - z_1\big)} = 0. \end{aligned}$$

Notice that, since $z_1 \notin \mathrm{Fix}(\mathrm{T}_1) \cup \mathrm{Fix}(\mathrm{T}_3)$, the denominator of $f(z_1)$ is not equal to zero. So condition (1) holds if and only if $\big(z - T_2(z)\big)\big(T_1(z) - T_3(z)\big) = 0$ for any $z \in \widehat{\mathbb{C}} - C$. Now, this is equivalent to either T_2 being equal to the identity map or $T_1 = T_3$, but this situation contradicts the assumption of the triple T_1, T_2, T_3. If the point z_1 is fixed by T_i but not fixed by T_j and T_k, we apply the same argument to the function $[z, T_j(z), T_i(z), T_k(z)]$.

So condition (1) never holds in case (i).

Case (ii). The orbit of a point z by an elliptic element lies on a circle whose centre is the fixed point of the elliptic element. Then in this case we have that, for any $z \in H^2$, the four points z, $T_1(z)$, $T_2(z)$ and $T_3(z)$ lie on a hyperbolic circle. Since this circle is actually a Euclidean circle in $\mathbb{C}$, then Condition (1) always holds in this case.

Case (iii). The orbit of a point by the action of a parabolic element lies on a horosphere with centre the fixed point of the parabolic. Then in this case we have that, for any $z \in H^2$, the four points z, $T_1(z)$, $T_2(z)$ and $T_3(z)$ lie on a horosphere with centre the fixed point of the parabolic elements T_i. Since horospheres are Euclidean circles, we have that Condition (1) always holds in this case.

Case (iv). Without loss of generality, we can assume that T_2 is parabolic and T_1 is hyperbolic fixing the point in $\mathrm{Fix}(\mathrm{T}_2)$. Then, by the assumption of (ii), the other fixed point of T_1 must be fixed by T_3.

We first suppose that T_3 is parabolic. Then, conjugating by an element in $\mathrm{PSL}_2(\mathbb{R})$, we can assume that the three elements T_1, T_2 and T_3 are the following:

$$T_1(z) = bz, \quad T_2(z) = z + a, \quad T_3(z) = \frac{z}{c\,z + 1},$$

where $a, c \in \mathbb{R} - \{0\}$ and $b \in \mathbb{R} - \{0,1\}$, so that $\mathrm{Fix(T_1)} = \{0, \infty\}$, $\mathrm{Fix(T_2)} = \{\infty\}$ and $\mathrm{Fix(T_3)} = \{0\}$. Then we have

$$\begin{aligned} f(z) &= \frac{\big(z - T_2(z)\big)\big(T_1(z) - T_3(z)\big)}{\big(z - T_3(z)\big)\big(T_1(z) - T_2(z)\big)} = \frac{\big(z - (z+a)\big)\big(b\,z - \frac{z}{c\,z+1}\big)}{\big(z - \frac{z}{c\,z+1}\big)\big(b\,z - (z+a)\big)} \\ &= \frac{-a\,(b\,c\,z + b - 1)}{c\,z\,((b-1)\,z - a)}. \end{aligned}$$

Notice that, in the previous simplification we had the common factor z in the numerator and denominator, corresponding to the fact that $z = 0$ is a fixed point of both T_1 and T_3. We will prove by contradiction that the condition (1) does not hold in this case; so, assume that (1) holds. Then, as in Case (i), f is a real constant function on $\widehat{\mathbb{C}} - C$, i.e., there exists $D \in \mathbb{R}$ such that $\frac{-a(b\,c\,z+b-1)}{c\,z((b-1)z-a)} = D$ for all $z \in \widehat{\mathbb{C}} - C$. This implies that

$$(b-1)\,c\,D\,z^2 + a\,c\,(b - D)\,z + a\,(b-1) = 0$$

for all $z \in \widehat{\mathbb{C}} - C$, so that all coefficients in the previous equation must be zero. In particular, it is $b = 1$ or $a = 0$. Now, if $b = 1$, then $T_1(z)$ is the trivial element, while if $a = 0$, then $T_2(z)$ is the trivial element, and both cases contradict the assumptions.

On the other hand, suppose that $T_3(z)$ is hyperbolic. Necessarily it has the same axis as T_1 and then, conjugating as before, we have that $T_3(z) = c\,z$ for some $c \in \mathbb{R} - \{0,1\}$. Assume (1), so that for any $z \in H^2$ the four points z, $T_1(z)$, $T_2(z)$ and $T_3(z)$ are on a circle. In particular, if we take z on the imaginary axis, then this circle coincides with the imaginary axis (because $z, T_1(z), T_3(z)$ are on that axis), but $T_2(z)$ is not there, so we arrive at a contradiction.

Case (v). The orbit of a point by a hyperbolic element lies on a curve equidistant to the axis of this element. Since this equidistant curve is an arc of a Euclidean circle, in the same way as in cases (ii) and (iii), we have that Condition (1) always holds in this case.

Case (vi). Without loss of generality, we can assume that $\mathrm{Fix(T_1)}= \{0, \infty\}$, $\mathrm{Fix(T_2)} = \{1, \infty\}$ and $\mathrm{Fix(T_3)} = \{0, 1\}$. Then the three elements T_1, T_2 and T_3 are as follows:

$$T_1(z) = a\,z, \quad T_2(z) = b\,z + 1 - b, \quad T_3(z) = \frac{z}{(1-c)\,z + c},$$

where $a, b, c \in \mathbb{R} - \{0, 1\}$. Then we have

$$f(z) = \frac{\big(z - T_2(z)\big)\big(T_1(z) - T_3(z)\big)}{\big(z - T_3(z)\big)\big(T_1(z) - T_2(z)\big)} = \frac{(b-1)(a(c-1)z - ac + 1)}{(c-1)((b-a)z - b + 1)}.$$

Here, we had the common factors $z(z-1)$ in the numerator and denominator, which corresponds to the fact that $z = 0$ is a fixed point of both T_1, T_3 and $z = 1$ is a fixed point of both T_2, T_3. We first suppose that (1) holds. Then, as we saw before, this condition is equivalent to the condition that f is a constant real function. So there is a constant $D \in \mathbb{R}$ such that $f(z) = D$ for all $z \in \mathbb{C} - C$. So we have

$$(b-1)(a(c-1)z - ac + 1) - D(c-1)((b-a)z - b + 1) = 0$$

for $z \in \mathbb{C} - C$, and therefore, it must be

$$\begin{aligned} &(c-1)\left((b-1)a - D(b-a)\right) = 0 \\ &(b-1)\left(-ac + 1 + D(c-1)\right) = 0. \end{aligned}$$

Since $c \neq 1$, we have $D = \dfrac{1 - ac}{c - 1}$ from the second equation. Putting this value into the first equation and taking into account that $a, c \neq 1$ and $a \neq 0$, we get that $c = b/a$. So $T_3(z) = \dfrac{a\,z}{(a-b)\,z + b}$. Hence, under Case (vi) we have obtained that Condition (1) is equivalent to Condition (2)(b). Finally, to prove the last statement in (2)(b), notice that the Möbius transformation T_1 is represented by the matrix

$$M_1 = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix} = \begin{pmatrix} e^{r_1/2} & 0 \\ 0 & e^{-r_1/2} \end{pmatrix},$$

where $a = e^{r_1}$, and r_1 is the translation distance of T_1. Notice that, if $r_1 > 0$, then 0 is the repelling fixed point of T_1 and ∞ the attracting fixed point, agreeing with the orientation given to the axis of T_1. Similarly, T_2 is represented by

$$M_2 = \begin{pmatrix} \sqrt{b} & \frac{1-b}{\sqrt{b}} \\ 0 & \frac{1}{\sqrt{b}} \end{pmatrix} = \begin{pmatrix} e^{r_2'/2} & e^{-r_2'/2} - e^{r_2'/2} \\ 0 & e^{-r_2'/2} \end{pmatrix},$$

where $b = e^{r_2'}$. Now, if $r_2' > 0$, then 1 is the repelling fixed point and ∞ the attracting fixed point. Therefore, the signed translation distance of T_2 is $r_2 = -r_2'$. Finally, T_3 is represented by

$$M_3 = \begin{pmatrix} \frac{1}{\sqrt{c}} & 0 \\ \frac{1-c}{\sqrt{c}} & \frac{c}{\sqrt{c}} \end{pmatrix} = \begin{pmatrix} e^{-r_3/2} & 0 \\ e^{-r_3/2} - e^{r_3/2} & e^{r_3/2} \end{pmatrix},$$

where $c = e^{r_3}$, and r_3 is the signed translation distance of T_3 (we check that, if $r_3 > 0$, then 1 is the repelling fixed point of T_3 and 0 is the attracting one). Since $c = b/a$, we have that $e^{r_3} = e^{-r_2}/e^{r_1}$, and therefore $r_1 + r_2 + r_3 = 0$.

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