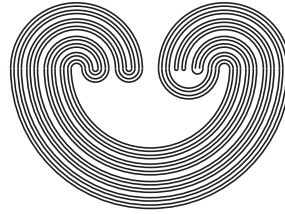


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ULTRAFILTERS AND HIGSON  
COMPACTIFICATIONS

by

MASARU KADA

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**Mail:** Topology Proceedings  
Department of Mathematics & Statistics  
Auburn University, Alabama 36849, USA

**E-mail:** [topolog@auburn.edu](mailto:topolog@auburn.edu)

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## HOW MANY MILES TO $\beta\omega$ ? II — ULTRAFILTERS AND HIGSON COMPACTIFICATIONS

MASARU KADA

**ABSTRACT.** We prove the following theorem: If there is a base  $\mathcal{F}$  of a non-rapid ultrafilter on  $\omega$ , then we can approximate  $\beta\omega$  by  $|\mathcal{F}|$ -many Higson compactifications of  $\omega$  in a nontrivial way.

### 1. INTRODUCTION

In this paper we give a partial answer to a question which was posed by Kada, Tomoyasu and Yoshinobu [3].

We refer the reader to the book [1] for undefined set-theoretic notions. For  $X, Y \in [\omega]^\omega$ , we write  $X \subseteq^* Y$  (or  $Y \supseteq^* X$ ) if  $X \setminus Y$  is finite. The symbol  $\omega^{1\omega}$  denotes the set of all strictly increasing functions in  $\omega^\omega$ . For  $f, g \in \omega^\omega$ , we write  $f \leq^* g$  if  $f(n) \leq g(n)$  holds for all but finitely many  $n \in \omega$ . A *dominating family* is a cofinal subset of  $\omega^\omega$  with respect to  $\leq^*$ .

For compactifications  $\alpha X$  and  $\gamma X$  of a completely regular Hausdorff space  $X$ , we write  $\alpha X \leq \gamma X$  if there is a continuous surjection from  $\gamma X$  onto  $\alpha X$  which fixes the points from  $X$ , and  $\alpha X \simeq \gamma X$  if  $\alpha X \leq \gamma X \leq \alpha X$ . The Stone-Čech compactification  $\beta X$  of  $X$  is the maximal compactification of  $X$  in the sense of the order relation  $\leq$  among compactifications of  $X$  modulo the equivalence relation  $\simeq$ .

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For a compactification  $\alpha X$  of  $X$  and disjoint closed subsets  $A, B$  of  $X$ , we write  $A \parallel B$  ( $\alpha X$ ) if  $\text{cl}_{\alpha X} A \cap \text{cl}_{\alpha X} B = \emptyset$ , and otherwise we write  $A \not\parallel B$  ( $\alpha X$ ). If  $X$  is a normal space,  $\alpha X \leq \gamma X$  is equivalent to the assertion that  $A \parallel B$  ( $\alpha X$ ) implies  $A \parallel B$  ( $\gamma X$ ). Note that, for a normal space  $X$ ,  $A \parallel B$  ( $\beta X$ ) holds for any pair  $A, B$  of disjoint closed subsets of  $X$ .

A metric  $d$  on a space  $X$  is *proper* if each  $d$ -bounded subset of  $X$  has a compact closure. We say a metric space is proper if its metric is proper. For a proper metric space  $(X, d)$  and disjoint closed subsets  $A, B$  of  $X$ , we say  $A$  and  $B$  *diverge* if for every  $R > 0$  there is a compact subset  $K$  of  $X$  such that  $d(A \setminus K, B \setminus K) > R$  holds.

The *Higson compactification*  $\overline{X}^d$  of  $(X, d)$  is uniquely characterized by the property that  $A \parallel B$  ( $\overline{X}^d$ ) if and only if  $A$  and  $B$  diverge. Note that Higson compactifications are metric-dependent. For example, if we define a metric  $d_1$  on the countable discrete space  $\omega$  by letting  $d_1(x, y) = |x^2 - y^2|$  for  $x, y \in \omega$ , any two disjoint sets diverge with respect to  $d_1$  and hence  $\overline{\omega}^{d_1} \simeq \beta\omega$  holds. But if we let  $d_2(x, y) = |x - y|$ , then  $\overline{\omega}^{d_2} \not\simeq \beta\omega$ , because the set of even numbers and the set of odd numbers do not diverge with respect to  $d_2$ . Note that the usual discrete metric  $\rho$  on  $\omega$  (that is,  $\rho(x, y) = 1$  whenever  $x \neq y$ ) does not admit a Higson compactification because it is not a proper metric.

In the paper [3] the authors introduced the following cardinal characteristics to investigate approximability of  $\beta\omega$  by sets of Higson compactifications of  $\omega$ . Let  $\text{PM}'(\omega)$  be the set of proper metrics  $d$  on  $\omega$  such that  $\overline{\omega}^d \not\simeq \beta\omega$  holds. For  $d_1, d_2 \in \text{PM}'(\omega)$ , we write  $d_1 \sqsubseteq d_2$  if  $\overline{\omega}^{d_1} \leq \overline{\omega}^{d_2}$ .

**Definition 1.1.**  $\mathfrak{hp}'$  is the smallest cardinality of a subset  $D$  of  $\text{PM}'(\omega)$  such that  $D$  is directed with respect to the order relation  $\sqsubseteq$  and  $\sup\{\overline{\omega}^d : d \in D\} \simeq \beta\omega$  (where the supremum is in the sense of the order relation  $\leq$  among compactifications of  $\omega$ ).

Throughout the present paper, an *ultrafilter* means a nonprincipal ultrafilter on  $\omega$ . The cardinal  $\mathfrak{u}$  is the smallest cardinality of a subset of  $[\omega]^\omega$  which generates an ultrafilter.

In the paper [3] the authors asked the following question.

**Question 1.2.**  $\mathfrak{hp}' \leq \mathfrak{u}$ ?

This question is still open. In Section 2 we prove that, if a subset  $\mathcal{F}$  of  $[\omega]^\omega$  generates a *non-rapid ultrafilter*, then  $\mathfrak{hp}' \leq |\mathcal{F}|$  holds. We say a filter  $\mathcal{F}$  on  $\omega$  is *rapid* if for all  $h \in \omega^{\uparrow\omega}$  there is a set  $X \in \mathcal{F}$  such that for all  $n < \omega$  we have  $|X \cap h(n)| \leq n$ , or equivalently, if the set of increasing enumerations of sets in  $\mathcal{F}$  is a dominating family. As a consequence, we see that  $\mathfrak{u} < \mathfrak{d}$  implies  $\mathfrak{hp}' \leq \mathfrak{u}$ , where  $\mathfrak{d}$  is the smallest cardinality of a dominating family. This statement gives a partial answer to the above question.

*Remark 1.3.* It is known that non-rapid ultrafilters can be constructed in ZFC, but we do not know if we can find a non-rapid ultrafilter which is generated by a subset of  $[\omega]^\omega$  of size  $\mathfrak{u}$  under ZFC. See Section 3 for further discussion.

## 2. THE MAIN RESULT

First we prove a simple combinatorial lemma.

**Lemma 2.1.** *Suppose that a subset  $\mathcal{F}$  of  $\omega^{\uparrow\omega}$  is not a dominating family. Then there is a function  $h \in \omega^{\uparrow\omega}$  such that, for all  $f \in \mathcal{F}$  there are infinitely many  $m < \omega$  such that the interval  $[h(m), h(m+1))$  contains two consecutive values of  $f$ .*

*Proof.* Suppose that  $\mathcal{F} \subseteq \omega^{\uparrow\omega}$ ,  $g \in \omega^{\uparrow\omega}$  and for all  $f \in \mathcal{F}$  there are infinitely many  $n < \omega$  which satisfy  $f(n) < g(n)$ . Define  $h \in \omega^{\uparrow\omega}$  by letting  $h(n) = g(2n)$  for each  $n$ . We show that  $h$  satisfies the requirement. Suppose not. Find an  $f \in \mathcal{F}$  such that, for all but finitely many  $m < \omega$ , the interval  $[h(m), h(m+1))$  contains at most one value of  $f$ . Then we can find a  $k < \omega$  such that for all  $n < \omega$  we have  $f(n+k) > h(n)$ . Since  $h(n) = g(2n)$  and  $g$  is increasing, for all  $n > k$  we have  $f(n+k) > h(n) = g(2n) > g(n+k)$ . But it is impossible by the choice of  $g$ .  $\square$

Now we are going to prove the main theorem.

**Theorem 2.2.** *Suppose that there is a subset  $\mathcal{F}$  of  $[\omega]^\omega$  of size  $\kappa$  such that  $\mathcal{F}$  generates a non-rapid ultrafilter on  $\omega$ . Then  $\mathfrak{hp}' \leq \kappa$ .*

*Proof.* Let  $\mathcal{F}$  be a subset of  $[\omega]^\omega$  of size  $\kappa$  which generates a non-rapid ultrafilter. Then the set of increasing enumerations of sets in  $\mathcal{F}$  is not a dominating family. By the previous lemma, find a function  $h \in \omega^{\uparrow\omega}$  such that, for every  $X \in \mathcal{F}$ , for infinitely many  $m < \omega$  we have  $|X \cap [h(m), h(m+1))| \geq 2$ .

We may assume that  $h(0) = 0$ . Define a function  $\pi \in \omega^\omega$  by letting  $\pi(k) = m + 1$  if  $h(m) \leq k < h(m + 1)$ .

For each  $X \in \mathcal{F}$ , we define a function  $\rho_X$  with domain  $\omega \times \omega$  by the following:

$$\rho_X(k, l) = \begin{cases} 0 & \text{if } k = l \\ 1 & \text{if } k, l \in X, k \neq l \text{ and } \pi(k) = \pi(l) \\ \pi(k) + \pi(l) & \text{otherwise.} \end{cases}$$

It is easily checked that  $\rho_X$  is a metric on  $\omega$  and any  $\rho_X$ -bounded subset of  $\omega$  is finite, and so  $\rho_X$  is a proper metric on  $\omega$ .

By the choice of  $h$ , there are infinitely many pairs  $k, l \in \omega$  for which  $\rho_X(k, l) = 1$  holds, and so we can construct a pair  $A, B$  of disjoint infinite subsets of  $\omega$  such that  $A \parallel B$  ( $\bar{\omega}^{\rho_X}$ ) holds. This ensures that  $\rho_X \in \text{PM}'(\omega)$  for all  $X \in \mathcal{F}$ .

Note that, for  $X, Y \in \mathcal{F}$ ,  $X \supseteq^* Y$  implies  $\rho_X \sqsubseteq \rho_Y$ . Since  $\mathcal{F}$  generates an ultrafilter,  $\mathcal{F}$  is  $\supseteq^*$ -directed (even  $\supseteq$ -directed), and so the set  $\{\rho_X : X \in \mathcal{F}\}$  is  $\sqsubseteq$ -directed.

We can easily see that, for  $B \subseteq \omega$ , if  $X \subseteq^* B$  or  $X \subseteq^* \omega \setminus B$ , then  $B \parallel \omega \setminus B$  ( $\bar{\omega}^{\rho_X}$ ). Since  $\mathcal{F}$  generates an ultrafilter, for each  $B \subseteq \omega$  we can find an  $X \in \mathcal{F}$  such that  $X \subseteq^* B$  or  $X \subseteq^* \omega \setminus B$ . This implies that, for any pair  $A, B$  of disjoint subsets of  $\omega$ , there is an  $X \in \mathcal{F}$  such that  $A \parallel B$  ( $\bar{\omega}^{\rho_X}$ ) holds, which means that  $\sup\{\bar{\omega}^{\rho_X} : X \in \mathcal{F}\} \simeq \beta\omega$ . By the definition of  $\mathfrak{hp}'$ , we have  $\mathfrak{hp}' \leq |\mathcal{F}| = \kappa$ .  $\square$

In the paper [3] the authors also introduced the following variant of the cardinal  $\mathfrak{hp}'$ .

**Definition 2.3.**  $\mathfrak{ht}$  is the smallest cardinality of a subset  $D$  of  $\text{PM}'(\omega)$  such that  $D$  is well-ordered by  $\sqsubseteq$  and  $\sup\{\bar{\omega}^d : d \in D\} \simeq \beta\omega$  (if such a set  $D$  exists; otherwise we write  $\mathfrak{ht} = \infty$ ).

An ultrafilter is called a *simple  $p_\kappa$ -point*, where  $\kappa$  is a regular uncountable cardinal, if it is generated by a subset of  $[\omega]^\omega$  which is well-ordered by  $\supseteq^*$  in order type  $\kappa$ . The following result is obtained as a corollary of the previous theorem.

**Corollary 2.4.** *Suppose that there is a subset  $\mathcal{F}$  of  $[\omega]^\omega$  of size  $\kappa$  such that  $\mathcal{F}$  is well-ordered by  $\supseteq^*$  and generates a non-rapid ultrafilter on  $\omega$  (so  $\mathcal{F}$  generates a simple  $p_\kappa$ -point). Then  $\mathfrak{ht} \leq \kappa$ .*

3. CONSEQUENCES OF THE MAIN RESULT

The cardinal  $\mathfrak{pp}$ , which was introduced in [3], is the smallest cardinal  $\kappa$  for which a simple  $\mathfrak{p}_\kappa$ -point exists (if such a  $\kappa$  exists; otherwise we write  $\mathfrak{pp} = \infty$ ). Here we introduce more cardinal characteristics.

**Definition 3.1.**  $u(\text{non-rapid})$  is the smallest cardinality of a subset  $\mathcal{F}$  of  $[\omega]^\omega$  which generates a non-rapid ultrafilter.

$\mathfrak{pp}(\text{non-rapid})$  is the smallest cardinality of a subset  $\mathcal{F}$  of  $[\omega]^\omega$  which is well-ordered by  $\supseteq^*$  and generates a non-rapid ultrafilter (if such a set  $\mathcal{F}$  exists; otherwise we write  $\mathfrak{pp}(\text{non-rapid}) = \infty$ ).

Using the above cardinal characteristics, Theorem 2.2 and Corollary 2.4 are represented as follows.

**Corollary 3.2.**  $\mathfrak{hp}' \leq u(\text{non-rapid})$  and  $\mathfrak{ht} \leq \mathfrak{pp}(\text{non-rapid})$ .

It is clear that  $u \leq \mathfrak{pp}$ ,  $u \leq u(\text{non-rapid})$  and  $\mathfrak{pp} \leq \mathfrak{pp}(\text{non-rapid})$ . Also it is easily observed that  $u < \mathfrak{d}$  implies  $u(\text{non-rapid}) = u$ , and  $\mathfrak{pp} < \mathfrak{d}$  implies  $\mathfrak{pp}(\text{non-rapid}) = \mathfrak{pp}$ . So we obtain the following result, which partially answers Question 1.2.

**Corollary 3.3.** *If  $u < \mathfrak{d}$ , then  $\mathfrak{hp}' \leq u$ . If  $\mathfrak{pp} < \mathfrak{d}$ , then  $\mathfrak{hp}' \leq \mathfrak{pp}$ .*

It is known that CH implies the existence of a simple  $\mathfrak{p}_{\aleph_1}$ -point. Since the Miller forcing preserves  $\mathfrak{p}$ -points [1, Lemma 7.3.48] and the preservation of  $\mathfrak{p}$ -points is preserved under countable support iteration [1, Theorem 6.2.6], a generating set of a simple  $\mathfrak{p}_{\aleph_1}$ -point in the ground model still generates an ultrafilter in the forcing model by iterated Miller forcing. On the other hand,  $\mathfrak{d} = \aleph_2$  holds in the model obtained by a countable support iteration of Miller forcing of length  $\omega_2$  over a model for CH. Hence  $\mathfrak{pp} < \mathfrak{d}$  is consistent with ZFC.

But the following question is still open.

**Question 3.4.**  $u(\text{non-rapid}) = u?$      $\mathfrak{pp}(\text{non-rapid}) = \mathfrak{pp}?$

In the paper [3], another upper bound for  $\mathfrak{hp}'$  is given.

**Definition 3.5** ([2, Section 5]). For a function  $h \in \omega^\omega$ ,  $l_h$  is the smallest size of a subset  $\Phi$  of  $\prod_{n < \omega} [\omega]^{\leq 2^n}$  such that for every  $f \in \prod_{n < \omega} h(n)$  there is a  $\varphi \in \Phi$  such that  $f(n) \in \varphi(n)$  for all but finitely many  $n$ . Let  $l = \sup\{l_h : h \in \omega^\omega\}$ .

**Theorem 3.6** ([3, Theorem 6.11]).  $\mathfrak{hp}' \leq \mathfrak{l}$ .

Now we can see that the above inequality is consistently strict.

**Corollary 3.7.**  $\mathfrak{hp}' < \mathfrak{l}$  (moreover,  $\mathfrak{ht} < \mathfrak{l}$ ) is consistent with ZFC.

*Proof.* We know that there is a proper forcing notion  $\mathbb{P}$  which satisfies the following two properties (see Remark 3.8).

- $\mathbb{P}$  preserves p-points.
- In the forcing model by  $\mathbb{P}$ , for any function  $H \in \omega^\omega \cap \mathbf{V}$ , there is a function  $g \in \prod_{n < \omega} H(n)$  such that, for every function  $x \in \prod_{n < \omega} H(n) \cap \mathbf{V}$  there are infinitely many  $n < \omega$  with  $x(n) = g(n)$ , where  $\mathbf{V}$  denotes a ground model.

We consider a forcing model obtained by a countable support iteration of alternation of Miller forcing and the above forcing notion  $\mathbb{P}$  of length  $\omega_2$  over a model for CH.

Since every iterand preserves p-points and the preservation of p-points is preserved under countable support iteration, a generating set of a simple  $\mathfrak{p}_{\aleph_1}$ -point in the ground model still generates an ultrafilter in our forcing model, and so  $\mathfrak{pp} = \aleph_1$  holds. On the other hand, it is easily observed that  $\mathfrak{d} = \mathfrak{l} = \aleph_2 = \mathfrak{c}$  holds in the same model. By Corollary 3.3,  $\aleph_1 = \mathfrak{hp}' = \mathfrak{ht} < \mathfrak{l} = \aleph_2$  holds in this model.  $\square$

*Remark 3.8.* The book [1] tells us in Subsection 7.4.C that the *infinitely equal forcing*  $\mathbb{EE}$  meets the requirements which appear in the proof of Corollary 3.7. But Brendle pointed out (in private communication) that  $\mathbb{EE}$  does not preserve p-points, and the following “tree-like infinitely equal forcing”  $\mathbb{TEE}$  is what we actually need.

$p \in \mathbb{TEE}$  if:

- (1)  $p$  is a subtree of  $\bigcup_{m < \omega} \prod_{n < m} 2^n$  without endpoints,
- (2) there is a  $C \in [\omega]^\omega$  such that, for  $s \in p$ , if  $|s| = n \in C$  then  $\text{succ}_p(s) = 2^n$ ,

and  $\mathbb{TEE}$  is ordered by inclusion.

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GRADUATE SCHOOL OF SCIENCE, OSAKA PREFECTURE UNIVERSITY. SAKAI,  
OSAKA 599–8531 JAPAN  
*E-mail address:* `kada@mi.s.osakafu-u.ac.jp`