TOPOLOGY PROCEEDINGS

Volume 33, 2009

Pages 139–152

http://topology.auburn.edu/tp/

On some applications of equivalence relations in topological dynamics

by

Jerzy Szymański

Electronically published on August 12, 2008

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 33 (2009)
Pages 139-152

E-Published on August 12, 2008

ON SOME APPLICATIONS OF EQUIVALENCE RELATIONS IN TOPOLOGICAL DYNAMICS

JERZY SZYMAŃSKI

Abstract. We discuss some applications of equivalence relations in topological dynamics. We look at their use in defining some topological objects (topological determinism and Kolmogorov property) and their applications in proofs of properties of these objects.

1. Introduction

The inspiration to apply closed equivalence relations to investigation of topological dynamical systems was the Rokhlin-Sinai theory of invariant partitions which had important applications in ergodic theory. In topological setting the object corresponding to an invariant partition is an invariant closed equivalence relation. In the Rokhlin-Sinai theory an important role play extreme partitions, perfect partitions and also Pinsker partitions. In the paper [19] Rokhlin and Sinai proved that there exists an extreme partition (Rokhlin-Sinai theorem). These partitions were applied, among other things, to investigation of determinism and Kolmogorov property for measure-theoretic dynamical systems.

The relative version of the Rokhlin-Sinai theorem was applied to solve the problem of the spectrum of Kolmogorov $\mathbb{Z}^d$-actions ([11]).

2000 *Mathematics Subject Classification.* Primary 37B05; Secondary 37B40, 37-02, 37-06.

Key words and phrases. Equivalence relations, extreme relations, deterministic extensions, Kolmogorov extensions, asymptotic pairs.

The author was supported by MNiSW Grant N N201 384834.

Topological analogues of the mentioned above measure-theoretic systems and results were considered in the series of papers [12], [13], [14]. The present paper is mainly the survey on results of these papers. At the end of the article we give some open questions.

Through the paper we assume (X, T) is a topological flow where X is a compact metric space equipped with metric d and T is a homeomorphism.

For a given $x \in X$, the sets $O_T^+(x) = \{T^n x,\ n \geq 0\}$, $O_T^-(x) = \{T^n x,\ n \leq 0\}$ and $O_T(x) = \{T^n x,\ n \in \mathbb{Z}\}$ are called the positive semiorbit, the negative semiorbit and the orbit of x, respectively.

Let $\mathbf{A}(T)$ be the set of all asymptotic pairs for T. Recall that $(x, x') \in \mathbf{A}(T)$ if $\lim_{n\to+\infty}\ d(T^n x, T^n x') = 0$. Obviously $T\mathbf{A}(T) = \mathbf{A}(T)$.

By $REL(X)$ we denote the set of all relations in $X \times X$. By $CER(X)$ we denote the subset of $REL(X)$ consisting of all closed equivalence relations and by Δ the diagonal relation. A relation $R \in CER(X)$ is said to be positively invariant (resp. invariant) with respect to T if $(T \times T)(R) \subset R$ (resp. $(T \times T)(R) = R$). The symbol $ICER^+(X)$ ($ICER(X)$) stands for the set of all positively invariant (invariant) relations. For a subset $F \subset X \times X$, the smallest invariant relation $R \in CER(X)$ containing F is denoted by $\langle F \rangle$. For a family $\{R_i\} \subset CER(X)$, the symbol $\bigvee_i R_i$ means the smallest closed invariant equivalence relation containing all R_i's.

2. Closed equivalence relations induced by pairs of points

We consider here some properties of a special kind of relations, i.e. relations induced by certain pairs of points. Such kind of relations are important in our further considerations.

For any $R, S \in REL(X)$ the symbol $R \circ S$ denotes the composition of R and S.

It is well-known that the operation $\circ$ has the following properties:

(i) For any relation $R \in REL(X)$ it holds $R \circ \Delta = R = \Delta \circ R$.
(ii) $\forall_{P,R,S \in REL(X)}\ P \circ (R \cup S) = (P \circ R) \cup (P \circ S)$.
(iii) For any $R_1, R_2, \ldots \in REL(X)$ and any $S_1, S_2, \ldots \in REL(X)$ we have $\bigcup_{i=1}^{\infty} R_i \circ \bigcup_{j=1}^{\infty} S_j = \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty} R_i \circ S_j$.

(iv) Let $R \in REL(X)$. Relation R is transitive $\iff$ $R \circ R \subset R$.
(v) If $R \in REL(X)$ is reflexive then R is transitive $\iff$ $R \circ R = R$.

Let $(x, x') \in X \times X$ where $x \neq x'$. We assume in the sequel that the orbits $O_T(x)$ and $O_T(x')$ are infinite.

We define relation $\mathbf{R} \in REL(X)$ in the following way:

$$\mathbf{R} = \mathbf{R}(x, x') = \bigcup_{n=0}^{\infty} (T \times T)^n (x, x') \cup \bigcup_{n=0}^{\infty} (T \times T)^n (x', x) \cup \Delta.$$

In the sequel we investigate properties of the above relation.

From the definition it follows at once that the relation $\mathbf{R}$ is reflexive, symmetric and positively invariant.

Denote

$$R_1 = \bigcup_{n=0}^{\infty} (T \times T)^n (x, x'), \quad R_2 = \bigcup_{n=0}^{\infty} (T \times T)^n (x', x).$$

Proposition 2.1. *If (x, x') is asymptotic then $\mathbf{R}$ is closed.*

Proof. Since $\overline{\mathbf{R}} = \overline{R_1} \cup \overline{R_2} \cup \Delta$ it is enough to observe that $\overline{R_i} \subset R_i \cup \Delta$ for $i = 1, 2$. Let (x_1, x_2) be the limit point of the set R_1. Since (x, x') is asymptotic, then in any neighborhood of (x_1, x_2) there are points of the form $(T^n x, T^n x')$, $n \geq 0$ for which the distance $d(T^n x, T^n x')$ is as small as requested. It follows immediately that $x_1 = x_2$ and so $(x_1, x_2) \in \Delta$. Since $X \times X$ is compact we have $\overline{R_1} = R_1 \cup (x, x)$ for some $x \in X$. Applying the same argument we obtain that for the same x we have $\overline{R_2} = R_2 \cup (x, x)$. In consequence

$$\mathbf{R} \subset \overline{\mathbf{R}} = R_1 \cup R_2 \cup (x, x) \subset \mathbf{R},$$

and so $\mathbf{R}$ is closed. □

Proposition 2.2. *Relation $\mathbf{R}$ is transitive $\iff$ $O_T(x) \cap O_T(x') = \emptyset$.*

Proof. Assume that $O_T(x) \cap O_T(x') = \emptyset$. Using the property (iv) in order to show that $\mathbf{R}$ is transitive it is enough to show that $\mathbf{R} \circ \mathbf{R} \subset \mathbf{R}$.

Applying the properties (i)-(ii) it is easy to check that

$$\mathbf{R} \circ \mathbf{R} = [(R_1 \cup R_2) \circ (R_1 \cup R_2)] \cup \mathbf{R}.$$

Therefore, according to (iv), we have to show that $(R_1 \cup R_2) \circ (R_1 \cup R_2) \subset \mathbf{R}$.

We have

$$(R_1 \cup R_2) \circ (R_1 \cup R_2) = (R_1 \circ R_1) \cup (R_1 \circ R_2) \cup (R_2 \circ R_1) \cup (R_2 \circ R_2).$$

Since for any $x_1, x_2, x_3, x_4 \in X$ from the definition of operation $\circ$ we have

$$(x_1, x_2) \circ (x_3, x_4) = \begin{cases} (x_1, x_4) & x_2 = x_3, \\ \emptyset & x_2 \neq x_3, \end{cases}$$

then applying the property (iii) and the assumption we get $R_1 \circ R_1 = R_2 \circ R_2 = \emptyset$. Next, using the fact that the orbits of points x and x' are infinite it follows $R_2 \circ R_1 = R_2$ and $R_1 \circ R_2 = R_1$, respectively. Thus $\mathbf{R} \circ \mathbf{R} \subset \mathbf{R}$ and so $\mathbf{R}$ is transitive.

Assume now that $\mathbf{R}$ is transitive, i.e. $\mathbf{R} \circ \mathbf{R} \subset \mathbf{R}$. We show that $O_T(x) \cap O_T(x') = \emptyset$. If it is not the case, since $x \neq x'$, it would exist $k \geq 1$ such that $T^k x = x'$ or $T^k x' = x$. Assume first that $T^k x = x'$. Then using the considerations from the previous part of the proof we would obtain $(T^k x', x) \in \mathbf{R}$ because

$$(T^k x', x) = (T^k x', T^k x) \circ (x', x) \in R_2 \circ R_2 \subset \mathbf{R} \circ \mathbf{R},$$

and from the assumption $\mathbf{R} \circ \mathbf{R} \subset \mathbf{R}$. The fact that $(T^k x', x) \in \mathbf{R}$ means that $(T^k x', x) \in R_1$ or $(T^k x', x) \in R_2$ or $(T^k x', x) \in \Delta$. It is impossible because in any of the cases applying $T^k x = x'$ we obtain a contradition with the asumed infiniteness of the oribit x or x'.

Similarly, assuming that $T^k x' = x$ we get the pair $(T^k x, x') \in R_1 \circ R_1 \subset \mathbf{R}$ - contradiction as before. □

Corollary 2.3. *If (x, x') is asymptotic and the orbits $O_T(x)$ and $O_T(x')$ are infinite and disjoint then $\mathbf{R} \in ICER^+(X) \setminus ICER(X)$.*

Proof. It remains to show that $(T \times T)\mathbf{R} \neq \mathbf{R}$. If it is not the case we would get the equality

$$\bigcup_{n=0}^{\infty} (T \times T)^n (x, x') \cup \bigcup_{n=0}^{\infty} (T \times T)^n (x', x) \cup \Delta = \mathbf{R} = (T \times T)\mathbf{R} = \bigcup_{n=1}^{\infty} (T \times T)^n (x, x') \cup \bigcup_{n=1}^{\infty} (T \times T)^n (x', x) \cup \Delta.$$

It follows that it would hold in one of the following three situations: $(x, x') \in (T \times T)R_1$ or $(x, x') \in (T \times T)R_2$ or $x = x'$, which in any case leads to a contradiction with the assumption. □

3. Extreme relations

In the present section we recall the definition of an extreme relation and the statement of the topological analogue of the Rokhlin-Sinai theorem - both in the absolute and the relative case. We present also a sketch of the proof of the relative version of the theorem to present the method of the construction of a relation with desired properties.

Let $M(X, T)$ be the set of all invariant probability measures on X.

Let us recall that a measurable partition ξ is called *extreme* ([18]) if it satisfies the following conditions

(i) $T^{-1}\xi \leq \xi$,

(ii) $\bigvee_{n=0}^{\infty} T^n \xi = \varepsilon$,

(iii) $\bigwedge_{n=0}^{\infty} T^{-n} \xi = \pi_\mu(T)$,

where $\pi_\mu(T)$ is the Pinsker partition of T, $\mu \in M(X, T)$ and ε is the partition on points.

In the paper [13] the topological analogue of extreme partition - extreme relation - was introduced.

Definition 3.1 ([13])**.** We say that a relation $R \in CER(X)$ is *extreme* (*extreme with respect to* μ) if

(i) $(T \times T)(R) \subset R$,

(ii) $\bigcap_{n=0}^{\infty} (T \times T)^n(R) = \Delta$,

(iii) $\bigvee_{n=0}^{\infty} (T \times T)^{-n}(R) = \Pi(T)\ (\Pi_\mu(T))$,

where $\Pi(T)$ $(\Pi_\mu(T))$ is the Pinsker relation (the Pinsker relation with respect to μ).

The main result concerning extreme relations is the analogue of the Rohklin-Sinai theorem:

Theorem 3.2 ([13]). *For any ergodic measure $\mu \in M(X,T)$, there exists a relation $R = R_\mu \in CER(X)$ with*

(i) $(T \times T)(R) \subset R$,

(ii) $\bigcap_{n=0}^{\infty} (T \times T)^n(R) = \Delta$,

(iii) $E_\mu(X,T) \cup S(\mu) \subset \overline{\bigcup_{n=0}^{\infty} (T \times T)^{-n}(R)} \subset \Pi_\mu(T)$,

where $S(\mu) = \{(x,x) \in X \times X;\ x \in \mathrm{Supp}\ \mu\}$ and $E_\mu(X,T)$ is the set of entropy pairs with respect to μ.

It follows from the theorem that for any ergodic measure $\mu \in M(X,T)$ there exists an extreme relation with respect to μ. If μ is the only ergodic measure then there exists an extreme relation.

The definition of extreme relation and the above theorem were then generalized in the recent paper [14] to topological extensions. Let us write them down in terms of extensions.
Let $\Sigma \in CER(X)$ be fixed and let σ be the invariant measurable partition associated with Σ. The symbol $h_\mu(T|\sigma)$ denotes the σ-relative entropy of T. By $E_\mu(X,T|\sigma)$ we denote the set of σ-relative entropy pairs with respect to μ. Let $\Pi_\mu(T|\sigma) = \langle E_\mu(X,T|\sigma)\rangle$ be the relative Pinsker relation with respect to μ.

Definition 3.3 ([14]). We say that a relation $R \in CER(X)$ is Σ-*relatively extreme with respect to* μ if

(i) $(T \times T)(R) \subset R \subset \Sigma$,

(ii) $\bigcap_{n=0}^{\infty} (T \times T)^n(R) = \Delta$,

(iii) $\bigvee_{n=0}^{\infty} (T \times T)^{-n}(R) = \Pi_\mu(T|\sigma)$.

Similarly as in the absolute case it is known ([14]) that for any ergodic measure $\mu \in M(X,T)$ there exists a Σ-relatively extreme relation with respect to μ. This is a corollary from the following theorem.

Theorem 3.4 ([14]). *For any ergodic measure $\mu \in M(X,T)$ and invariant relation $\Sigma \in CER(X)$, there exists a relation $R = R_\mu \in CER(X)$ with*

(i) $(T \times T)(R) \subset R \subset \Sigma$,

(ii) $\bigcap_{n=0}^{\infty}(T\times T)^n(R)=\Delta$,

(iii) $E_\mu(X,T|\sigma)\cup S(\mu)\subset\overline{\bigcup_{n=0}^{\infty}(T\times T)^{-n}(R)}\subset\Pi_\mu(T|\sigma)$.

Proof. (sketch) If $h_\mu(T|\sigma)=0$ then the relation $R=\Delta$ satisfies conditions (i)-(iii). Therefore we can assume that $h_\mu(T|\sigma)>0$. Now we are going to construct the desired relation induced by a pair of points (as described in section 2) but the pair has to be especially chosen. We prove first that there exists σ-relative extreme partition ζ such that its graph is included in the set of asymptotic pairs, i.e. $\Delta_\zeta\subset\mathbf{A}(T)$.

In the next step, similarly as in the proof of Proposition 5 ([1]), one constructs a set $G\subset X\times X$ such that for any point $(x,x')\in G$, the orbit $O_{T\times T}(x,x')$ is dense in $\Lambda_\mu^\sigma=\mathrm{Supp}\ \lambda_\mu^\sigma$, where $\lambda_\mu^\sigma=(\mu\times_{\pi_\mu(T|\sigma)}\mu)$ is the relative square of measure μ over the σ-relative Pinsker partition $\pi_\mu(T|\sigma)$. The set G is of measure 1 w.r. to $(\mu\times_\zeta\mu)$.

After removing some sets of measure zero w.r. to $(\mu\times_\zeta\mu)$ the properties of ζ imply the inclusions:

$$\Delta_\zeta\subset\Pi_\mu(T|\sigma), \tag{3.1}$$

$$\Delta_\zeta\subset\Sigma. \tag{3.2}$$

Now, by definition, $(\mu\times_\zeta\mu)$ is concentrated on Δ_ζ and since $h_\mu(T|\sigma)>0$ the measure is not concentrated on Δ. Therefore we can choose a proper pair $(x,x')\in\Delta_\zeta\cap G$ and define a relation R as follows

$$R=O^+_{T\times T}(x,x')\cup O^+_{T\times T}(x',x)\cup\Delta.$$

This relation is, of course, reflexive, symmetric and positively invariant. Since $(x,x')\in\mathbf{A}(T)$, it is closed (Proposition 2.1) and the equality (ii) is satisfied.

Applying similar method as in the proof of Theorem 1 ([6]), one can show the following relative version of this theorem:

$$\Lambda_\mu^\sigma=E_\mu(X,T|\sigma)\cup S(\mu).$$

The density of $O_{T\times T}(x,x')$ in Λ_μ^σ implies

$$E_\mu(X,T|\sigma)\cup S(\mu)\subset\overline{\bigcup_{n=0}^{\infty}(T\times T)^{-n}(R)}.$$

The fact that $x \neq x'$ and the assumption $h_\mu(T|\sigma) > 0$ imply that the orbits $O_T(x)$ and $O_T(x')$ are infinite and disjoint. Therefore from Proposition 2.2 we obtain that R is transitive.

From (3.1), we get $(x, x') \in \Pi_\mu(T|\sigma)$, hence, $R \subset \Pi_\mu(T|\sigma)$. Since $\Pi_\mu(T|\sigma)$ is closed and invariant, we get

$$\overline{\bigcup_{n=0}^{\infty} (T \times T)^{-n}(R)} \subset \Pi_\mu(T|\sigma).$$

Since Σ is closed and invariant, then applying (3.2) we get $R \subset \Sigma$, i.e. R satisfies all desired properties. □

The special relation constructed in the above theorem proved to be very useful in investigation both deterministic and Kolmogorov flows (and corresponding extensions).

4. Deterministic flows

Definition 4.1. A flow (X, T) is said to be *deterministic* ([12]) if every positively invariant relation $R \in CER(X)$ is invariant.

It is known that the dynamics of the measure-theoretic deterministic systems is not dependent on the direction of time. However in the topological case it was shown by Hochman ([9]) that there exists a deterministic flow (X, T) such that T^{-1} is not deterministic.

Let $\pi : X \to Y$ be a homomorphism defining an extension (X, T) of a flow (Y, S) and let $\Sigma = \Sigma_\pi \in ICER(X)$ be the relation associated with π.

Definition 4.2. The homomorphism $\pi : X \to Y$ is called *deterministic* (or (X, T) is called *a deterministic extension* of (Y, S)) ([14]) if for every relation $R \in CER(X)$ such that $(T \times T)(R) \subset R \subset \Sigma_\pi$ we have $(T \times T)(R) = R$.

The application of the analogue of the Rokhlin-Sinai theorem and its relative version allowed to show that deterministic systems have zero topological entropy and deterministic extensions have zero relative topological entropy (cf. [13], [14]).

Now we are going to present a new proof of theorem that deterministic extensions preserve topological entropy ([14]). In the proof we give also another proof of the fact that deterministic extensions have zero relative topological entropy which involves more classical methods than these used in [14].

The notion of the relative topological entropy which we use here was introduced by Downarowicz and Serafin in [5]. In the paper there are proved the following theorems which we are going to apply in further considerations:

$$h(X|Y) = \sup_{y\in Y} h(X|y) = \sup_{\nu\in M(Y,S)} h(X|\nu), \tag{4.1}$$

where in the case of compact space $h(X|y)$ is Bowen's entropy $h(T,\pi^{-1}y)$ of T where $\pi^{-1}y$ is a compact set and $h(X|\nu)$ can be expressed as

$$h(X|\nu) = \int_Y h(X|y)d\nu. \tag{4.2}$$

In the paper of Lemańczyk and Siemaszko ([16]) it is proved that

$$\forall_{\mu\in M(X,T)}\ h_\mu(T) = h_{\pi\mu}(S) \iff \forall_{\nu\in M(Y,S)}\ h(X|y) = 0 \text{ for } \nu\text{-a.e. } y\in Y.$$

The authors of [16] also noticed that the property in the left hand side of the above equivalence (P_2 property) in the case of compact metric space is equivalent to the following property

$$\forall_{\nu\in M(Y,S)}\forall_{y\in Y}\ h(X|y) = 0,$$

which in fact follows directly from (4.2) and (4.1). It implies at once that

$$\forall_{\mu\in M(X,T)}\ h_\mu(T) = h_{\pi\mu}(S) \iff h(X|Y) = 0. \tag{4.3}$$

Theorem 4.3. *Deterministic extensions preserve topological entropy, i.e. if (X,T) is a deterministic extension of (Y,S) then $h(T) = h(S)$.*

Proof. We prove first that $h(X|Y) = 0$. Let (X,T) be a deterministic extension of (Y,S) and assume that $h(X|Y) > 0$. From the property (4.3) it follows that $h_\mu(T) > h_{\pi\mu}(S)$ for any measure $\mu \in M(Y,S)$. Using the well-known formula (cf. [10], [5]):

$$h_\mu(T) = h_{\pi\mu}(S) + h_\mu(T|\sigma),$$

where $\sigma = \pi^{-1}\varepsilon_Y$ and ε_Y is the partition on points of the space Y, we obtain that $h_\mu(T|\sigma) > 0$. For the measure μ by τ denote the unique measure on the space $M(X,T)$ which is concentrated on the set of all ergodic invariant measures $M^e(X,T)$ and has the property

$$\forall_{f\in C(X)}\ \int_X f(x)d\mu(x)=\int_{M^e(X,T)}\left(\int_X f(x)dm(x)\right)d\tau(m).$$

Then from the relative Jacobs theorem

$$h_\mu(T|\sigma)=\int_{M^e(X,T)} h_m(T|\sigma)d\tau(m),$$

which proof can be found in [15], it follows directly that

$$\forall_{\mu\in M(X,T)}\ h_\mu(T|\sigma)=0 \iff \forall_{\mu\in M^e(X,T)}\ h_\mu(T|\sigma)=0.$$

We obtain therefore that $h_\mu(T|\sigma)>0$ for some ergodic measure $\mu\in M(X,T)$.

Let $R\in CER(X)$ be a relation satisfying the conditions (i)-(iii) of Theorem 3.4. Since from the assumption we get $(T\times T)(R)=R$, thus from the above properties we obtain $R=\Delta$. Therefore we have

$$E_\mu(X,T|\sigma)\cup S(\mu)\subset\Delta,$$

which is a contradiction with the condition $h_\mu(T|\sigma)>0$. Therefore $h(X|Y)=0$.

Since we always have $h(S)\le h(T)$ it is enough to observe that $h(T)\le h(S)$. We have $h(X|Y)=0$. Therefore the fact that inequality $h(T)\le h(S)$ holds is a straightforward consequence of the formula (4.1) and the theorem of Bowen ([3]):

$$h(T)\le h(S)+\sup_{y\in Y} h(X|y).$$

□

It was shown in [14] that in deterministic flow there is no proper asymptotic pairs. In the proof one constructs a positively invariant relation which is not invariant assuming only the existence of an asymptotic pair. Since such pairs exist in any expansive flow (cf. [2]) we obtain that expansive flows cannot be deterministic.

5. Kolmogorov flows

Definition 5.1. A flow (X, T) is called *a topological Kolmogorov flow* (*K-flow*) ([12]) if there exists a relation $R \in CER(X)$ with

(i) $(T \times T)(R) \subset R$,

(ii) $\bigcap_{n=0}^{\infty} (T \times T)^n(R) = \Delta$,

(iii) $\overline{\bigcup_{n=0}^{\infty} (T \times T)^{-n}(R)} = X \times X$.

In the relative case the definition has the form.

Definition 5.2. The homomorphism $\pi : (X, T) \longrightarrow (Y, S)$ is called *Kolmogorov* (or (X, T) is called *a Kolmogorov extension* of (Y, S)) ([14]) if there exists a relation $R \in CER(X)$ such that

(i) $(T \times T)(R) \subset R \subset \Sigma_\pi$,

(ii) $\bigcap_{n=0}^{\infty} (T \times T)^n(R) = \Delta$,

(iii) $\overline{\bigcup_{n=0}^{\infty} (T \times T)^{-n}(R)} = \Sigma_\pi$.

It follows from the definition that topological K-flows cannot be deterministic.

The relation satisfying conditions (i)-(iii) of Definition 5.1 as well as extreme relations were recently considered in the paper [8].

In the following example we can see that for the mixing topological Markov chains there exists a relation satisfying conditions (i)-(iii), i.e. topological Markov chains are K-flows.

Example 5.3. Let $(\Lambda, \sigma_{|\Lambda})$ be a mixing topological Markov chain with the transition matrix L. Since $(\Lambda, \sigma_{|\Lambda})$ is mixing, then the matrix L is aperiodic (cf. [4], Proposition 17.10).

We define the relation $R \subset \Lambda \times \Lambda$ in the following way: If $x = (x_n),\ y = (y_n) \in \Lambda$, then $(x, y) \in R \iff \forall_{n \geq 0}\ x_n = y_n$.

Now one checks that the relation satisfies desired conditions (i)-(iii) of Definition 5.1.

Theorems 3.2 and 3.4 allowed to prove the following proposition. The analogous fact holds in the relative case.

Proposition 5.4 ([13]). *If a topological flow admits a K-measure with full support then it is a topological K-flow.*

The following proposition relates K-flows to asymptotic pairs. Let us state it only in the relative version.

Proposition 5.5 ([14]). *If π is a K-extension then $\mathbf{A}(T) \cap \Sigma_\pi$ is dense in Σ_π.*

At the end we give the following description of the relative Pinsker relation $\Pi(T|\Sigma)$ different from those described in [16] and [17].

Proposition 5.6 ([14]).

$$\Pi(T|\Sigma) = \left\langle \overline{\bigcup_{\mu} \bigcup_{n=0}^{\infty} (T \times T)^{-n}(R_\mu)} \right\rangle,$$

where $\mu \in M(X,T)$ runs over all ergodic measures with $h_\mu(T|\sigma) > 0$.

6. Some open qestions

1. Recall that a flow (X,T) is said to be *rigid* ([7]) if there exists an increasing sequence (n_k) of positive integers with $T^{n_k}x \to x$ for any $x \in X$. It is clear that rigid flows are deterministic. In the mentioned paper there is also a notion of weak rigidity. A flow (X,T) is *weakly rigid* if the identity homeomorphism $I : X \to X$ is a limit point of the collection $\{T^n;\ n \in \mathbb{Z}\}$ in the topology of pointwise convergence. In other words, if I is not an isolated point in the enveloping semigrup $E(X)$. Clearly every infinite minimal distal flow is weakly rigid and it is known that every distal flow is deterministic. However, the question is whether weakly rigid flows are deterministic? What is known is the following.

One can define *positive weak rigidity* by the condition $I \in \overline{\{T^n; n \geq 1\}}$. Every positively weakly rigid flow is deterministic (cf. [14]). In fact the proof shows that every positively doubly recurrent flow (cf. [20]), i.e. a flow such that every point in the product $X \times X$ is positively recurrent under $T \times T$, is deterministic. Since for a weakly rigid flow either (X,T) or (X,T^{-1}) are positively weakly rigid it follows that for a weakly rigid flow at least one of (X,T) or (X,T^{-1}) is deterministic.

2. It is known ([12]) that there exist K-flows with zero topological entropy. It would be interesting to characterize all K-flows with zero entropy.

3. It is shown in [12] that any minimal K-flow is weakly mixing. Does the the relative analogue this theorem holds? In other words, does the minimality of (X, T) imply that any K-extension π is relatively weakly mixing, i.e. the dynamical system $(\Sigma_\pi, T \times T)$ is transitive?

References

[1] F. Blanchard, B. Host, S. Ruette, *Asymptotic pairs in positive-entropy systems*, Ergodic Theory Dynam. Systems **22** (2002), 671–686.

[2] B. F. Bryant, P. Walters *Asymptotic properties of expansive homeomorphisms*, Math. Systems Theory **3** (1969), 60–66.

[3] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. **153** (1971), 401–414.

[4] M. Denker, C. Grillenberger and K. Sigmund, *Ergodic theory on compact spaces*, Lecture Notes in Math. **527**, Springer-Verlag, Berlin, Heidelberg, New York 1976.

[5] T. Downarowicz, J. Serafin, *Fiber entropy and conditional variational principles in compact non-metrizable spaces*, Fund. Math. **172** (2002), 217-247.

[6] E. Glasner, *A simple characterization of the set of μ-entropy pairs and applications*, Israel J. Math. **102** (1997), 13–27.

[7] S. Glasner and D. Maon, *Rigidity in topological dynamics*, Ergodic Theory Dynam. Systems **9** (1989), 309-320.

[8] E. Glasner, X. Ye, *Local entropy theory*, Preprint.

[9] M. Hochman, Private communication.

[10] B. Kamiński, *On regular generators of $\mathbb{Z}^2$-actions in exhaustive partitions*, Studia Math. **85** (1987), 17–26.

[11] B. Kamiński, E. Sąsiada, *Spectrum of abelian groups of transformation with completely positive entropy*, Bull. Acad. Polon. Sci. Sr. Sci. Math. Astronom. Phys. **24**, No. 9 (1976), 683–689.

[12] B. Kamiński, A. Siemaszko, J. Szymański, *The determinism and the Kolmogorov property in topological dynamics*, Bull. Pol. Acad. Sci. Math. **51**, No. 4 (2003), 401–417.

[13] B. Kamiński, A. Siemaszko, J. Szymański, *Extreme relations for topological flows*, Bull. Pol. Acad. Sci. Math. **53**, No. 1 (2005), 17–24.

[14] B. Kamiński, A. Siemaszko, J. Szymański, *On deterministic and Kolmogorov extensions for topological flows*, Topol. Methods Nonlinear Anal. **32**, No. 1 (2008), 191–204.

[15] F. Ledrappier, P. Walters, *A relativised variational principle for continuous transformations*, J. London Math. Soc. **16** (1977), 568–576.

[16] M. Lemańczyk, A. Siemaszko, *A note on the existence of a largest topological factor with zero entropy*, Proc. Amer. Math. Soc. **129**, No. 2 (2001), 475–482.

[17] K. K. Park, A. Siemaszko, *Relative topological Pinsker factors and entropy pairs*, Monatsh. Math. **134**, No. 1 (2001), 67–79.

[18] V. A. Rokhlin, *Lectures on the entropy theory of transformations with invariant measure* (Russian), Uspekhi Mat. Nauk **22** (5) (1967), 3–56.

[19] V. A. Rokhlin, Y. G. Sinai, *Construction and properties of invariant measurable partitions* (Russian), Dokl. Akad. Nauk SSSR **141** (1961), 1038–1041.

[20] B. Weiss, *Multiple recurrence and doubly minimal systems*, Contemp. Math. **215** (1998), 189–196.

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, (Poland)

E-mail address: `jerzy@mat.uni.torun.pl`