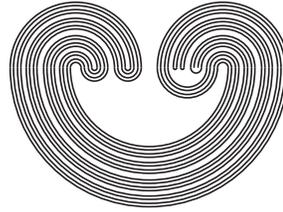


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## COARSE EMBEDDABILITY INTO BANACH SPACES

by

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## COARSE EMBEDDABILITY INTO BANACH SPACES

M.I. OSTROVSKII

**ABSTRACT.** The main purposes of this paper are (1) To survey the area of coarse embeddability of metric spaces into Banach spaces, and, in particular, coarse embeddability of different Banach spaces into each other; (2) To present new results on the problems: (a) Whether coarse non-embeddability into  $\ell_2$  implies presence of expander-like structures? (b) To what extent  $\ell_2$  is the most difficult space to embed into?

### 1. INTRODUCTION

**1.1. Basic definitions.** Let  $A$  and  $B$  be metric spaces with metrics  $d_A$  and  $d_B$ , respectively.

**Definition 1.1.** A mapping  $f : A \rightarrow B$  is called a *coarse embedding* (or a *uniform embedding*) if there exist functions  $\rho_1, \rho_2 : [0, \infty) \rightarrow [0, \infty)$  such that

1.  $\forall x, y \in A \rho_1(d_A(x, y)) \leq d_B(f(x), f(y)) \leq \rho_2(d_A(x, y))$ .
2.  $\lim_{r \rightarrow \infty} \rho_1(r) = \infty$ .

*Remark 1.2.* We prefer to use the term *coarse embedding* because in the Nonlinear Functional Analysis the term *uniform embedding* is used for uniformly continuous injective maps whose inverses are uniformly continuous on their domains of definition, see [6, p. 3].

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In some of the papers cited below the term *uniform embedding* is used.

**Definition 1.3.** A mapping  $f : A \rightarrow B$  is called *Lipschitz* if there exists a constant  $0 \leq L < \infty$  such that

$$(1.1) \quad d_B(f(x), f(y)) \leq L \cdot d_A(x, y).$$

The infimum of all  $L > 0$  for which the inequality in (1.1) is valid is called the *Lipschitz constant* of  $f$  and is denoted by  $\text{Lip}(f)$ . A Lipschitz mapping is called a *Lipschitz embedding* if it is one-to-one, and its inverse, defined as a mapping from the image of  $f$  into  $A$ , is also a Lipschitz mapping.

**Definition 1.4.** A metric space  $A$  is said to have *bounded geometry* if for each  $r > 0$  there exist a positive integer  $M(r)$  such that each ball  $B(x, r) = \{y \in A : d_A(x, y) \leq r\}$  of radius  $r$  contains at most  $M(r)$  elements.

**Definition 1.5.** A metric space is called *locally finite* if all balls in it have finitely many elements.

Our terminology and notation of Banach space theory follows [6] and [27].

**1.2. Some history and motivation.** M. Gromov [20] suggested to use coarse embeddings of Cayley graphs of infinite groups with finitely many generators and finitely many relations (with their graph-theoretical metric) into a Hilbert space or into a uniformly convex Banach space as a tool for working on such well-known conjectures as the Novikov conjecture and the Baum–Connes conjecture (discussion of these conjectures is beyond the scope of this paper). G. Yu [58] and G. Kasparov and G. Yu [32] have shown that this is indeed a very powerful tool. G. Yu [58] used the condition of coarse embeddability of metric spaces with bounded geometry into a Hilbert space; G. Kasparov and G. Yu [32] used the condition of coarse embeddability of metric spaces with bounded geometry into a uniformly convex Banach space. These results made the following problem posed by M. Gromov in [20, Problem (4)] very important:

*“Does every finitely generated or finitely presented group admit a uniformly metrically proper Lipschitz embedding into a Hilbert space? Even such an embedding into a reflexive uniformly convex Banach space would be interesting. This seems hard.”*

Also, they attracted attention to the following generalized version of the problem:

*Whether each metric space with bounded geometry is coarsely embeddable into a uniformly convex Banach space?*

The result of G. Kasparov and G. Yu [32] also made it interesting to compare classes of metric spaces embeddable into different Banach spaces (with particular interest to spaces with bounded geometry).

## 2. OBSTRUCTIONS TO EMBEDDABILITY OF SPACES WITH BOUNDED GEOMETRY

M. Gromov [19, Remark (b), p. 218] wrote: “There is no known geometric obstruction for uniform embeddings into infinite dimensional Banach spaces.” Writing this M. Gromov was unaware of P. Enflo’s work [16] in which it was shown that there is no uniformly continuous embedding with uniformly continuous inverse of the Banach space  $c_0$  into a Hilbert space. A.N. Dranishnikov, G. Gong, V. Lafforgue, and G. Yu [13, Section 6] adjusted the construction of P. Enflo [16] in order to prove that there exist locally finite metric spaces which are not coarsely embeddable into Hilbert spaces. After [13] was written, M. Gromov (see [21, p. 158]) observed that expanders provide examples of spaces with bounded geometry which are not coarsely embeddable into a Hilbert space and into  $\ell_p$  for  $1 \leq p < \infty$ . Recall the definition (see [11] for an accessible introduction to the theory of expanders).

**Definition 2.1.** For a finite graph  $G$  with vertex set  $V$  and a subset  $F \subset V$  by  $\partial F$  we denote the set of edges connecting  $F$  and  $V \setminus F$ . The *expanding constant* (also known as *Cheeger constant*) of  $G$  is

$$h(G) = \inf \left\{ \frac{|\partial F|}{|F|} : F \subset V, 0 < |F| \leq |V|/2 \right\}.$$

A sequence  $\{G_n\}$  of graphs is called a *family of expanders* if all of  $G_n$  are finite, connected,  $k$ -regular for some  $k \in \mathbb{N}$  (that is, each vertex is adjacent to exactly  $k$  other vertices), their expanding constants  $h(G_n)$  are bounded away from 0 (that is, there exists  $\varepsilon > 0$  such that  $h(G_n) \geq \varepsilon$  for all  $n$ ), and their orders (numbers of vertices) tend to  $\infty$  as  $n \rightarrow \infty$ .

We consider (vertex sets of) connected graphs as metric spaces, with their standard graph-theoretic distance: the distance between two vertices is the number of edges in the shortest path joining them.

Let  $A$  be a metric space containing isometric copies of all graphs from some family of expanders. The Gromov's observation is:  $A$  does not embed coarsely into  $\ell_p$  for  $1 \leq p < \infty$  (see [52, pp. 160–161] for a detailed proof, it is worth mentioning that the result can be proved using the argument which is well known in the theory of Lipschitz embeddings of finite metric spaces, see [37, pp. 192–193]).

M. Gromov [21] suggested to use random groups in order to prove that there exist Cayley graphs of finitely presented groups which are not coarsely embeddable into a Hilbert space. Many details on this approach were given in the paper M. Gromov [22] (some details were explained in [18], [44], and [55]). However, to the best of my knowledge, the work on clarification of all of the details of the M. Gromov's construction has not been completed (as of now).

The posed above problem about the existence of coarse embeddings of spaces with bounded geometry into uniformly convex Banach spaces was recently solved in the negative by V. Lafforgue [34], his construction is also expander-based.

N. Ozawa [47, Theorem A.1] proved that a metric space  $A$  containing isometric copies of all graphs from some family of expanders does not embed coarsely into any Banach space  $X$  such that  $B_X$  (the unit ball of  $X$ ) is uniformly homeomorphic to a subset of a Hilbert space. See [6, Chapter 9, Section 2] for results on spaces  $X$  such that  $B_X$  is uniformly homeomorphic to  $B_{\ell_2}$ .

It would be very interesting to find out whether each metric space with bounded geometry which is not coarsely embeddable into a Hilbert space contains a substructure similar to a family of expanders. A version of this problem was posed in [23] using the following terminology:

**Definition 2.2.** A metric space  $X$  *weakly contains* a family  $\{G_n\}_{n=1}^{\infty}$  of expanders with vertex sets  $\{V_n\}_{n=1}^{\infty}$  if there are maps  $f_n : V_n \rightarrow X$  satisfying

- (i)  $\sup_n \text{Lip}(f_n) < \infty$ ,
- (ii)  $\lim_{n \rightarrow \infty} \sup_{v \in V_n} \frac{|f_n^{-1}(v)|}{|V_n|} = 0$ .

*Problem 2.3.* [23, p. 261] Let  $\Gamma$  be a finitely presented group whose Cayley graph  $G$  with its natural metric is not coarsely embeddable into  $\ell_2$ . Does it follow that  $G$  weakly contains a family of expanders?

The following theorem can help with search of an expander-like structure in metric spaces with bounded geometry which are not coarsely embeddable into a Hilbert space. In the theorem we consider coarse embeddability into  $L_1 = L_1(0, 1)$ . For technical reasons it is more convenient to work with  $L_1$ . As we shall see in section 4 coarse embeddability into  $L_1$  is equivalent to coarse embeddability into a Hilbert space.

**Theorem 2.4.** *Let  $M$  be a locally finite metric space which is not coarsely embeddable into  $L_1$ . Then there exists a constant  $D$ , depending on  $M$  only, such that for each  $n \in \mathbb{N}$  there exists a finite set  $B_n \subset M \times M$  and a probability measure  $\mu$  on  $B_n$  such that*

- $d_M(u, v) \geq n$  for each  $(u, v) \in B_n$ .
- For each Lipschitz function  $f : M \rightarrow L_1$  the inequality

$$(2.1) \quad \int_{B_n} \|f(u) - f(v)\|_{L_1} d\mu(u, v) \leq D \text{Lip}(f)$$

holds.

**Lemma 2.5.** *Let  $M$  be metric space which is not coarsely embeddable in  $L_1$ . There exists a constant  $C$  depending on  $M$  only such that for each Lipschitz function  $f : M \rightarrow L_1$  there exists a subset  $B_f \subset M \times M$  such that*

$$\sup_{(x,y) \in B_f} d_M(x, y) = \infty, \text{ but } \sup_{(x,y) \in B_f} \|f(x) - f(y)\|_{L_1} \leq C \text{Lip}(f).$$

*Proof.* Assume the contrary. Then, for each  $n \in \mathbb{N}$ , the number  $n^3$  cannot serve as  $C$ . This means, that for each  $n \in \mathbb{N}$  there exists a Lipschitz mapping  $f_n : M \rightarrow L_1$  such that for each subset  $U \subset M \times M$  with

$$\sup_{(x,y) \in U} d_M(x, y) = \infty,$$

we have

$$\sup_{(x,y) \in U} \|f_n(x) - f_n(y)\| > n^3 \text{Lip}(f_n).$$

We choose a point in  $M$  and denote it by  $O$ . Without loss of generality we may assume that  $f_n(O) = 0$ . Consider the mapping

$$f : M \rightarrow \left( \sum_{n=1}^{\infty} \oplus L_1 \right)_1 \subset L_1$$

given by

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{Kn^2} \cdot \frac{f_n(x)}{\text{Lip}(f_n)},$$

where  $K = \sum_{n=1}^{\infty} \frac{1}{n^2}$ . It is clear that the series converges and  $\text{Lip}(f) \leq 1$ .

Let us show that  $f$  is a coarse embedding. We need an estimate from below only (the estimate from above is satisfied because  $f$  is Lipschitz).

Now we show that the assumption made at the beginning of the proof implies that for each  $n \in \mathbb{N}$  there is  $N \in \mathbb{N}$  such that

$$d_M(x, y) \geq N \Rightarrow \|f_n(x) - f_n(y)\| > n^3 \text{Lip}(f_n).$$

In fact, if for some  $n \in \mathbb{N}$  there is no such  $N$ , we can find a sequence of pairs  $(x_k, y_k) \in M \times M$  satisfying  $d_M(x_k, y_k) \geq k$  and  $\|f_n(x_k) - f_n(y_k)\| \leq n^3 \text{Lip}(f_n)$ . Then the set  $U = \{(x_k, y_k) : k \in \mathbb{N}\}$  violates the assumption.

On the other hand,

$$\|f_n(x) - f_n(y)\| > n^3 \text{Lip}(f_n) \Rightarrow \|f(x) - f(y)\| > \frac{n}{K}$$

Hence  $f : M \rightarrow L_1$  is a coarse embedding and we get a contradiction.  $\square$

**Lemma 2.6.** *Let  $M$  be a locally finite metric space which is not coarsely embeddable in  $L_1$ . Let  $C$  be the constant whose existence is proved in Lemma 2.5 and let  $\varepsilon$  be an arbitrary positive number. For each  $n \in \mathbb{N}$  we can find a finite subset  $M_n \subset M$  such that for each Lipschitz mapping  $f : M \rightarrow L_1$  there is a pair  $(u_{f,n}, v_{f,n}) \in M_n \times M_n$  such that*

- $d_M(u_{f,n}, v_{f,n}) \geq n$ .
- $\|f(u_{f,n}) - f(v_{f,n})\| \leq (C + \varepsilon) \text{Lip}(f)$ .

*Proof.* We choose a point in  $M$  and denote it by  $O$ . The ball in  $M$  of radius  $R$  centered at  $O$  will be denoted by  $B(R)$ . It is clear that it suffices to prove the result for fixed  $n$  and for 1-Lipschitz mappings satisfying  $f(O) = 0$ .

Assume the contrary. Since  $M$  is locally finite, this implies that for each  $R \in \mathbb{N}$  there is a 1-Lipschitz mapping  $f_R : M \rightarrow L_1$  such that  $f_R(O) = 0$  and, for  $u, v \in B(R)$ , the inequality  $d_M(u, v) \geq n$  implies  $\|f_R(u) - f_R(v)\|_{L_1} > C + \varepsilon$ .

We refer to [9], [24], or [12, Chapter 8] for results on ultraproducts, our terminology and notation follows [12]. We form an ultraproduct of the mappings  $\{f_R\}_{R=1}^\infty$ , that is, a mapping  $f : M \rightarrow (L_1)^\mathcal{U}$ , given by  $f(m) = \{f_R(m)\}_{R=1}^\infty$ , where  $\mathcal{U}$  is a non-trivial ultrafilter on  $\mathbb{N}$  and  $(L_1)^\mathcal{U}$  is the corresponding ultrapower. Each ultrapower of  $L_1$  is isometric to an  $L_1$  space on some measure space (see [12, Theorem 8.7], [9], [24]), and its separable subspaces are isometric to subspaces of  $L_1(0, 1)$  (see [14, p. 168], [27, pp. 14–15], and [53, pp. 399 & 416]). Therefore we can consider  $f$  as a mapping into  $L_1(0, 1)$ . It is easy to verify that  $\text{Lip}(f) \leq 1$  and that  $f$  satisfies the condition

$$d_M(u, v) \geq n \Rightarrow \|f(u) - f(v)\|_{L_1} \geq (C + \varepsilon).$$

We get a contradiction with the definition of  $C$ . □

*Proof of Theorem 2.4.* Let  $D$  be a number satisfying  $D > C$ , and let  $B$  be a number satisfying  $C < B < D$ .

We fix  $n \in \mathbb{N}$ . According to Lemma 2.6, there is a finite subset  $M_n \subset M$  such that for each 1-Lipschitz function  $f$  on  $M$  there is a pair  $(u, v)$  in  $M_n$  such that  $d_M(u, v) \geq n$  and  $\|f(u) - f(v)\| \leq B$ .

Let  $\alpha_n$  be the cardinality of  $M_n$ , we choose a point in  $M_n$  and denote it by  $O$ . Since both sides in (2.1) are shift-invariant and positive-homogeneous, it suffices to prove the theorem in the case when  $f : M_n \rightarrow L_1$  is a 1-Lipschitz function satisfying  $f(O) = 0$ .

Each  $\alpha_n$ -element subset of  $L_1$  is isometric to a subset in  $\ell_1^{\alpha_n(\alpha_n-1)/2}$  (see [57], [3]). Therefore it suffices to prove the result for 1-Lipschitz embeddings into  $\ell_1^{\alpha_n(\alpha_n-1)/2}$ . Now we show that it suffices to prove the existence of a probability measure  $\mu$  on  $B_n$  such that

$$(2.2) \quad \int_{B_n} \|f(u) - f(v)\| d\mu(u, v) \leq B$$

for each  $f$  from a  $\left(\frac{D-B}{2}\right)$ -net in the set of all functions satisfying the conditions mentioned above, endowed with the metric

$$\tau(f, g) = \max_{m \in M_n} \|f(m) - g(m)\|.$$

In fact, assume that we have proved the existence of a probability measure  $\mu$  such that (2.2) holds for each function  $f$  from such a  $\left(\frac{D-B}{2}\right)$ -net. Let  $g : M_n \rightarrow \ell_1^{\alpha_n(\alpha_n-1)/2}$  be a 1-Lipschitz function satisfying  $g(O) = 0$ . Let  $f$  be an element of the net satisfying  $\tau(f, g) \leq \left(\frac{D-B}{2}\right)$ . Since  $\mu$  is a probability measure, we have

$$\begin{aligned} & \int_{B_n} \|g(u) - g(v)\| d\mu(u, v) \\ & \leq \int_{B_n} \|f(u) - f(v)\| d\mu(u, v) + 2 \left( \frac{D-B}{2} \right) \\ & \leq B + (D-B) = D; \end{aligned}$$

that is, we have proved the inequality (2.1) for the 1-Lipschitz function  $g$ .

By compactness there exists a finite net satisfying the condition. Let  $N$  be such a net. We are going to use the minimax theorem, see, e.g. [56, p. 344]. In particular, we use the notation similar to the one used in [56].

Let  $A$  be the matrix whose columns are labelled by functions from  $N$ , whose rows are labelled by pairs  $(u, v)$  of elements of  $M_n$  satisfying  $d_M(u, v) \geq n$ , and whose entry on the intersection of the column corresponding to  $f$ , and the row corresponding to  $(u, v)$  is  $\|f(u) - f(v)\|$ .

Then, for each column vector  $x = \{x_f\}_{f \in N}$  with  $x_f \geq 0$  and  $\sum_{f \in N} x_f = 1$ , the entries of the product  $Ax$  are the differences

$$\|F(u) - F(v)\|, \text{ where } F : M \rightarrow \left( \sum_{f \in N} \oplus \ell_1^{\alpha_n(\alpha_n-1)/2} \right)_1 \text{ is given}$$

by  $F(m) = \sum_{f \in N} x_f f(m)$ . The function  $F$  can be considered as a

function into  $L_1$ . It satisfies  $\text{Lip}(F) \leq 1$ . Hence there is a pair  $(u, v)$  in  $M_n$  satisfying  $d_M(u, v) \geq n$  and  $\|F(u) - F(v)\| \leq B$ .

Therefore we have

$$\max_x \min_\mu \mu Ax \leq B,$$

where the minimum is taken over all vectors  $\mu = \{\mu(u, v)\}$ , indexed by  $u, v \in M_n$ ,  $d_M(u, v) \geq n$ , and satisfying the conditions  $\mu(u, v) \geq 0$  and  $\sum \mu(u, v) = 1$ .

By the von Neumann minimax theorem [56, p. 344], we have

$$\min_\mu \max_x \mu Ax \leq B,$$

which is exactly the inequality we need to prove because  $\mu$  can be regarded as a probability measure on the set of pairs from  $M_n$  with distance  $\geq n$ .  $\square$

### 3. COARSE EMBEDDABILITY INTO REFLEXIVE BANACH SPACES

The first result of this nature was obtained by N. Brown and E. Guentner [8, Theorem 1]. They proved that for each metric space  $A$  having bounded geometry there is a sequence  $\{p_n\}$ ,  $p_n > 1$ ,  $\lim_{n \rightarrow \infty} p_n = \infty$  such that  $A$  embeds coarsely into the Banach space  $(\sum_{n=1}^{\infty} \oplus \ell_{p_n})_2$ , which is, obviously, reflexive.

This result was strengthened in [5], [31], and [45]. (Observe that the space  $(\sum_{n=1}^{\infty} \oplus \ell_{p_n})_2$  has no cotype.)

**Theorem 3.1.** [45] *Let  $X$  be a Banach space with no cotype and let  $A$  be a locally finite metric space. Then  $A$  embeds coarsely into  $X$ .*

**Theorem 3.2.** [5] *Let  $X$  be a Banach space with no cotype and let  $A$  be a locally finite metric space. Then there exists a Lipschitz embedding of  $A$  into  $X$ .*

*Remark 3.3.* Interested readers can reconstruct the proof from [5] by applying Proposition 6.1 (see below) to  $Z = c_0$  in combination with the result of I. Aharoni mentioned in Section 4.1.

**Definition 3.4.** A metric space  $(X, d)$  is called *stable* if for any two bounded sequences  $\{x_n\}$  and  $\{y_n\}$  in  $X$  and for any two non-trivial ultrafilters  $\mathcal{U}$  and  $\mathcal{V}$  on  $\mathbb{N}$  the condition

$$\lim_{n, \mathcal{U}} \lim_{m, \mathcal{V}} d(x_n, y_m) = \lim_{m, \mathcal{V}} \lim_{n, \mathcal{U}} d(x_n, y_m)$$

holds.

**Theorem 3.5.** [31] *Let  $A$  be a stable metric space. Then  $A$  embeds coarsely into a reflexive Banach space.*

*Remark 3.6.* It is easy to see that locally finite metric spaces are stable.

N.J. Kalton [31] found examples of Banach spaces which are not coarsely embeddable into reflexive Banach spaces,  $c_0$  is one of the examples of such spaces. Apparently his result provides the first example of a metric spaces which is not coarsely embeddable into reflexive Banach spaces. (See [48, Introduction].)

#### 4. COARSE CLASSIFICATION OF BANACH SPACES

As we already mentioned the result of G. Kasparov and G. Yu [32] makes it very interesting to compare the conditions of coarse embeddability into a Banach space  $X$  for different spaces  $X$ . Since compositions of coarse embeddings are coarse embeddings, one can approach this problem by studying coarse embeddability of Banach space into each other. In this subsection we describe the existing knowledge on this matter.

**4.1. Essentially nonlinear coarse embeddings.** There are many examples of pairs  $(X, Y)$  of Banach spaces such that  $X$  is coarsely embeddable into  $Y$ , but the Banach-space-theoretical structure of  $X$  is quite different from the Banach-space-theoretical structure of each subspace of  $Y$ :

- A result which goes back to I.J. Schoenberg [54] (see [38, p. 385] for a simple proof) states that  $L_1$  with the metric  $\sqrt{\|x - y\|_1}$  is isometric to a subset of  $L_2$ . Hence  $L_1$  and all of its subspaces, in particular,  $L_p$  and  $\ell_p$  ( $1 \leq p \leq 2$ ) (see [29] and [7]) embed coarsely into  $L_2 = \ell_2$ .
- This result was generalized by M. Mendel and A. Naor [39, Remark 5.10]: For every  $1 \leq q < p$  the metric space  $(L_q, \|x - y\|_{L_q}^{q/p})$  is isometric to a subspace of  $L_p$ .
- The well-known result of I. Aharoni [1] implies that each separable Banach space is coarsely embeddable into  $c_0$  (although its Banach space theoretical properties can be quite different from those of any subspace of  $c_0$ ). A simpler proof of this result was obtained in [2], see, also, [6, p. 176].

- N.J. Kalton [30] proved that  $c_0$  embeds coarsely into a Banach space with the Schur property.
- P. Nowak [42] proved that  $\ell_2$  is coarsely embeddable into  $\ell_p$  for all  $1 \leq p \leq \infty$ .

#### 4.2. Obstructions to coarse embeddability of Banach spaces.

The list of discovered obstructions to coarse embeddability also constantly increases:

- Only minor adjustments of the argument of Y. Raynaud [51] (see, also [6, pp. 212–215]) are needed to prove the following results:
  - (1) Let  $A$  be a Banach space with a spreading basis which is not an unconditional basis. Then  $A$  does not embed coarsely into a stable metric space. (See [6, p. 429] for the definition of a spreading basis and [33] for examples of stable Banach spaces. Examples of stable Banach spaces include  $L_p$  ( $1 \leq p < \infty$ ).)
  - (2) Let  $A$  be a nonreflexive Banach space with non-trivial type. Then  $A$  does not embed coarsely into a stable metric space. (Examples of nonreflexive Banach spaces with non-trivial type were constructed in [25], [26], [49].)
- A.N. Dranishnikov, G. Gong, V. Lafforgue, and G. Yu [13] adjusted the argument of P. Enflo [16] to prove that Banach spaces with no cotype are not coarsely embeddable into  $\ell_2$ .
- W. B. Johnson and L. Randrianarivony [28] proved that  $\ell_p$  ( $p > 2$ ) is not coarsely embeddable into  $\ell_2$ .
- M. Mendel and A. Naor [40] proved (for  $K$ -convex spaces) that cotype of a Banach space is an obstruction to coarse embeddability, in particular,  $\ell_p$  is not coarsely embeddable into  $\ell_q$  when  $p > q \geq 2$ .
- L. Randrianarivony [50] strengthened the result from [28] to a characterization of quasi-Banach spaces which embed coarsely into a Hilbert space, and proved: a separable Banach space is coarsely embeddable into a Hilbert space if and only if it is isomorphic to a subspace of  $L_0(\mu)$ .
- N.J. Kalton [31] found some more obstructions to coarse embeddability. In particular, N.J. Kalton discovered an invariant, which he named the  $\mathcal{Q}$ -property, which is necessary for coarse embeddability into reflexive Banach spaces.

**4.3. To what extent is  $\ell_2$  the most difficult space to embed into?** Because  $\ell_2$  is, in many respects, the ‘best’ space, and because of Dvoretzky’s theorem (see [15] and [41]) it is natural to expect that  $\ell_2$  is among the most difficult spaces to embed into. The strongest possible result in this direction would be a positive solution to the following problem.

*Problem 4.1.* Does  $\ell_2$  embed coarsely into an arbitrary infinite dimensional Banach space?

This problem is still open, but the coarse embeddability of  $\ell_2$  is known for wide classes of Banach spaces. As was mentioned above, P.W. Nowak [42] proved that  $\ell_2$  embeds coarsely into  $\ell_p$  for each  $1 \leq p \leq \infty$ . In Section 5 we prove that  $\ell_2$  embeds coarsely into a Banach space containing a subspace with an unconditional basis which does not contain  $\ell_\infty^n$  uniformly (Theorem 5.1). This result is a generalization of P.W. Nowak’s result mentioned above because the spaces  $\ell_p$  ( $1 \leq p < \infty$ ) satisfy the condition of Theorem 5.1, but the spaces satisfying the condition of Theorem 5.1 do not necessarily contain subspaces isomorphic to  $\ell_p$  (see [17], and [36, Section 2.e]).

In all existing applications of coarse embeddability results, the most important is the case when we embed spaces with bounded geometry into Banach spaces. In this connection the following result from [46] is of interest.

**Theorem 4.2** ([46]). *Let  $A$  be a locally finite metric space which embeds coarsely into a Hilbert space, and let  $X$  be an infinite dimensional Banach space. Then there exists a coarse embedding  $f : A \rightarrow X$ .*

In this paper we use an idea of F. Baudier and G. Lancien [5], and prove this result in a stronger form, for Lipschitz embeddings (see Section 6):

**Theorem 4.3.** *Let  $M$  be a locally finite subset of a Hilbert space. Then  $M$  is Lipschitz embeddable into an arbitrary infinite dimensional Banach space.*

5. COARSE EMBEDDINGS OF  $\ell_2$ 

**Theorem 5.1.** *Let  $X$  be a Banach space containing a subspace with an unconditional basis which does not contain  $\ell_\infty^n$  uniformly. Then  $\ell_2$  embeds coarsely into  $X$ .*

*Proof.* We use the criterion for coarse embeddability into a Hilbert space due to M. Dadarlat and E. Guentner [10, Proposition 2.1] (see [35] and [42] for related results). We state it as a lemma (by  $S(X)$  we denote the unit sphere of a Banach space  $X$ ).

**Lemma 5.2** ([10]). *A metric space  $A$  admits a coarse embedding into  $\ell_2$  if and only if for every  $\varepsilon > 0$  and every  $R > 0$  there exists a map  $\zeta : A \rightarrow S(\ell_2)$  such that*

- (i)  $d_A(x, y) \leq R$  implies  $\|\zeta(x) - \zeta(y)\| \leq \varepsilon$ .
- (ii)  $\lim_{t \rightarrow \infty} \inf\{\|\zeta(x) - \zeta(y)\| : x, y \in A, d_A(x, y) \geq t\} = \sqrt{2}$ .

We assume without loss of generality that  $X$  has an unconditional basis  $\{e_i\}_{i \in \mathbb{N}}$ . Let  $\mathbb{N} = \cup_{i=1}^\infty \mathbb{N}_i$  be a partition of  $\mathbb{N}$  into infinitely many infinite subsets. Let  $X_i = \text{cl}(\text{span}\{e_i\}_{i \in \mathbb{N}_i})$ . By the theorem of E. Odell and T. Schlumprecht [43] (see, also, [6, Theorem 9.4]), for each  $i \in \mathbb{N}$  there exists a uniform homeomorphism  $\varphi_i : S(\ell_2) \rightarrow S(X_i)$ . We apply Lemma 5.2 in the case when  $A = \ell_2$ . By the uniform continuity of  $\varphi_i$  and  $\varphi_i^{-1}$  we get: for each  $i \in \mathbb{N}$  there exists  $\delta_i > 0$  and a map  $\zeta_i : \ell_2 \rightarrow S(X_i)$  such that

$$(5.1) \quad \liminf_{t \rightarrow \infty} \{\|\zeta_i(x) - \zeta_i(y)\|_{X_i} : \|x - y\|_{\ell_2} \geq t\} \geq \delta_i.$$

$$(5.2) \quad \|x - y\|_{\ell_2} \leq i \text{ implies } \|\zeta_i(x) - \zeta_i(y)\|_{X_i} \leq \frac{\delta_i}{i2^i}.$$

(We use the conclusion (ii) of Lemma 5.2 to get (5.1), and (i) to get (5.2).)

Fix  $x_0 \in \ell_2$ . Let  $f : \ell_2 \rightarrow X$  be the map defined as the direct sum of the maps  $\frac{i}{\delta_i}(\zeta_i(x) - \zeta_i(x_0))$ . We claim that it is a coarse embedding (the fact that it is a well-defined map follows from (5.2)).

Let  $\|x - y\| = r$ , then for  $i \geq r$  we get  $\|\frac{i}{\delta_i}\zeta_i(x) - \frac{i}{\delta_i}\zeta_i(y)\|_{X_i} \leq \frac{1}{2^i}$ . Hence  $\|f(x) - f(y)\| \leq \sum_{i=1}^{\lceil r \rceil - 1} \frac{2i}{\delta_i} + \sum_{i=\lceil r \rceil}^\infty \frac{1}{2^i} =: \rho_2(r)$ . We proved an estimate from above.

To prove an estimate from below, it is enough, for a given  $h \in \mathbb{R}$ , to find  $t \in \mathbb{R}$  such that  $\|x - y\|_{\ell_2} \geq t$  implies  $\|f(x) - f(y)\|_X \geq h$ . Observe that we can find  $i \in \mathbb{N}$  and  $t \in \mathbb{R}$  such that  $\|x - y\|_{\ell_2} \geq t$  implies  $\|\frac{i}{\delta_i}\zeta_i(x) - \frac{i}{\delta_i}\zeta_i(y)\|_{X_i} \geq h$ . In fact, choose an arbitrary  $i > h$ . The conclusion follows from the condition (5.1). On the other hand, each space with an unconditional basis has an equivalent norm in which it is unconditionally monotone (see [27, p. 9]). The unconditional monotonicity implies  $\|x_i\|_{X_i} \leq \|\sum_{k=1}^{\infty} x_k\|$  for  $x_k \in X_k$ ,  $k \in \mathbb{N}$ . This gives the desired estimate from below:

$$\|f(x) - f(y)\| \geq \left\| \frac{i}{\delta_i}\zeta_i(x) - \frac{i}{\delta_i}\zeta_i(y) \right\|_{X_i} \geq h. \quad \square$$

## 6. LIPSCHITZ EMBEDDINGS OF LOCALLY FINITE METRIC SPACES

The purpose of this section is to prove Theorem 4.3. We prove the main step in our argument (Proposition 6.1) in a somewhat more general context than is needed for Theorem 4.3, because it can be applied in some other situations (see, in this connection, the paper [4] containing two versions of Proposition 6.1). The coarse version of this result was proved in [46], in the proof of the Lipschitz version we use an idea from [5].

**Proposition 6.1.** *Let  $A$  be a locally finite subset of a Banach space  $Z$ . Then there exists a sequence of finite dimensional linear subspaces  $Z_i$  ( $i \in \mathbb{N}$ ) of  $Z$  such that  $A$  is Lipschitz embeddable into each Banach space  $Y$  having a finite dimensional Schauder decomposition  $\{Y_i\}_{i=1}^{\infty}$  with  $Y_i$  linearly isometric to  $Z_i$ .*

See [36, Section 1.g] for information on Schauder decompositions. It is clear that we may restrict ourselves to the case when the Schauder decomposition satisfies

$$(6.1) \quad \|y_i\| \leq \left\| \sum_{i=1}^{\infty} y_i \right\| \quad \text{when } y_i \in Y_i \text{ for each } i \in \mathbb{N}.$$

*Proof.* Let  $Z_i$  be the linear subspace of  $Z$  spanned by  $\{a \in A : \|a\|_Z \leq 2^i\}$  and let  $S_i = \{a \in A : 2^{i-1} \leq \|a\|_Z \leq 2^i\}$ . Let  $T_i : Z_i \rightarrow Y_i$  be some linear isometries and let  $E_i : Z_i \rightarrow Y$  be compositions of these linear isometries with the natural embeddings  $Y_i \rightarrow Y$ . We define an embedding  $\varphi : A \rightarrow Y$  by

$$\varphi(a) = \frac{2^i - \|a\|_Z}{2^{i-1}} E_i(a) + \frac{\|a\|_Z - 2^{i-1}}{2^{i-1}} E_{i+1}(a) \text{ for } a \in S_i.$$

One can check that there is no ambiguity for  $\|a\|_Z = 2^i$ .

*Remark 6.2.* The mapping  $\varphi$  is a straightforward generalization of the mapping constructed in [5].

It remains to verify that  $\varphi$  is a Lipschitz embedding. We consider three cases.

- (1)  $a, b$  are in the same  $S_i$ ;
- (2)  $a, b$  are in consecutive sets  $S_i$ , that is,  $b \in S_i, a \in S_{i+1}$ ;
- (3)  $a, b$  are in 'distant' sets  $S_i$ , that is,  $b \in S_i, a \in S_k, k \geq i+2$ .

Everywhere in the proof we assume  $\|a\| \geq \|b\|$ .

CASE (1). The inequality (6.1) implies that the number

$$\|\varphi(a) - \varphi(b)\|_Y$$

is between the maximum and the sum of the numbers

$$(6.2) \quad \left\| \frac{2^i - \|a\|_Z}{2^{i-1}} E_i(a) - \frac{2^i - \|b\|_Z}{2^{i-1}} E_i(b) \right\|,$$

$$(6.3) \quad \left\| \frac{\|a\|_Z - 2^{i-1}}{2^{i-1}} E_{i+1}(a) - \frac{\|b\|_Z - 2^{i-1}}{2^{i-1}} E_{i+1}(b) \right\|.$$

It is clear that the norm in (6.2) is between the numbers

$$\frac{2^i - \|a\|_Z}{2^{i-1}} \|E_i(a) - E_i(b)\| \mp \frac{\|a\|_Z - \|b\|_Z}{2^{i-1}} \|E_i(b)\|,$$

and the norm in (6.3) is between the numbers

$$\frac{\|a\|_Z - 2^{i-1}}{2^{i-1}} \|E_{i+1}(a) - E_{i+1}(b)\| \mp \frac{\|a\|_Z - \|b\|_Z}{2^{i-1}} \|E_{i+1}(b)\|.$$

Therefore

$$\begin{aligned} \frac{1}{2} \left( \|a - b\|_Z - \frac{\|a\|_Z - \|b\|_Z}{2^{i-2}} \|b\|_Z \right) &\leq \|\varphi(a) - \varphi(b)\|_Y \\ &\leq \|a - b\|_Z + \frac{\|a\|_Z - \|b\|_Z}{2^{i-2}} \|b\|_Z. \end{aligned}$$

This inequality implies a suitable estimate from above for the Lipschitz constant of  $\varphi$ , and an estimate for the Lipschitz constant of its inverse in the case when  $\|a - b\|_Z$  is much larger than

$\|a\|_Z - \|b\|_Z$ , for example, if  $\|a - b\|_Z \geq 5(\|a\|_Z - \|b\|_Z)$ . To complete the proof in the case (1) it suffices to estimate  $\|\varphi(a) - \varphi(b)\|$  from below in the case when  $\|a\|_Z - \|b\|_Z \geq \frac{\|a - b\|_Z}{5}$ . In this case we use the observation that for  $a, b \in S_i$  satisfying  $\|a\|_Z \geq \|b\|_Z$  the sum of (6.2) and (6.3) can be estimated from below by

$$\begin{aligned} & \left( \frac{2^i - \|a\|_Z}{2^{i-1}} \|a\|_Z - \frac{2^i - \|b\|_Z}{2^{i-1}} \|b\|_Z \right) \\ & + \left( \frac{\|a\|_Z - 2^{i-1}}{2^{i-1}} \|a\|_Z - \frac{\|b\|_Z - 2^{i-1}}{2^{i-1}} \|b\|_Z \right) \\ & = \|a\|_Z - \|b\|_Z \geq \frac{\|a - b\|_Z}{5}. \end{aligned}$$

This completes our proof in the case (1).

CASE (2). The inequality (6.1) implies that the number

$$\|\varphi(a) - \varphi(b)\|_Y$$

is between the maximum and the sum of the numbers

$$(6.4) \quad \left\| \frac{2^i - \|b\|_Z}{2^{i-1}} E_i(b) \right\|,$$

$$(6.5) \quad \left\| \frac{2^{i+1} - \|a\|_Z}{2^i} E_{i+1}(a) - \frac{\|b\|_Z - 2^{i-1}}{2^{i-1}} E_{i+1}(b) \right\|,$$

$$(6.6) \quad \left\| \frac{\|a\|_Z - 2^i}{2^i} E_{i+2}(a) \right\|.$$

Both (6.4) and (6.6) are estimated from above by  $2(\|a\|_Z - \|b\|_Z)$ . As for (6.5), we have

$$(6.7) \quad \begin{aligned} & \left\| \frac{2^{i+1} - \|a\|_Z}{2^i} E_{i+1}(a) - \frac{\|b\|_Z - 2^{i-1}}{2^{i-1}} E_{i+1}(b) \right\| \\ & = \left\| \frac{2^i - (\|a\|_Z - 2^i)}{2^i} a + \frac{(2^i - \|b\|_Z) - 2^{i-1}}{2^{i-1}} b \right\|_Z \end{aligned}$$

$$\leq \|a - b\|_Z + 2(\|a\|_Z - 2^i) + 2(2^i - \|b\|_Z) \leq 3\|a - b\|_Z.$$

We turn to estimate from below. From (6.4) and (6.6) we get

$$\|\varphi(a) - \varphi(b)\| \geq \max\{(2^i - \|b\|_Z), (\|a\|_Z - 2^i)\}.$$

Therefore it suffices to find an estimate in the case when

$$(6.8) \quad \max\{(2^i - \|b\|_Z), (\|a\|_Z - 2^i)\} \leq \frac{\|a - b\|_Z}{5}.$$

Rewriting (6.5) in the same way as in (6.7), we get

$$\|\varphi(a) - \varphi(b)\|_Y \geq \left\| (a - b) + \frac{2^i - \|b\|_Z}{2^{i-1}}b - \frac{\|a\|_Z - 2^i}{2^i}a \right\|$$

In the case when (6.8) is satisfied, we can continue this chain of inequalities with

$$\geq \|a - b\|_Z - \frac{4}{5}\|a - b\|_Z = \frac{1}{5}\|a - b\|_Z.$$

CASE (3). In this case the number  $\|\varphi(a) - \varphi(b)\|_Y$  is between the maximum and the sum of the four numbers:

$$\frac{2^i - \|b\|_Z}{2^{i-1}}\|b\|_Z, \frac{\|b\|_Z - 2^{i-1}}{2^{i-1}}\|b\|_Z,$$

$$\frac{2^k - \|a\|_Z}{2^{k-1}}\|a\|_Z, \frac{\|a\|_Z - 2^{k-1}}{2^{k-1}}\|a\|_Z.$$

Hence  $\|\varphi(a) - \varphi(b)\|_Y$  is between  $\frac{\|a\|_Z}{2}$  (= the average of the last two numbers) and  $\|a\|_Z + \|b\|_Z$  (=the sum of all four numbers).

On the other hand,

$$\frac{1}{2}\|a\|_Z \leq \|a\|_Z - \|b\|_Z \leq \|a - b\|_Z \leq \|a\|_Z + \|b\|_Z \leq 2\|a\|_Z.$$

These inequalities immediately imply estimates for Lipschitz constants.  $\square$

*Proof of Theorem 4.3.* Each finite dimensional subspace of  $\ell_2$  is isometric to  $\ell_2^k$  for some  $k \in \mathbb{N}$ . By Proposition 6.1 there exists a sequence  $\{n_i\}_{i=1}^\infty$  such that  $A$  embeds coarsely into each Banach space  $Y$  having a Schauder decomposition  $\{Y_i\}$  with  $Y_i$  isometric to  $\ell_2^{n_i}$ . On the other hand, using Dvoretzky's theorem ([15], see, also, [41, Section 5.8]) and the standard techniques of constructing basic sequences (see [36, p. 4]), it is easy to prove that for an arbitrary sequence  $\{n_i\}_{i=1}^\infty$  an arbitrary infinite dimensional Banach space  $X$  contains a subspace isomorphic to a space having such Schauder decomposition.  $\square$

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**Added in proof.** (August 2008) The following recent papers are closely related to the subject of this paper: (1) G. Pisier, Complex Interpolation between Hilbert, Banach and Operator spaces, arXiv:0802.0476; (2) R. Tessera, Coarse embeddings into a Hilbert space, Haagerup Property and Poincare inequalities, arXiv:0802.2541.

#### REFERENCES

- [1] I. Aharoni, *Every separable metric space is Lipschitz equivalent to a subset of  $c_0^+$* , Israel J. Math., **19** (1974), 284–291.
- [2] P. Assouad, *Remarques sur un article de Israel Aharoni sur les prolongements lipschitziens dans  $c_0$* , Israel J. Math., **31** (1978), no. 1, 97–100.
- [3] K. Ball, *Isometric embedding in  $l_p$ -spaces*, European J. Combin., **11** (1990), no. 4, 305–311.
- [4] F. Baudier, *Metrical characterization of super-reflexivity and linear type of Banach spaces*, Archiv Math., **89** (2007), no. 5, 419–429.
- [5] F. Baudier, G. Lancien, *Embeddings of locally finite metric spaces into Banach spaces*, Proc. Amer. Math. Soc., **136** (2008), 1029–1033.
- [6] Y. Benyamini, J. Lindenstrauss, *Geometric nonlinear functional analysis*, volume 1, Providence, R.I., AMS, 2000.
- [7] J. Bretagnolle, D. Dacuhna-Castelle, J.-L. Krivine, *Lois stables et espaces  $L^p$* , Ann. Inst. H. Poincaré Sect. B (N.S.), **2** (1965/1966) 231–259.
- [8] N. Brown, E. Guentner, *Uniform embeddings of bounded geometry spaces into reflexive Banach space*, Proc. Amer. Math. Soc., **133** (2005), no. 7, 2045–2050.
- [9] D. Dacuhna-Castelle, J.-L. Krivine, *Applications des ultraproducts à l'étude des espaces et des algèbres de Banach*, Studia Math., **41** (1972), 315–334.
- [10] M. Dadarlat, E. Guentner, *Constructions preserving Hilbert space uniform embeddability of discrete groups*, Trans. Amer. Math. Soc., **355** (2003), no. 8, 3253–3275.
- [11] G. Davidoff, P. Sarnak, and A. Valette, *Elementary Number Theory, Group Theory and Ramanujan Graphs*, Cambridge University Press, 2003.
- [12] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*, Cambridge Studies in Advanced Mathematics, **43**, Cambridge University Press, Cambridge, 1995.
- [13] A. N. Dranishnikov, G. Gong, V. Lafforgue, G. Yu, *Uniform embeddings into Hilbert space and a question of Gromov*, Canad. Math. Bull., **45** (2002), no. 1, 60–70.

- [14] N. Dunford and J. T. Schwartz, *Linear Operators. Part I: General Theory*, New York, Interscience Publishers, 1958.
- [15] A. Dvoretzky, *Some results on convex bodies and Banach spaces*, in: Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), pp. 123–160, Jerusalem Academic Press, Jerusalem; Pergamon, Oxford, 1961.
- [16] P. Enflo, *On a problem of Smirnov*, Arkiv Mat., **8** (1969), 107–109.
- [17] T. Figiel, W. B. Johnson, *A uniformly convex Banach space which contains no  $\ell_p$* , Compositio Math., **29** (1974), 179–190.
- [18] É. Ghys, *Groupes aléatoires (d’après Misha Gromov, ...)*, Astérisque, No. 294 (2004), viii, 173–204.
- [19] M. Gromov, *Asymptotic invariants of infinite groups*, in: A. Niblo, M. Roller (Eds.) Geometric group theory, London Math. Soc. Lecture Notes, **182**, 1–295, Cambridge University Press, 1993.
- [20] M. Gromov, *Problems (4) and (5)*, p. 67 in: Novikov conjectures, index theorems and rigidity. Vol. 1. Including papers from the conference held at the Mathematisches Forschungsinstitut Oberwolfach, Oberwolfach, September 6–10, 1993, Edited by S. C. Ferry, A. Ranicki, J. Rosenberg, London Mathematical Society Lecture Note Series, **226**, Cambridge University Press, Cambridge, 1995.
- [21] M. Gromov, *Spaces and questions*, GAFA 2000 (Tel Aviv, 1999), Geom. Funct. Anal. 2000, Special Volume, Part I, 118–161.
- [22] M. Gromov, *Random walk in random groups*, Geom. Funct. Anal., **13** (2003), no. 1, 73–146.
- [23] E. Guentner, J. Kaminker, *Geometric and analytic properties of groups*, in: Noncommutative geometry, Ed. by S. Doplicher and R. Longo, Lecture Notes Math., **1831** (2004), 253–262.
- [24] S. Heinrich, *Ultraproducts in Banach space theory*, J. Reine Angew. Math., **313** (1980), 72–104.
- [25] R. C. James, *A nonreflexive Banach space that is uniformly nonoctahedral*, Israel J. Math., **18** (1974), 145–155.
- [26] R. C. James and J. Lindenstrauss, *The octahedral problem for Banach spaces*, in: Proceedings of the Seminar on Random Series, Convex Geometry and Geometry of Banach Spaces, Aarhus, October 14 - October 20, 1974; Aarhus Universitet, 1975, pp. 100–120.
- [27] W. B. Johnson and J. Lindenstrauss, *Basic concepts in the geometry of Banach spaces*, in: Handbook of the geometry of Banach spaces (W.B. Johnson and J. Lindenstrauss, Eds.) Vol. **1**, Elsevier, Amsterdam, 2001, pp. 1–84.
- [28] W. B. Johnson, L. Randrianarivony,  *$\ell_p(p > 2)$  does not coarsely embed into a Hilbert space*, Proc. Amer. Math. Soc., **134** (2006), no. 4, 1045–1050.
- [29] M. I. Kadets, *Linear dimension of the spaces  $L_p$  and  $l_q$*  (Russian), Uspehi Mat. Nauk, **13** (1958), no. 6, 95–98.
- [30] N. J. Kalton, *Spaces of Lipschitz and Hölder functions and their applications*, Collect. Math., **55** (2004), 171–217.

- [31] N. J. Kalton, *Coarse and uniform embeddings into reflexive spaces*, Quart. J. Math. Oxford, **58** (2007), 393–414.
- [32] G. Kasparov, G. Yu, *The coarse geometric Novikov conjecture and uniform convexity*, Adv. Math., **206** (2006), no. 1, 1–56.
- [33] J.-L. Krivine, B. Maurey, *Espaces de Banach stables*, Israel J. Math., **39** (1981), 273–295.
- [34] V. Lafforgue, *Un renforcement de la propriété (T)*, Duke Math. J., **143** (2008), no. 3, 559–602.
- [35] J. Li, Q. Wang, *Remarks on coarse embeddings of metric spaces into uniformly convex Banach spaces*, J. Math. Anal. Appl., **320** (2006), 892–901.
- [36] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces*, v. I, Berlin, Springer-Verlag, 1977.
- [37] J. Matoušek, *On embedding expanders into  $\ell_p$  spaces*, Israel J. Math., **102** (1997), 189–197.
- [38] J. Matoušek, *Embedding finite metric spaces into normed spaces*, an updated version of Chapter 15 of the book: J. Matoušek, Lectures on Discrete Geometry, Springer-Verlag, 2002, available at <http://kam.mff.cuni.cz/~matousek/>.
- [39] M. Mendel, A. Naor, *Euclidean quotients of finite metric spaces*, Adv. Math., **189** (2004), no. 2, 451–494.
- [40] M. Mendel, A. Naor, *Metric cotype*, Ann. Math., **168** (2008), 247–298.
- [41] V. D. Milman, G. Schechtman, *Asymptotic theory of finite dimensional normed spaces*, Berlin, Springer-Verlag, 1986.
- [42] P. W. Nowak, *On coarse embeddability into  $\ell_p$ -spaces and a conjecture of Dranishnikov*, Fund. Math., **189** (2006), no. 2, 111–116.
- [43] E. Odell, T. Schlumprecht, *The distortion problem*, Acta Math., **173** (1994), no. 2, 259–281.
- [44] Y. Ollivier, *A January 2005 invitation to random groups*, Sociedade Brasileira de Matematica, available at <http://www.umpa.ens-lyon.fr/~yollivie/>
- [45] M. I. Ostrovskii, *Coarse embeddings of locally finite metric spaces into Banach spaces without cotype*, Comptes rendus de l'Academie bulgare des Sciences, **59** (2006), no. 11, 1113–1116, available at <http://facpub.stjohns.edu/ostrovsm>
- [46] M. I. Ostrovskii, *On comparison of the coarse embeddability into a Hilbert space and into other Banach spaces*, unpublished manuscript, 2006, available at <http://facpub.stjohns.edu/ostrovsm>
- [47] N. Ozawa, *A note on non-amenability of  $\mathcal{B}(\ell_p)$  for  $p = 1, 2$* , Internat. J. Math. **15** (2004), no. 6, 557–565.
- [48] V. G. Pestov, *A theorem of Hrushovski–Solecki–Vershik applied to uniform and coarse embeddings of the Urysohn metric space*, to appear in Proceedings of the International Workshop on the Urysohn metric space (Ben-Gurion University, Israel, May 21–14, 2006), a Special Volume of *Topology and its Applications*, available at <http://front.math.ucdavis.edu/>

- [49] G. Pisier and Q. Xu, *Random series in the real interpolation spaces between the spaces  $v_p$* , in: Geometrical Aspects of Functional Analysis. Israel Seminar, 1985-86, J. Lindenstrauss and V. D. Milman (Eds.), *Lecture Notes Math.*, **1267** (1987), 185–209.
- [50] L. Randrianarivony, *Characterization of quasi-Banach spaces which coarsely embed into a Hilbert space*, Proc. Amer. Math. Soc., **134** (2006), no. 5, 1315–1317.
- [51] Y. Raynaud, *Espaces de Banach superstables, distances stables et homéomorphismes uniformes*, Israel J. Math., **44** (1983), 33–52.
- [52] J. Roe, *Lectures on coarse geometry*, University Lecture Series, **31**, American Mathematical Society, Providence, R.I., 2003.
- [53] H. L. Royden, *Real Analysis*, Third edition, Mcmillan, New York, 1988.
- [54] I. J. Schoenberg, *Metric spaces and positive definite functions*, Trans. Amer. Math. Soc., **44** (1938), 522–553.
- [55] L. Silberman, *Addendum to: “Random walk in random groups” by M. Gromov*, Geom. Funct. Anal., **13** (2003), no. 1, 147–177.
- [56] G. Strang, *Linear algebra and its applications*, Second edition, New York, Academic Press, 1980.
- [57] H. S. Witsenhausen, *Minimum dimension embedding of finite metric spaces*, J. Combin. Theory, Ser. A **42** (1986), no. 2, 184–199.
- [58] G. Yu, *The Coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Inventiones Math., **139** (2000), 201–240.

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