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EXTREME NON-ARENS REGULARITY OF SEMIGROUP ALGEBRAS

MAHMOUD FILALI[1] AND TERO VEDENJUOKSU

Abstract. Let S be an infinite, discrete, weakly cancellative semigroup, $\ell^\infty(S)$ the space of bounded functions on S, and $WAP(S)$ the subspace of weakly almost periodic functions. Then the quotient space $\ell^\infty(S)/WAP(S)$ contains an isometric linear copy of $\ell^\infty(S)$. This implies that the semigroup algebra $\ell^1(S)$ is extremely non-Arens regular.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{A}^{**}$ be its second dual. In [1], R. Arens introduced two Banach algebra products on $\mathcal{A}^{**}$ extending that of $\mathcal{A}$ and making $\mathcal{A}^{**}$ a Banach algebra. The products of $a'', b'' \in \mathcal{A}^{**}$ are defined by the following three steps:

$$\langle a'a, b\rangle = \langle a', ab\rangle \qquad \langle a.a', b\rangle = \langle a', ba\rangle$$
$$\langle b''a', a\rangle = \langle b'', a'a\rangle \qquad \langle a'.a'', a\rangle = \langle a'', a.a'\rangle$$
$$\langle a''b'', a'\rangle = \langle a'', b''a'\rangle \qquad \langle a''.b'', a'\rangle = \langle b'', a'.a''\rangle$$

for every $a' \in \mathcal{A}^*$, and $a, b \in \mathcal{A}$. When these products coincide, the Banach algebra $\mathcal{A}$ is called *Arens regular*.

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The C^*-algebras are the known examples of Arens regular Banach algebras. The algebra ℓ^1 with pointwise multiplication is Arens regular, which was first considered in [4, Theorem 4.2], see also [11]. But when ℓ^1 is given the convolution as multiplication, then it is not Arens regular; this is due to Arens, see also [7, Corollary 6.7].

In [17], Pym considered the space $WAP(\mathcal{A}^*)$ of weakly almost periodic functionals on $\mathcal{A}$, this is the space of all elements $a' \in \mathcal{A}^*$ such that $\{a'a : a \in \mathcal{A},\ \|a\| \leq 1\}$ is relatively weakly compact in $\mathcal{A}^*$, and proved that $\mathcal{A}$ is Arens regular if and only if $WAP(\mathcal{A}^*) = \mathcal{A}^*$, i.e., when the quotient $\mathcal{A}^*/WAP(\mathcal{A}^*)$ is trivial.

In [20], Young proved that the group algebra $L^1(G)$ is Arens regular if and only if G is finite, see also Ülger [19] for a proof based on [17]. Moreover, in [21], based again on Pym's criterion [17], Young proved that the semigroup algebra $\ell^1(S)$ of a discrete cancellative semigroup S is Arens regular if and only if S is finite. But unlike group algebras, semigroup algebras can be Arens regular when S is infinite: take for example the multiplication defined on an infinite set by $xy = x$ if $x = y$ $xy = 0$ if $x \neq y$. For more information, the reader is directed to [5], [6], [11].

On the other side, Granirer introduced in [13] the concept of extreme non-Arens regularity by stating that $\mathcal{A}$ is *extremely non-Arens regular* (enAr for short) when the quotient $\mathcal{A}^*/WAP(\mathcal{A}^*)$ contains a closed linear subspace which has $\mathcal{A}^*$ as a continuous linear image. That is, the quotient $\mathcal{A}^*\big/WAP(\mathcal{A}^*)$ is at least as big as $\mathcal{A}^*$. Granirer proved that the Fourier algebra $A(G)$ is enAr in some special cases such as $\mathbb{R}$ and $\mathbb{T}$. In [15], improving Granirer's results, Hu proved that $A(G)$ is enAr for all locally compact groups satisfying $b(G) \geq \kappa(G)$, where $b(G)$ is the smallest cardinality of an open basis of the identity of G and $\kappa(G)$ is the smallest cardinality of a compact covering of G.

In [12], Fong and Neufang considered the group algebra $L^1(G)$ and proved that it is enAr when G is non-compact $\sigma-$compact or it contains an open $\sigma-$compact subgroup H which is either normal or has $|H| < |G|$.

In [3], we proved the full dual of Hu's result, that is, the group algebra is enAr whenever $b(G) \leq \kappa(G)$.

In this paper we prove that the semigroup algebra $\ell^1(S)$ is enAr for all infinite, discrete, weakly cancellative semigroups S. Our main tool is the slowly oscillating functions; this notion, taken from coarse geometry [18], was first used by Protasov in his study of the Stone-Čech compactification of a countable discrete group [16].

2. Prerequisites

Let S be a discrete semigroup. We can identify the Banach algebras $\ell^1(S)^*$ and $\ell^\infty(S)$ and the three stages defining the first Arens product on $\ell^1(S)^{**}$ are simplified to the following steps:

$$f_s(t) = f(st), \quad f_y(s) = y(f_s), \quad xy(f) = x(f_y)$$

for all $f \in \ell^\infty(S)$ and $s, t \in S$.

Let βS be the Stone-Čech compactification of S. As known, using the universal property of the Stone-Čech compactification, the multiplication on S can be extended to βS such that for every $y \in \beta S$ and $s \in S$, the shifts

$$x \mapsto xy : \beta S \mapsto \beta S, \qquad x \mapsto sx : \beta S \to \beta S$$

are continuous. This product may also be seen as the restriction to βS of the first Arens product defined on $\ell^1(S)^{**}$.

Note also that with the Riesz representation theorem we may identify the Banach space $\ell^\infty(S)^*$ with the Banach space $M(\beta S)$ of bounded regular Borel measures on βS.

For more information about the compact right topological semigroup βS, see [14] and [9].

For every $x \in \beta S$, we define the *height* of x by

$$\rho(x) = \min\{|A| : x \in \overline{A},\, A \subseteq S\}.$$

This is often refered to as the norm of x and denoted by $\|x\|$, but to avoid confusion with the norms of the Banach spaces in this paper, we shall use this new notation. The set of the *uniform ultrafilters* in βS is defined by

$$\mathcal{U}(S) = \{x \in \beta S : \rho(x) = |S|\}.$$

A bounded function $f\colon S \to \mathbb{C}$ is called *slowly oscillating* if for every $\epsilon > 0$ and $s \in S$, there exists $A \subseteq S$ such that

(i) $|A| < |S|$,
(ii) $|f(st) - f(t)| < \epsilon$ and $|f(ts) - f(t)| < \epsilon$ for every $t \in S \setminus A$.

We shall denote the continuous extension of a function $f \in \ell^\infty(S)$ to βS also by f. Slowly oscillating functions satisfy the following identities

$$f(yx) = f(x) \quad \text{and} \quad f(xs) = f(x)$$

for all $x \in \mathcal{U}(S)$, $y \in \beta S$ and $s \in S$.

These are obtained by noting first that $f(sx) = f(x)$ and $f(xs) = f(x)$ for every $s \in S$ and $x \in \mathcal{U}(S)$. If $y \in \beta S$, then, by taking a net (s_α) in S converging to y in βS, we see that

$$f(yx) = \lim_\alpha f(s_\alpha x) = \lim_\alpha f(x) = f(x).$$

Recall that a semigroup S is called *weakly cancellative* if the sets

$$s^{-1}t = \{u \in S : su = t\} \quad \text{and} \quad ts^{-1} = \{u \in S : us = t\}$$

are finite for every $s, t \in S$. We shall use the notations

$$s^{-1}B = \{t \in S : st \in B\} \quad \text{and} \quad A^{-1}B = \bigcup_{s \in A} s^{-1}B,$$

where $s \in S$ and $A, B \subseteq S$. The sets Bs^{-1} and BA^{-1} are defined similarly.

3. $\ell^1(S)$ is enAr

We start by separating uniform ultrafilters by slowly oscillating functions.

Lemma 3.1. *Let S be an infinite, discrete, weakly cancellative semigroup with $|S| = \kappa$. There exists $T \subseteq S$ such that $|T| = \kappa$ and every distinct points $x, y \in \overline{T}$ can be separated with a slowly oscillating function.*

Proof. To simplify the notation, we start by joining an identity e to S and denote this new discrete semigroup by S_e. We deal the countable and uncountable cases separately.

Suppose that $|S| = \omega$. Since S_e is weakly cancellative, there exists a cover $\{S_n\}_{n<\omega}$ of S_e consisting of finite sets starting with $S_1 = \{e\}$ and satisfying

$$S_n^2 \cup S_n^{-1}S_n \cup S_nS_n^{-1} \subseteq S_{n+1}$$

for every $n < \omega$. Therefore we have

$$(S_nS_m) \cup (S_n^{-1}S_m) \cup (S_nS_m^{-1}) \subseteq S_{n+m}$$

for each $n < \omega$.

Furthermore, there exists a subset $T = \{t_n\}_{n<\omega}$ of S such that

$$S_n t_n S_n \cap S_m t_m S_m = \emptyset \tag{3.1}$$

whenever $n \neq m$. To see this, suppose that the points $t_1, t_2, \ldots, t_{n-1}$ have been chosen. Then

$$\bigcup_{m=1}^{n-1} \left(S_n^{-1} S_m t_m S_m S_n^{-1}\right)$$

is finite because S_e is weakly cancellative. Thus we can choose

$$t_n \in S \setminus \left(\bigcup_{m=1}^{n-1} \left(S_n^{-1} S_m t_m S_m S_n^{-1}\right) \right)$$

which clearly satisfies our condition.

Let $x, y \in \overline{T}$ be distinct points and partition T into disjoint sets $A = \{t_n \in T : n \in I\}$ and $B = \{t_n \in T : n \notin I\}$ such that $x \in \overline{A}$ and $y \in \overline{B}$, and where $I \subseteq \omega$. This is possible since βS is extremely disconnected. Now, for every $n \in I$ $(n \geq 2)$, define

$$f_n = \begin{cases} 1 & \text{on } S_1 t_n S_1 = \{t_n\} \\ 1 - \frac{1}{n-1} & \text{on } S_2 t_n S_2 \setminus (S_1 t_n S_1) \\ 1 - \frac{2}{n-1} & \text{on } S_3 t_n S_3 \setminus (S_2 t_n S_2) \\ \vdots & \\ \frac{1}{n-1} & \text{on } S_{n-1} t_n S_{n-1} \setminus (S_{n-1} t_n S_{n-2}) \\ 0 & \text{off } S_n t_n S_n. \end{cases}$$

For every $n \notin I$ $(n \geq 2)$, define

$$f_n = \begin{cases} -1 & \text{on } S_1 t_n S_1 = \{t_n\} \\ -1 + \frac{1}{n-1} & \text{on } S_2 t_n S_2 \setminus (S_1 t_n S_1) \\ -1 + \frac{2}{n-1} & \text{on } S_3 t_n S_3 \setminus (S_2 t_n S_2) \\ \vdots & \\ -\frac{1}{n-1} & \text{on } S_{n-1} t_n S_{n-1} \setminus (S_{n-1} t_n S_{n-2}) \\ 0 & \text{off } S_{n-1} t_n S_{n-1}. \end{cases}$$

Define a function

$$f = \sum_{n<\omega} f_n.$$

By (3.1), the supports of the functions f_n are disjoint, and hence the function f is well-defined. Clearly $f \in \ell^\infty(S)$. Moreover, the function f separates the points $x, y \in \overline{T}$ since

$$f(t_n) = \begin{cases} 1, & \text{if } n \in I \\ -1, & \text{if } n \notin I. \end{cases}$$

We check now that f is slowly oscillating. Let $0 < \epsilon < 1$, $s \in S$ and choose S_m such that $s \in S_m$. Moreover, choose $\ell \in \mathbb{N}$ such that $\frac{m}{\ell} < \epsilon$ and let

$$A := \bigcup_{j=1}^{\ell} \big((S_m^{-1} S_j t_j S_j) \cup (S_j t_j S_j S_m^{-1}) \big).$$

Fix $t \in S_e \setminus A$ and let us show that st and t belong to $S_n t_n S_n$ for some $n > \ell$. Clearly, this is the case whenever $f(st) = f(t) = 0$.

Now if $f(st) \neq 0$, then $st \in S_{n-1} t_n S_{n-1}$ for some $n < \omega$, and so $t \in S_m^{-1} S_{n-1} t_n S_{n-1}$. The choice of the set A implies that $n > \ell > m$, and so $t \in S_m^{-1} S_{n-1} t_n S_{n-1} \subseteq S_n t_n S_n$.

Similarly, if $f(t) \neq 0$, then $t \in S_{n-1} t_n S_{n-1} \subseteq S_n t_n S_n$ for some $n > m$, and so $st \in S_m S_{n-1} t_n S_{n-1} \subseteq S_n t_n S_n$.

For each $t \in S_n t_n S_n$, there is a unique $k \in \{0, 1, 2, \ldots, n-1\}$ such that $t \in S_{k+1} t_n S_{k+1} \setminus (S_k t_n S_k)$. Therefore

$$\begin{aligned} st &\in S_{m+k+1} t_n S_{k+1} \setminus (S_{k-m} t_n S_k) \\ &\subseteq S_{m+k+1} t_n S_{m+k+1} \setminus (S_{k-m} t_n S_{k-m}). \end{aligned}$$

(Here we are assuming that $S_p = \emptyset$ when $p \leq 0$.)

If $n \in I$, then

$$\begin{cases} 1 - \frac{m+k}{n-1} \leq f(st) \leq 1 - \frac{k-m}{n-1}, \\ f(t) = 1 - \frac{k}{n-1}, \end{cases}$$

and so we have

$$-\frac{m}{n-1} \leq f(st) - f(t) \leq \frac{m}{n-1}. \tag{3.2}$$

Similarly, if $n \notin I$, then analogous argument gives us again inequality (3.2). That is, for $s \in S_e$ and $t \in S_e \setminus A$.

$$|f(st) - f(t)| \leq \frac{m}{n-1} \leq \frac{m}{\ell} < \epsilon.$$

Similar argument shows also that $|f(ts) - f(t)| < \epsilon$. Therefore the function f is slowly oscillating on S_e, and so its restriction to S is slowly oscillating with $f(x) = 1$ and $f(y) = -1$.

Suppose now that $\kappa > \omega$. Consider an enumeration $\{s_\alpha\}_{\alpha<\kappa}$ of S_e with $s_1 = e$. For each $\alpha < \kappa$, define a set S_α as follows: let $Y_1 = \{s_\beta\}_{\beta<\alpha}$ and

$$Y_{n+1} = Y_n^2 \cup Y_n^{-1} Y_n \cup Y_n Y_n^{-1}$$

for every $n < \omega$, and define $S_\alpha = \cup_{n<\omega} Y_n$. Then an easy calculation shows that

$$\begin{cases} S_\alpha^2 \cup (S_\alpha^{-1} S_\alpha) \cup (S_\alpha S_\alpha^{-1}) \subseteq S_\alpha, \\ |S_\alpha| \leq \max\{\omega, |\alpha|\} \text{ and} \\ \bigcup_{\beta<\alpha} S_\beta \subseteq S_\alpha \end{cases} \tag{3.3}$$

for every $\alpha < \kappa$. By transfinite induction, we can construct a subset $T = \{t_\alpha\}_{\alpha<\kappa}$ of S for which

$$S_\alpha t_\alpha S_\alpha \cap S_\beta t_\beta S_\beta = \emptyset$$

whenever $\alpha \neq \beta$. As before, let $x, y \in \overline{T}$ be distinct points and partition T into the sets $A = \{t_\alpha : \alpha \in I\}$ and $B = T \setminus A = \{t_\alpha : \alpha \notin I\}$ with $x \in \overline{A}$ and $y \in \overline{B}$, and where $|I| < \kappa$.

For each $\alpha < \kappa$, define a function

$$f_\alpha = \begin{cases} \chi_{S_\alpha t_\alpha S_\alpha} & \text{if } \alpha \in I, \\ -\chi_{S_\alpha t_\alpha S_\alpha} & \text{if } \alpha \notin I, \end{cases}$$

where $\chi_{S_\alpha t_\alpha S_\alpha}$ is a characteristic function of a set $S_\alpha t_\alpha S_\alpha$. Furthermore, let

$$f = \sum_{\alpha \in I} f_\alpha.$$

Then, clearly $f(x) = 1$ and $f(y) = -1$. So it remains to show that f is slowly oscillating.

Fix a point $s \in S_e$. Choose $\beta < \kappa$ such that $s \in S_\beta$ and let

$$A := \bigcup_{\gamma<\beta} \big((S_\beta^{-1} S_\gamma t_\gamma S_\gamma) \cup (S_\gamma t_\gamma S_\gamma S_\beta^{-1}) \big).$$

Now $|A| \leq \max\{\omega, |\beta|\} < \kappa$. Let $t \in S_e \setminus A$ and $\epsilon > 0$. If $f(st) = 1$, then $st \in S_\alpha t_\alpha S_\alpha$ for some $\alpha \in I$. Therefore $t \in S_\beta^{-1} S_\alpha t_\alpha S_\alpha$.

The choice of the set A implies that $\alpha > \beta$, and hence $t \in S_\alpha t_\alpha S_\alpha$ for $\alpha \geq \beta$. That is, we also have $f(t) = 1$.

Conversely, if $f(t) = 1$, then $t \in S_\alpha t_\alpha S_\alpha$ for some $\alpha \in I$ and $\alpha \geq \beta$ due to the choice of A. Thus $st \in S_\alpha t_\alpha S_\alpha$ showing that $f(st) = 1$. Similarly, $f(st) = -1$ if and only if $f(t) = -1$.

Thus we have $|f(st) - f(t)| < \epsilon$. A similar argument shows that $|f(ts) - f(t)| < \epsilon$, and completes the proof. $\square$

In [19, Theorem 4], Ülger proved that $WAP(L^1(G)) = WAP(G)$ for a locally compact topological group G. The following lemma shows that the result holds also for discrete weakly cancellative semigroups. Our proof is inspired from that of Ülger but it is simpler since S is discrete.

For $f \in \ell^\infty(S)$, denote $S.f = \{f^t : t \in S\}$, and recall that the function f is weakly almost periodic on S when $S.f$ is weakly relatively compact in $\ell^\infty(S)$. Denote the space of the weakly almost periodic functions on S by $WAP(S)$.

A functional $f \in \ell^\infty(S)$ on $\ell^1(S)$ is called weakly almost periodic on $\ell^1(S)$ if the set $Q_f = \{f_\varphi : \varphi \in \ell^1(S),\ \|\varphi\| \leq 1\}$ is weakly relatively compact in $\ell^\infty(S)$. The space of the weakly almost periodic functionals on $\ell^1(S)$ is denoted by $WAP(\ell^1(S))$.

For every $\varphi \in \ell^1(S)$ and $f \in \ell^\infty(S)$, let f_φ be the function in $\ell^\infty(S)$ given by $f_\varphi(s) = \varphi(f_s)$. Hence

$$f_\varphi(s) = \varphi(f_s) = \sum_{t \in S} \varphi(t) f(st) = \sum_{t \in S} \varphi(t) f^t(s).$$

The convex hull $co(F)$ of a subset F of a linear space X is the intersection of all convex subsets of X that contain F. It is known that $co(F)$ is the set of all convex combinations of elements in F. If X is a normed space, the closure of $co(F)$ is denoted by $\overline{co}(A)$.

Lemma 3.2. *Let S be a discrete semigroup. Then $WAP(\ell^1(S)) = WAP(S)$.*

Proof. The inclusion $WAP(\ell^1(S)) \subseteq WAP(S)$ is clear since $S.f$ is contained in $co(S.f)$ and hence also in Q_f.

For the converse, let $f \in WAP(S)$. We need to show that the set Q_f is weakly relatively compact in $\ell^\infty(S)$. Since $f \in WAP(S)$, the set $F = S.f \cup \{0\}$ is weakly relatively compact in $\ell^\infty(S)$. By Krein-Šmulian Theorem (see for example [8]), also $co(F)$ is weakly relatively compact.

Let $\varphi = \sum_{t\in S}\varphi(t)\delta_t \in \ell^1(S)$ be arbitrary. Since the set of measures of finite support is norm-dense in $\ell^1(S)$, there exists a sequence (φ_n) of measures of finite support, where

$$\varphi_n = \sum_{k=1}^{n} \varphi(t_k)\delta_{t_k},$$

converging to φ in norm. With no loss of generality, we may assume that the measure $\varphi \in \ell^1(S)$ is real.

Suppose that $\varphi \in \ell^1(S)$ has a finite support. Then

$$f_\varphi = \sum_{k=1}^{n} \varphi(t_k) f^{t_k} + \left(1 - \sum_{k=1}^{n} \varphi(t_k)\right) \cdot 0 \in co(F).$$

Now, if $\varphi \in \ell^1(S)$ is arbitrary (still real), then $\lim_{n\to\infty} \|\varphi - \varphi_n\|_1 = 0$, and so also

$$\lim_{n\to\infty} \|f_{\varphi_n} - f_\varphi\|_1 = 0.$$

Therefore $f_\varphi \in \overline{co}(F)$, and so we have $Q_f \subseteq \overline{co}(F)$. That is, Q_f is weakly relatively compact in $\ell^\infty(S)$, as required. □

Theorem 3.3. *Let S be a discrete weakly cancellative semigroup with $|S| = \kappa$. Then $\ell^\infty(S)/WAP(S)$ contains a linear isometric copy of $\ell^\infty(S)$. In particular, the quotient space $\ell^\infty(S)/WAP(S)$ is non-separable.*

Proof. Let us enumerate S as $S = \{s_\lambda : \lambda < \kappa\}$. Let T be the subset of S constructed in Lemma 3.1 and partition T into κ subsets $T_\lambda = \{t_\alpha : \alpha \in I_\lambda\}$ such that $|T_\lambda| = \kappa$ for each $\lambda < \kappa$. For each $\lambda < \kappa$, fix two distinct uniform ultrafilters x and y on T_λ, i.e., $\rho(x) = \rho(y) = \kappa$ together with a slowly oscillating function $f_\lambda = \sum_{\alpha\in I_\lambda} f_\alpha$ as constructed in Lemma 3.1 such that $f_\lambda(x) = -1$ and $f_\lambda(y) = 1$. Now correspond to each function c in $\ell_\infty(S)$ the function

$$f_c = \sum_{\lambda<\kappa} c(s_\lambda) f_\lambda.$$

Since the support of each f_λ is contained in $\cup_{\alpha\in I_\lambda} S_\alpha t_\alpha S_\alpha$, we see that the supports of the functions f_λ are non-overlapping, and so the function f_c is well-defined. Clearly, f_c is bounded and $\|f_c\| = \|c\|$.

The mapping $c \mapsto f_c + WAP(S)$ is linear, so we only need to prove that it is an isometry from $\ell^\infty(S)$ into the quotient space $\ell^\infty(S)/WAP(S)$.

Since the quotient mapping is bounded, $\|f_c + WAP(S)\| \leq \|f_c\|$, and we only need to check that $\|f_c + h\| \geq \|c\|$ for every $h \in WAP(S)$. With no loss of generality, we may assume that $\|c\| = 1$. Suppose, otherwise, that $\|f_c + h\| < 1$ for some $h \in WAP(S)$, pick $\epsilon > 0$ such that $\|f_c + h\| < 1 - \epsilon$, and let $\lambda < \kappa$ such that $1 - \epsilon < |c(s_\lambda)|$.

Let (x_i) and (y_j) be nets in T_λ converging in βS to x and y, respectively. Then $|f_c(x_i y_j) + h(x_i y_j)| < 1 - \epsilon$ for all i and j. Remember that since f_c is slowly oscillating and x and y are uniform ultrafilters, $f_c(xy) = f_c(y)$ and $f_c(x) = f_c(xs)$ for every $s \in S$. Since $h \in WAP(S)$, we have by [2, Corollary 4.2.7],

$$h(xy) = \lim_j h(xy_j) = \lim_i \lim_j h(x_i y_j) = \lim_j \lim_i h(x_i y_j).$$

Accordingly,

$$\begin{aligned} |f_c(y) + h(xy)| &= |f_c(xy) + h(xy)| \\ &= \lim_i \lim_j |f_c(x_i y_j) + h(x_i y_j)| < 1 - \epsilon \end{aligned}$$

and

$$\begin{aligned} |f_c(x) + h(xy)| &= \lim_j |f_c(xy_j) + h(xy_j)| \\ &= \lim_j \lim_i |f_c(x_i y_j) + h(x_i y_j)| < 1 - \epsilon. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} 2 - 2\epsilon < 2|c(s_\lambda)| &= |c(s_\lambda) f_\lambda(y) - c(s_\lambda) f_\lambda(x)| \\ &= |f_c(y) - f_c(x)| \\ &\leq |f_c(y) + h(xy)| + |f_c(x) + h(xy)| \\ &< 2 - 2\epsilon, \end{aligned}$$

which is absurd. Thus, $\|f_c + h\| \geq \|c\|$ for every $h \in WAP(S)$, and so

$$\|f_c + WAP(S)\| = \inf\{\|f_c + h\| : h \in WAP(S)\} = \|c\|,$$

as required. □

Corollary 3.4. *Let S be an infinite, weakly cancellative, discrete semigroup. Then $\ell^1(S)$ is extremely non-Arens regular.*

Proof. By Theorem 3.3 and Lemma 3.2, $\ell^\infty(S)/WAP(S)$ contains a linear isometric copy of $\ell^\infty(S)$, showing that $\ell^1(S)$ is extremely non-Arens regular. □

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