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A NOTE ON DITOPOLOGICAL TEXTURE SPACES

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Abstract. In this study, we present the hereditary separation properties of plain ditopological texture spaces for induced subtextures. Brown and his team proved that the complete biregularity is productive. Here, using the concept of induced subtexture, we show that the converse of this result is true for plain texture spaces, namely if a product of plain ditopological texture spaces is completely biregular, then all factor spaces are also completely biregular.

1. Introduction

From the motivational point of view, textures were first considered only as a point–base setting for fuzzy sets [see e.g. 9], but in recent papers [5,12] it is observed that they are in fact C–spaces [17], and equivalently T_0 core spaces [16] or T_0 topological spaces with injective hulls [4] as objects of the category **dfTex** where the morphisms are difunctions. Further, the highly economic structures of textures can be used in obtaining well–known constructions as Wallman or Alexandroff compactifications and this may play an important role to determine the structures of principal examples in main stream topology [2,15]. On the other hand, in a textural discussion, the symmetry property of uniform spaces corresponds to a kind of complementation of direlations [19]. The basic separation properties of ditopological textures are studied extensively in [11] and it is proved that the complete biregularity is productive.

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In this study, we present the hereditary separation properties of ditopological texture spaces and using induced subtexture, we show that if a product of plain ditopological texture spaces is completely biregular, then all factor spaces are also completely biregular in the category of **dfPDitop**. Now let us recall some basic concepts on textures from [9] and [10]. A texturing on a set S is a point separating, complete, completely distributive lattice $\mathcal{S}$ of subsets of S with respect to inclusion which contains $S, \emptyset$ and, for which arbitrary meet coincides with intersection and finite joins coincide with union. Then $(S, \mathcal{S})$ is called a texture space. The sets

$$P_s = \bigcap \{A \mid s \in A \in \mathcal{S}\} \text{ and } Q_s = \bigvee \{P_t \mid s \notin P_t\}$$

are known as p–sets and q–sets and they are important tools for textures as we will see in the sequel. If for all $s \in S$, we have $P_s \not\subseteq Q_s$, then $(S, \mathcal{S})$ is called a *plain texture space*. This is equivalent to say that $(S, \mathcal{S})$ is closed under arbitrary unions. Now let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be texture spaces. Consider the product texture $\mathcal{P}(S) \otimes \mathcal{T}$ of the texture spaces $(S, \mathcal{P}(S))$ and $(T, \mathcal{T})$ and denote the p-sets and the q sets by $\overline{P}_{(s,t)}$ and $\overline{Q}_{(s,t)}$ respectively where $s \in S$ and $t \in T$ [See for products 9]. Here it is easy to see that $\overline{P}_{(s,t)} = \{s\} \times P_t$ and $\overline{Q}_{(s,t)} = (S \setminus \{s\} \times T) \cup (S \times Q_t)$. Now $r \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *relation* from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ if it satisfies

(i) $r \not\subseteq \overline{Q}_{(s,t)}, P_{s'} \not\subseteq Q_s \implies r \not\subseteq \overline{Q}_{(s',t)}$, and

(ii) $r \not\subseteq \overline{Q}_{(s,t)} \implies \exists s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',t)}$.

Dually, $R \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *corelation* from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ if the following conditions hold:

(1) $\overline{P}_{(s,t)} \not\subseteq R, P_s \not\subseteq Q_{s'} \implies \overline{P}_{(s',t)} \not\subseteq R$

(2) $\overline{P}_{(s,t)} \not\subseteq R \implies \exists s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}_{(s',t)} \not\subseteq R$.

A pair (r, R), where r is a relation and R a corelation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ is called a *direlation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$.
Let (f, F) be a direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then (f, F) is called a *difunction from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if

(1) for $s, s' \in S$, $P_s \not\subseteq Q_{s'} \implies \exists t \in T$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F$.

(2) for $t, t' \in T$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t')} \not\subseteq F \implies P_{t'} \not\subseteq Q_t$.

Further, a difunction (f, F) is

(i) *surjective* if for $t, t' \in T, P_t \not\subseteq Q_{t'} \implies \exists s \in S$ with $f \not\subseteq \overline{Q}_{(s,t')}$ and $\overline{P}_{(s,t)} \not\subseteq F$.

(ii) *injective* if for $s, s' \in S$ and $t \in T, f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F \implies P_s \not\subseteq Q_{s'}$.

For a convenient topological structure on textures, we consider that the open sets and closed sets are independent of each other. A *ditopology* on a texture space $(S, \mathcal{S})$ is a pair (τ, κ) of subsets of $\mathcal{S}$, where the family of *open sets* τ satisfies

(1) $S, \emptyset \in \tau$,
(2) $G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau$ and
(3) $G_i \in \tau, i \in I \implies \bigvee_i G_i \in \tau$,

and the family of *closed sets* κ satisfies

(1) $S, \emptyset \in \kappa$,
(2) $K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa$ and
(3) $K_i \in \kappa, i \in I \implies \bigcap K_i \in \kappa$.

In this respect, the closure and interior of a set $A \in \mathcal{S}$ in a ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$, is given by the equalites

$$[A] = \bigcap\{K \in \kappa \mid A \subseteq K\} \text{ and }]A[= \bigvee\{G \in \tau \mid G \subseteq A\}$$

respectively.

Recall that ditopological texture spaces and bicontinuous difunctions form a category which is denoted by **dfDitop**. For some results on ditopologies, we refer [3,14,15 and 20].

For the concepts on textures which are not explained here, see [4–9].

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2. Induced Subtextures

Substructures of textures are extensively discussed in [8] and [12]. One of the well known substructures of textures can be defined on elements of textures, that is if $(S, \mathcal{S})$ is a texture space and $A \in \mathcal{S}$, then the family $\mathcal{S}_A = \{A \cap B \mid B \in \mathcal{S}\}$ is a texture on A. In this case, the pair $(A, \mathcal{S}_A)$ is called the *principal subtexture* of $(S, \mathcal{S})$. Since the principal subtexture is a natural counterpart in the theory of texture spaces, then most of the hereditary properties of textures can be obtained in a natural way. For instance, the complete biregularity can be given in terms of restricted difunctions which is given in [2]. Here, we don't follow this line, and for any subset of S we discuss the substructures of plain texture spaces since we will use it in products in the last chapter.

Definition 2.1. Let $(S, \mathcal{S})$ be a texture space and $A \subseteq S$. If the family $\mathcal{S}_A = \{A \cap B \mid B \in \mathcal{S}\}$ is a texturing on A, then $\mathcal{S}_A$ is called an *induced structure on* A, and $(A, \mathcal{S}_A)$ is called an *induced subtexture* of $(S, \mathcal{S})$. If we define $\tau_A = \{G \cap A \mid G \in \tau\}$ and $\kappa_A = \{K \cap A \mid K \in \kappa\}$, then (τ_A, κ_A) is called the *induced* ditopology on A, and $(A, \mathcal{S}_A, \tau_A, \kappa_A)$ is called *an induced ditopological subtexture space* of $(S, \mathcal{S}, \tau, \kappa)$.

Theorem 2.2. *Let $(S, \mathcal{S})$ be a plain texture space and $A \subseteq S$. Then the pair $(A, \mathcal{S}_A)$ is an induced subtexture space of $(S, \mathcal{S})$ where $\mathcal{S}_A = \{U \cap A \mid U \in \mathcal{S}\}$ is a texturing on S.*

Proof. Since $(S, \mathcal{S})$ is plain, by [1, Theorem 2.1.4] arbitrary joins coincide with unions and so trivially, the family $\mathcal{S}_A$ is a completely distributive lattice with respect to set inclusion. The other conditions are straightforward. $\square$

Example 2.3. *Consider the real ditopological texture space $(\mathbb{R}, \mathcal{R}, \tau, \kappa)$ where*

$$\mathcal{R} = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{(-\infty, a] \mid a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$$

and $\tau = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}, \quad \{(-\infty, a] \mid a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$.
Since $(\mathbb{R}, \mathcal{R}, \tau, \kappa)$ is plain, then clearly, the pair $(\mathbb{N}, \mathcal{N})$ is an induced subtexture of $(\mathbb{R}, \mathcal{R})$ where $\mathbb{N} = \{0, 1, 2, ...\} \subseteq \mathbb{R}$ and

$$\mathcal{N} = \{\mathbb{N} \cap A \mid A \in \mathcal{R}\} = \{\{0, 1, 2, ..., a\} \mid a \in \mathbb{N}\} \cup \{\mathbb{N}, \emptyset\}.$$

If we consider the induced ditopology $(\tau_{\mathbb{N}}, \kappa_{\mathbb{N}})$ on $(\mathbb{N}, \mathcal{N})$, then we may write

$$\tau_{\mathbb{N}} = \kappa_{\mathbb{N}} = \mathcal{N}.$$

Since $(\mathbb{N}, \mathcal{N}, \tau_{\mathbb{N}}, \kappa_{\mathbb{N}})$ is a discrete codiscrete ditopological texture space, then it satisfies all separation axioms which will be mentioned in this paper.

Now let us denote the p-sets and the q-sets of an induced substructure $(A, \mathcal{S}_A)$ by P_s^A and Q_s^A respectively for some $s \in A$.

Lemma 2.4. *Let $(S, \mathcal{S})$ be a plain texture space and $(A, \mathcal{S}_A)$ be an induced subtexture space of $(S, \mathcal{S})$. If $s \in A$, then $P_s^A \subseteq P_s \cap A$ and $Q_s \cap A \subseteq Q_s^A$.*

Proof. Since $(A,\mathcal{A})$ is an induced subtexture space, then for all $s \in S$ we may write $P_s \cap A \in \mathcal{S}_A$ and so if $s \in A$, then clearly, we have $P^A_s \subseteq P_s \cap A$. For the second inclusion, let $s \in A$ and $t \in S$. Then we have $A \cap P_t \in \mathcal{S}_A$ and therefore, $A \cap P_t = \bigcup\{P^A_r \mid r \in A \cap P_t\}$. If $s \notin A \cap P_t$, then by the first inclusion we may write that $s \notin P^A_r$ where $r \in A \cap P_t$. This implies that $P^A_r \subseteq Q^A_s$, that is $A \cap P_t \subseteq Q^A_s$. On the other hand, $Q_s \cap A \in \mathcal{S}_A$ and then

$$\begin{aligned} Q_s \cap A &= \bigcup\{P_t \mid s \notin P_t\} \cap A = \bigcup\{A \cap P_t \mid s \notin P_t\} \\ &= \bigcup\{A \cap P_t \mid s \notin A \cap P_t\} \subseteq Q^A_s. \end{aligned}$$

□

Example 2.5. [8] *Consider the texture space* $(S,\mathcal{S})$ *where*

$$S = (0,1] \text{ and } \mathcal{S} = \{(0,r] \mid r \in [0,1]\}.$$

We know that $(S,\mathcal{S})$ *is not plain. Now let* $A = \{\frac{1}{2}, 1\}$ *and consider the induced texture* $\mathcal{S}_A = \{A, \emptyset, \{1/2\}\}$ *on* A. *Clearly,* $Q_1 = (0,1]$ *and* $Q^A_1 = \{\frac{1}{2}\}$. *However,* $Q_1 \cap A = A \not\subseteq \{\frac{1}{2}\} = Q^A_1$ *and hence, we cannot remove the condition of "to be plain" of* $(S,\mathcal{S})$ *in Lemma 1.4.*

3. Point separation properties

Now let us recall the following concepts.

Definition 3.1. [11] A ditopological texture space $(S,\mathcal{S},\tau,\kappa)$ is called

(a) T_0 if for $s,t \in S$, $Q_s \not\subseteq Q_t \implies (\exists H \in \tau \cup \kappa)(P_s \not\subseteq H \not\subseteq Q_t)$.
(b) T_1 if for $s,t \in S$, $Q_s \not\subseteq Q_t \implies (\exists K \in \kappa)(P_s \not\subseteq K \not\subseteq Q_t)$.
(c) *co*–T_1 if for $s,t \in S$, $Q_s \not\subseteq Q_t \implies (\exists G \in \tau)(P_s \not\subseteq G \not\subseteq Q_t)$.
(d) *bi*–T_1 if it is T_1 and co–T_1.
(e) T_2 if T_0 and R_1
(f) *co*–T_2 if T_0 and *co*–R_1
(g) *bi*–T_2 if it is T_2 and co–T_2.

However much the point separation properties is given in terms of the p-sets and the q-sets, there are some characterizations of them which are independent from p-sets and q-sets [11] and here we prefer to use these characterizations to obtain the following results.

Theorem 3.2. *Let $(S,\mathcal{S},\tau,\kappa)$ be a ditopological plain texture space and $(A,\mathcal{S}_A,\tau_A,\kappa_A)$ be a ditopological induced subtexture space. If $(S,\mathcal{S},\tau,\kappa)$ is bi–T_i for $i=0,1,2$, then $(A,\mathcal{S}_A,\tau_A,\kappa_A)$ is also bi–T_i for $i=0,1,2$.*

Proof. Let $B\in\mathcal{S}_A$. Then we have $B=A\cap C$ for some $C\in\mathcal{S}$. Then by [11, Theorem 4.7.(3)], we may write

$$C=\bigvee_{j\in J}\bigcap_{i\in I_j}C_i^j$$

where $C_i^j\in\tau\cup\kappa$. Since $(S,\mathcal{S})$ is plain, then

$$B=A\cap C=A\cap(\bigcup_{j\in J}\bigcap_{i\in I_j}C_i^j)=(\bigcup_{j\in J}\bigcap_{i\in I_j}(A\cap C_i^j)$$

where $D_i^j=A\cap C_i^j\in\tau_A\cup\kappa_A$ and hence, $(A,\mathcal{S}_A,\tau_A,\kappa_A)$ is also T_0. If $(S,\mathcal{S},\tau,\kappa)$ is T_1 and $B\in\mathcal{S}_A$, then by [11, Theorem 4.11 (1) (i)],

$$C=\bigvee_{i\in I}\in F_i$$

where $F_i\in\kappa, C\in\mathcal{S}$ and $B=A\cap C$. Therefore,

$$B=A\cap(\bigvee_{i\in I}F_i)=A\cap(\bigcup_{i\in I}F_i)=\bigcup_{i\in I}(A\cap F_i).$$

If $(S,\mathcal{S},\tau,\kappa)$ is co–T_1, using a similar argument and [11, Theorem 4.11 (2) (i)], it can be shown that $(A,\mathcal{S}_A)$ is also co–T_1.

Let $(S,\mathcal{S},\tau,\kappa)$ be bi–T_2 and $B\in\mathcal{S}_A$. Then we may write $B=A\cap E$ for some $E\in\mathcal{S}$. Since $(S,\mathcal{S},\tau,\kappa)$ is bi–T_2, then by [11, Theorem 4.17 (3)], there exists $H_i^j\in\tau,\ K_i^j\in\kappa,\ i\in I,\ j\in J_i$ with $H_i^j\subseteq K_i^j$ for all i,j and

$$E=\bigvee_{i\in I}\bigcap_{j\in J_i}H_i^j=\bigvee_{i\in I}\bigcap_{j\in J_i}K_i^j$$

and so

$$A\cap E=A\cap(\bigvee_{i\in I}\bigcap_{j\in J_i}H_i^j)=A\cap(\bigvee_{i\in I}\bigcap_{j\in J_i}K_i^j)$$

and since $(S,\mathcal{S},\tau,\kappa_A)$ is plain, then

$$A\cap E=(\bigcup_{i\in I}\bigcap_{j\in J_i}(A\cap H_i^j)=(\bigcup_{i\in I}\bigcap_{j\in J_i}(A\cap K_i^j).$$

Now if we observe that $A\cap H_i^j \in \tau_A$ and $A\cap K_i^j \in \kappa_A$, then we find that $(A, \mathcal{S}_A, \tau_A, \kappa_A)$ is also bi–T_2. □

Observing the above definitions, we may say that T_0 regular space is T_3, and T_0 co–regular space is co–T_3 and so T_3 and co–T_3 space is bi–T_3. Further, a T_1 normal space is T_4 and co–T_1 normal space is co–T_4.

4. Complete regularity and normality

Definition 4.1. [11] A ditopological texture spaces $(S, \mathcal{S}, \tau, \kappa)$ is

(a) *completely regular* if given $G \in \tau, G \not\subseteq Q_s$, there exists a bicontinuous difunction $(f, F) : (S, \mathcal{S}, \tau, \kappa) \to (\mathbb{I}, \mathcal{I}, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ satisfying $P_s \subseteq f^{\leftarrow}P_0$ and $F^{\leftarrow}Q_1 \subseteq G$.

(b) *completely co–regular* if given $K \in \kappa, P_s \not\subseteq K$, there exists a bicontinuous difunction $(f, F) : (S, \mathcal{S}, \tau, \kappa) \to (\mathbb{I}, \mathcal{I}, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ satisfying $K \subseteq f^{\leftarrow}P_0$ and $F^{\leftarrow}Q_1 \subseteq Q_s$.

(c) *completely biregular* if it is completely regular and completely co–regular.

(d) $T_{3\frac{1}{2}}$ if it is T_0 and completely biregular.

(e) *normal* if given $G \in \tau,\ K \in \kappa$ with $K \subseteq G$, there exist $H \in \tau$ and $M \in \kappa$ with $K \subseteq H \subseteq M \subseteq G$.

Now for general textures let us recall the following result.

Lemma 4.2. [10] *Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be textures and $\psi : S \to T$ be point function satisfying the following conditions:*

(a) $s, s' \in S, P_s \not\subseteq Q_{s'} \implies P_{\psi(s)} \not\subseteq Q_{\psi(s')}$.

(b) $P_{\psi(s)} \not\subseteq B, B \in \mathcal{T} \implies \exists s' \in S$ *with* $P_s \not\subseteq Q_{s'}$ *for which* $P_{\psi(s')} \not\subseteq B$.

(c) *For* $A \in \mathcal{T}$ *and* $s \in S^{\flat}$ *we have* $A \not\subseteq Q_{\psi(s)} \implies A \not\subseteq Q_{\psi(u)}$ *for some* $P_u \not\subseteq Q_s$.

Then the difunction (f_ψ, F_ψ) corresponding to ψ satisfies the equalities

$$f_\psi = \bigvee\{\overline{P}_{(s,\psi(s))} \mid s \in S\} \text{ and } G_\psi = \bigcap\{\overline{Q}_{(s,\psi(s))} \mid s \in S^{\flat}\}.$$

Further, $f_\psi^{\leftarrow}A = F_\psi^{\leftarrow}A = \psi^{-1}(A)$ for all $A \in \mathcal{T}$.

As it is known that ditopological texture spaces and bicontinuous point functions satisfying the conditions (a) and (b) in Lemma 4.2

form a category which is denoted by **fDitop**. Recall that if we get the objects as plain, then we obtain a category which is denoted by **fPDitop** whose morphisms are point functions satisfying the condition (a), and in this case note that the conditions (b)-(c) are automatically satisfied [10].

Now we need the following lemma.

Lemma 4.3. *Let* $(S,\mathcal{S})$, $(T,\mathcal{T})$ *be texture spaces and* $\psi : S \to T$ *be a point function satisfying the conditions (a)-(c) in Lemma 4.2. If* $(A,\mathcal{S}_A)$ *is a plain induced subtexture space of* $(S,\mathcal{S})$, *then the restriction function* $\psi_{|_A} : A \to T$ *also satisfies the conditions (a)-(c).*

Proof. Let $s, s' \in A$, $P_s^A \not\subseteq Q_{s'}^A$. Then by Lemma 1.4, we have $P_s \cap A \not\subseteq Q_{s'} \cap A$ and so we find $P_s \not\subseteq Q_{s'}$. Since the function ψ satisfies the condition (a), then we may write $P_{\psi(s)} \not\subseteq Q_{\psi(s')}$. Further, $s, s' \in A$ gives that $P_{\psi_{|_A}(s)} \not\subseteq Q_{\psi_{|_A}(s')}$. Therefore, restriction function $\psi_{|_A} : A \to T$ satisfies the condition (a). Since A is plain, then the conditions (b) and (c) are automatically satisfied for $\psi_{|A}$. □

The complete biregularity can be characterized in terms of point functions in the category of **fPDitop**.

Theorem 4.4. *Let* $(S,\mathcal{S},\tau,\kappa) \in \mathrm{Ob}\mathbf{fPDitop}$. *Then we have the following.*

(i) $(S,\mathcal{S},\tau,\kappa)$ *is completely regular if and only if given* $G \in \tau, G \not\subseteq Q_s$, *there exists a morphism* $\psi : S \to \mathbb{I}$ *in* **fPDitop** *satisfying* $\psi(P_s) = \{0\}$ *and* $\psi(S \setminus G) = \{1\}$.

(ii) $(S,\mathcal{S},\tau,\kappa)$ *is completely co–regular if and only if given* $K \in \kappa, P_s \not\subseteq K$, *there exists a morphism* $\psi : S \to \mathbb{I}$ *in* **fPDitop** *satisfying* $\psi(S \setminus Q_s) = \{1\}$ *and* $\psi(K) = \{0\}$.

Proof. (i) ($\Longleftarrow$:) Let $G \in \tau, G \not\subseteq Q_s$. Then for a morphism $\psi : S \to \mathbb{I}$ in **fPDitop** we have $\psi(P_s) = \{0\}$ and $\psi(S \setminus G) = \{1\}$. Therefore, by Lemma 5.1, there exists a difunction (f_ψ, F_ψ) corresponding to ψ satisfies the equalities

$$f_\psi = \bigvee\{\overline{P}_{(s,\psi(s))} \mid s \in S\} \text{ and } G_\psi = \bigcap\{\overline{Q}_{(s,\psi(s))} \mid s \in S\}.$$

Since ψ is bicontinuous and $f_\psi^{\leftarrow} A = F_\psi^{\leftarrow} A = \psi^{-1}(A)$ for all $A \in \mathcal{T}$, then (f_ψ, F_ψ) is also bicontinuous. Suppose that $P_s \not\subseteq f_\psi^{\leftarrow} P_0$.

Choose a point $s' \in S$ where $P_s \not\subseteq Q_{s'}$ and $P_{s'} \not\subseteq f_{\psi}^{\leftarrow}P_0$. Then there exists $t \in \mathbb{I}$ such that $f_\psi \not\subseteq \overline{Q}_{(s',t)}$ and $P_t \not\subseteq P_0$. By definition of f_ψ, we have $P_{\psi(s')} \not\subseteq Q_t$. Since $P_t \not\subseteq P_0$, then $P_{\psi(s')} \not\subseteq P_0$, namely $\psi(s') \notin P_0$ and so $\psi(s') \neq 0$. However, $\psi(P_s) = 0$ and $P_s \not\subseteq Q_{s'}$ implies that $\psi(s') = 0$ and this is a contradiction.

($\Longrightarrow$:) Now suppose that $F_{\psi}^{\leftarrow}Q_1 \not\subseteq G$. Take a point $s' \in S$ where $F_{\psi}^{\leftarrow}Q_1 \not\subseteq Q_{s'}$ and $P_{s'} \not\subseteq G$. Then there exists a point $t \in \mathbb{I}$ such that $\overline{P}_{(s',t)} \not\subseteq F_\psi$ and $Q_1 \not\subseteq Q_t$. By definition of F_ψ, we have $\overline{P}_{(s',t)} \not\subseteq \overline{Q}_{(r,\psi(r))}$ for some $r \in S$. Therefore, $P_t \not\subseteq Q_{\psi(s')}$ and so $Q_1 \not\subseteq Q_t$ gives that $Q_1 \not\subseteq Q_{\psi(s')}$, that is $\psi(s') < 1$. On the other hand, $s' \in S \setminus G$ and so $\psi(s') = 1$ is an immediate contradiction.

(ii) ($\Longleftarrow$:) Now let $P_s \not\subseteq K \in \kappa$. Suppose that $K \not\subseteq f_{\psi}^{\leftarrow}P_0$. Take a point $s' \in S$ where $K \not\subseteq Q_{s'}$ and $P_{s'} \not\subseteq f_{\psi}^{\leftarrow}P_0$. Then there is a point $t \in \mathbb{I}$ such that $f_\psi \not\subseteq \overline{Q}_{(s',t)}$ and $P_t \not\subseteq P_0$. It is easy to check that $P_{\psi(s')} \not\subseteq Q_t$ and so $P_{\psi(s')} \not\subseteq P_0$, that is $\psi(s') \neq 0$. However, $s' \in K$ and hence, $\psi(s') = 0$ is a contradiction.

($\Longrightarrow$:) Suppose that $F_{\psi}^{\leftarrow}Q_1 \not\subseteq Q_s$. Take a point $s' \in S$ where $F_{\psi}^{\leftarrow}Q_1 \not\subseteq Q_{s'}$ and $P_{s'} \not\subseteq Q_s$. Then there exists $t \in \mathbb{I}$ such that $\overline{P}_{(s',t)} \not\subseteq F_\psi$ and $Q_1 \not\subseteq Q_t$. Now we have $P_t \not\subseteq Q_{\psi(s')}$ and hence, $Q_{\psi(s')} \subseteq Q_t$. Since $Q_1 \not\subseteq Q_{\psi(s')}$, then $\psi(s') < 1$. Since $P_{s'} \not\subseteq Q_s$, then $s' \in S \setminus Q_s$ and so $\psi(s') = 1$ is a contradiction. □

Theorem 4.5. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological plain texture space and $(A, \mathcal{S}_A, \tau_A, \kappa_A)$ be a plain induced ditopological subtexture space.*

(i) *If $(S, \mathcal{S}, \tau, \kappa)$ is completely regular, then $(A, \mathcal{S}_A, \tau_A, \kappa_A)$ is also completely regular.*

(ii) *If $(S, \mathcal{S}, \tau, \kappa)$ is completely co–regular, then $(A, \mathcal{S}_A, \tau_A, \kappa_A)$ is also completely co–regular.*

Proof. (i) Let $(S, \mathcal{S}, \tau, \kappa)$ be completely regular and let $G \not\subseteq Q_s^A$ where $G \in \tau_A$. Since $G = A \cap H$ for some $H \in \tau$, then $H \not\subseteq Q_s$. Further, since S is completely regular, then by Theorem 4.4, there exists a morphism $\psi : S \to \mathbb{I}$ in **fPDitop** satisfying $\psi(P_s) = \{0\}$ and $\psi(S \setminus H) = \{1\}$. Then clearly, $\psi_{|A}(P_s^A) = \{0\}$ and $\psi_{|A}(A \setminus G) = \{1\}$. By Lemma 4.3, $\psi_{|A} : A \to \mathbb{I}$ is also a morphism in **fPDitop** satisfying the conditions (a)-(c). Therefore, by Theorem 4.4, $(A, \mathcal{S}_A, \tau_A, \kappa_A)$ is completely regular.

(ii) The proof is dual to (i) □

Corollary 4.6. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a bi–T_1 normal ditopological plain texture space and $A \in \mathcal{S}$. Then the plain ditopological induced subtexture space $(A, \mathcal{S}_A, \tau_A, \kappa_A)$ is completely biregular.*

Proof. By Corollary 5.24 in [11], $(S, \mathcal{S}, \tau, \kappa)$ is completely biregular and so by Theorem 4.5, $(A, \mathcal{S}_A, \tau_A, \kappa_A)$ is also completely biregular. $\square$

5. Product Texture Spaces

Now for $i \in I$, let $(S_i, \mathcal{S}_i, \tau_i, \kappa_i)$ be a ditopological texture space. Consider the product ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ where $S = \prod_{i \in I} S_i$. Take a point $s = (s_i)_{i \in I} \in S$. Let $D(s, j) = \prod_{i \in I} D(s, j)_i$ where

$$D(s,j)_i = \begin{cases} S_j & \text{if } i = j \\ \{s_i\} & \text{otherwise} \end{cases}$$

and let $\mathcal{D}_j$ be the product texture on $D(s, j)$.

Theorem 5.1. *$(D(s, j), \mathcal{D}_j)$ is an induced subtexture space of the product space $(S, \mathcal{S})$ where $S = \prod_{i \in I} S_i$.*

Proof. Immediate. $\square$

Theorem 5.2. [13] (i) *The function $\varphi : S_j \longrightarrow D(s, j)$ defined by $\varphi(a) = (a_i)_{i \in I}$, $a \in S_j$ where*

$$a_i = \begin{cases} a & \text{if } i = j \\ s_i & \text{otherwise} \end{cases}$$

satisfies the conditions (a)-(c) in Lemma 4.2. Further, the restriction $\pi_j|_{D(s,j)} : D(s, j) \longrightarrow S_j$ is inverse of φ.

(ii) *The mapping φ is a textural homeomorphism in* **fDitop**.

Proof. (i) For (a) let $s_j, s'_j \in S_j$ and $P_{s_j} \not\subseteq Q_{s'_j}$. By [9, Proposition 1.3], $P_{\varphi(s_j)} = \prod_{i \in I} P_{s_i}$ where $P_i = \{s_i\}$ for $i \neq j$ and $Q_{\varphi(s'_j)} = \bigcup_{i \in I} E(i, Q_{s'_i})$. Since we have $Q_{s'_i} = \emptyset$ for $i \neq j$, we may write $Q_{\varphi(s'_j)} = E(j, Q_{s'_j})$ and so clearly, $P_{\varphi(s_j)} \not\subseteq Q_{\varphi(s'_j)}$. Now let $P_{\varphi(s_j)} \not\subseteq B, B \in \mathcal{D}_j$. Choose a point $r = (r_i)_{i \in I} \in D(s, j)$ where $P_{\varphi(s_j)} \not\subseteq Q_r$ and $P_r \not\subseteq B$. Then, $P_{s_j} \not\subseteq Q_{r_j}$ and $P_{\varphi(r_j)} \not\subseteq B$ and this verifies (b). For (c) let $B \in \mathcal{D}_j$, $s_j \in S_j^\flat$ and $B \not\subseteq Q_{\psi(s_j)}$. Since $B \in \mathcal{D}_j$, then $B = \prod_{i \in I} B_i$ where $B_i = \{s_i\}$ for $i \neq j$. Therefore, $B_j \not\subseteq Q_{s_j}$

and so for some $u \in S_j$, we have $B_j \not\subseteq Q_{u_j}$ and $P_{u_j} \not\subseteq Q_{s_j}$ and so $B \not\subseteq Q_{\varphi(u_j)}$. Clearly, φ is the inverse of it and the proof of (i) is complete. (ii) Clearly, φ is bijective and it is bicontinuous [14, Lemma 2.10]. Further, since the inverse $\pi_j|_{D(s,j)}$ is also a projection function, it satisfies the conditions (a)-(c) in Lemma 4.2 and it is bicontinuous [10, Lemma 3.9]. □

Theorem 5.3. *For the function* $\pi_{j|_{D(s,j)}} : D(s,j) \longrightarrow S_j$*, the equalities*

$$f_\psi = \bigvee \{\overline{P}_{(s,\psi(s))} \mid s \in S_j\} \text{ and } F_\psi = \bigcap \{\overline{Q}_{(s,\psi(s))} \mid s \in S_j\}$$

define a dihomeomorphism (f_ψ, F_ψ) *from* $D(s,j)$ *to* S_j *where* $\psi = \pi_{j|_{D(s,j)}}$.

Proof. Since the function ψ is bijective, it is easy to see that corresponding difunction (f_ψ, F_ψ) is also bijective. Further, $\pi_j|_{D(s,j)}$ is a textural homeomorphism in **fDitop**, and so in view of Theorem 2.8 in [2], (f_ψ, F_ψ) is a dihomeomorphism in **dfDitop**. □

Theorem 5.4. *Let* $\{(S_i, \mathcal{S}_i, \tau_i, \kappa_i) : i \in I\}$ *be a family of non-empty ditopological plain texture spaces. If the product ditopological texture space* $(S, \mathcal{S}, \tau, \kappa)$ *is completely biregular, then the ditopological induced subtexture space* $(D(s,j), \mathcal{D}_j, \tau_{D(s,j)}, \kappa_{D(s,j)})$ *is also completely biregular.*

Proof. Since $(D(s,j), \mathcal{D}_j, \tau_{D(s,j)}, \kappa_{D(s,j)})$ is plain, the proof is immediate by Theorem 4.5. □

Theorem 5.5. *For* $i \in I$ *let* $(S_i, \mathcal{S}_i, \tau_i, \kappa_i)$ *be non-empty ditopological plain texture spaces and* $(S, \mathcal{S}, \tau, \kappa)$ *be their product. Then* $(S, \mathcal{S}, \tau, \kappa)$ *is completely biregular if and only if* $(S_i, \mathcal{S}_i, \tau_i, \kappa_i)$ *is completely biregular for all* $i \in I$.

Proof. By the preceeding theorem, the induced ditopological subtexture space $(D(s,j), \mathcal{D}_j, \tau_{D(s,j)}, \kappa_{D(s,j)})$ is also completely biregular and since dihomemomorphims preserves the complete biregularity [11, Proposition 5. 27], in view of Theorem 5.8, S_i is completely biregular for all $i \in I$. The proof of the second part of the theorem is proved for general textures [11, Theorem 5.16]. □

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