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ON SUPER SECOND COUNTABLE AND SUPER SEPARABLE METRIC SPACES

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Abstract. In the framework of ZF (i.e., Zermelo-Fraenkel set theory without the Axiom of Choice) we show that:
(1) *Every super second countable metric space is super separable.*
(2) *Every super second countable metric space is hereditarily super second countable.*

The above results answer related questions from Gutierres "On first and second countable spaces and the axiom of choice".

We also show that the axiom CAC($\mathbb{R}$) (i.e., the Axiom of Choice restricted to countable families of non-empty subsets of reals) is equivalent to the converse of (1) and to the corresponding statement of (2) for super separable metric spaces.

1. Introduction

It is well-known that in ZF every separable metric space is second countable. On the other hand, in [6] it has been established that the above two topological notions coincide for the class of metric spaces if and only if the axiom CAC($\mathbb{R}$) (i.e., the Axiom of Choice restricted to countable families of non-empty subsets of the real line $\mathbb{R}$) holds. In this paper, we consider two stronger notions than

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separability and second countability, namely super separability (SS) and super second countability (SSC) (see Definition 1 in Section 2; the notion of SSC is due to Goncalo Gutierres [2]). In particular, we study the corresponding problem of whether it is provable in ZF that a metric space is SSC if and only if it is SS. It turns out that the situation is strikingly different. In Theorem 3.7 (i) we prove that in ZF, every SSC metric space is SS. This answers in the negative the related question of Gutierres [2] of whether there exist non-separable SSC metric spaces. On the other hand, we establish in Theorem 3.7 (ii) that the statement *every SS metric space is SSC* is equivalent to CAC($\mathbb{R}$) and thus it is not provable in ZF set theory.

In this paper, we also examine whether the properties of SSC and SS are hereditary among metric spaces. In particular, we show in Theorem 3.5 that *every SSC metric space is hereditarily SSC* is provable within the axiom system of ZF. In contrast to this fact, we prove in Theorem 3.7 (ii) that the proposition *every SS metric space is hereditarily SS* is equivalent to CAC($\mathbb{R}$), hence it is not a theorem of ZF.

Note. We shall only focus on SS metric spaces since, in ZF, there exist separable topological spaces which are not SS. For example, consider the Tychonoff product $2^{\mathbb{R}}$, where $2 = \{0, 1\}$ is taken with the discrete topology. $2^{\mathbb{R}}$ is a separable (non-metrizable) space (in ZF) which has a dense subspace which is not separable. Indeed, let $A = \{f \in 2^{\mathbb{R}} : |f^{-1}(\{1\})| < \omega\}$. It is straightforward to verify that A is dense in $2^{\mathbb{R}}$. Assume that $D = \{d_n : n \in \omega\}$ is a dense subset of A. Clearly, the set $M = \bigcup\{d_n^{-1}(\{1\}) : n \in \omega\}$ is a countable subset of $\mathbb{R}$ (being a countable union of finite linearly ordered, thus well-ordered, sets). Let $x \in \mathbb{R} \setminus M$ and $O = \pi_x^{-1}(\{1\})$, where π_x is the canonical projection of $2^{\mathbb{R}}$ onto the xth-coordinate. Then $O \cap A \cap D = \emptyset$, and consequently D is not dense in A, a contradiction.

2. Notation, terminology, and some known results

Definition 2.1. (1) A topological space (X, T) is called *super second countable* (SSC) if every base for T has a countable subfamily which is a base.

(2) A topological space (X, T) is called *super separable* (SS) if every dense subspace of X is separable.

(3) A topological space (X, T) is called *dense-in-itself* iff X has no isolated points.
(4) If (X, T) is a topological space, then a set $Y \subset X$ is called *nowhere dense* if the closure of Y has empty interior.
(5) CAC($\mathbb{R}$): The axiom of choice restricted to countable families of non-empty subsets of $\mathbb{R}$.
(6) M(P,Q): Every metric space having the property P has also the property Q.
(7) For any topological space (X, T), let

$$\mathrm{Iso}(X) = \{x \in X : x \text{ is isolated in } X\}.$$

By transfinite recursion we define a decreasing sequence $(X_\alpha)_{\alpha \in \mathrm{Ord}}$ of closed subspaces of X as follows:

$$\begin{aligned} X_0 &= X, \\ X_{\alpha+1} &= X_\alpha \setminus \mathrm{Iso}(X_\alpha), \\ X_\alpha &= \bigcap \{X_\beta : \beta < \alpha\} \text{ for limit } \alpha. \end{aligned}$$

The set X_α, $\alpha \in \mathrm{Ord}$, is called the *αth Cantor-Bendixson derivative* of X.
(8) A topological space (X, T) is called *scattered* iff $\mathrm{Iso}(Y) \neq \emptyset$ for each non-empty closed subspace Y of X. Clearly, X is scattered iff there exists an ordinal α_0 such that $X_{\alpha_0} = \emptyset$. If X is scattered, then the ordinal number $\min\{\alpha : X_\alpha = \emptyset\}$ is called the *Cantor-Bendixson rank* of the scattered space X and it is denoted by $|X|_{CB}$.

In what follows, "S" stands for separable, "hSSC" stands for hereditarily super second countable, and "hSS" stands for hereditarily super separable. Furthermore, we shall denote every open (closed) disc of radius ϵ centered at x by $D(x, \epsilon)$ ($B(x, \epsilon)$, respectively).

Proposition 2.2. *([6]) The following statements are equivalent:*
(i) *CAC($\mathbb{R}$).*
(ii) *A metric space is second countable iff it is separable.*

Proposition 2.3. *([2]) The following statements are equivalent:*
(i) *CAC($\mathbb{R}$).*
(ii) *$\mathbb{R}$ is hereditarily separable.*
(iii) *$\mathbb{R}$ is SS.*

The crucial point of the proof of (iii) ⇒ (i) in Proposition 2.3 is that given the base of $\mathbb{R}$ with the standard topology consisting of open intervals $((q_n, r_n))_{n\in\omega}$ with rational endpoints, then for each $n \in \omega$, one may effectively define (i.e., without using any form of choice) a bijection $f_n : \mathbb{R} \to (q_n, r_n)$. This result however, does not remain valid in case we replace $\mathbb{R}$ with a proper subset A of $\mathbb{R}$.

Proposition 2.4. *([2]) The following statements are equivalent:*

(i) *CAC($\mathbb{R}$).*
(ii) *Every second countable topological space is SSC.*
(iii) $\mathbb{R}$ *(with the standard topology) is SSC.*

Proposition 2.5. *The following statements are equivalent:*

(i) *CAC($\mathbb{R}$).*
(ii) $\mathbb{R}$ *is hereditarily SSC.*
(iii) $\mathbb{R}$ *is hereditarily SS.*
(iv) *Every countable subspace of* $\mathbb{R}$ *is SSC.*

Proof. (i) ⇒ (ii) follows from the fact that every second countable space is hereditarily second countable and from Proposition 2.4.

(ii) ⇒ (iii) See the forthcoming Theorem 3.7 (i).

(iii) ⇒ (iv) In view of Proposition 2.3, our hypothesis implies CAC($\mathbb{R}$). Hence, (iv) holds.

(iv) ⇒ (i) This follows from the fact that $\mathbb{Q}$ (= the rationals) is SSC implies CAC($\mathbb{R}$) (the latter implication can be proved as in (iii) ⇒ (i) of Theorem 2.3 in [2]). □

Proposition 2.6. *([3]) CAC($\mathbb{R}$) iff every family $\mathcal{A} = \{A_n : n \in \omega\}$ of non-empty subsets of reals has a partial choice function (i.e., $\mathcal{A}$ has an infinite subfamily with a choice function).*

3. Main results

One of our primary concerns in this paper is the kinds of subspaces of metric spaces which inherit the SS or SSC property. In the next theorem we list such subspaces.

Theorem 3.1. *The following statements are provable in ZF:*

(i) *Every closed subspace of a SSC metric space is SSC.*
(ii) *If (X, d) is a SS metric space and O is an open subset of X, then the closure of O in X is SS.*
(iii) *Every dense subspace of a SS metric space is SS.*

(iv) *Every open subset of a SS metric space is SS. In particular, if X is a metric space and Y is a SS subspace of X, then the interior Y° of Y in X is SS.*

Proof. (i) Fix a SSC metric space (X, d) and let T_d be the topology on X induced by d. Let A be a closed subset of X and let $\mathcal{B}$ be a base for the subspace topology on A. Put $\mathcal{C} = \{O \in T_d : O \cap A \in \mathcal{B}\} \cup \{O \in T_d : O \cap A = \emptyset\}$. We assert that $\mathcal{C}$ is a base for T_d. Indeed, let $O \in T_d$ and let $x \in O$. We consider the following cases:

(a) $O \cap A = \emptyset$. Then $O \in \mathcal{C}$ and $x \in O \subset O$.

(b) $O \cap A \neq \emptyset$. If $x \notin A$, then $O \cap A^c \in T_d$ (since A is closed), $O \cap A^c \in \mathcal{C}$, and $x \in O \cap A^c \subset O$. If $x \in A$, then there exists $B \in \mathcal{B}$ such that $x \in B \subset O \cap A$. Let $V \in T_d$ be such that $B = V \cap A$ and let $W = V \cap O$. Then $W \in T_d$, $x \in W \subset O$ and $W \cap A = V \cap O \cap A = B \cap O = B \in \mathcal{B}$.

Thus, $\mathcal{C}$ is a base for T_d. Since X is SSC, let $\mathcal{D} = \{D_n : n \in \omega\} \subset \mathcal{C}$ be a base for T_d. Let $M = \{n \in \omega : D_n \cap A = \emptyset\}$. Then $\mathcal{D}' = \{D_n \cap A : n \in \omega \setminus M\} \subset \mathcal{B}$ is clearly a countable base for the subspace topology on A and A is SSC as required.

(ii) Fix a dense subset Y of $\overline{O}$. We show that Y is separable. Let $Z = (Y \cap O) \cup (X \backslash \overline{O})$. Clearly, Z is a dense subset of X, so there exists a countable subset D of Z which is dense in Z. Then $D \cap Y$ is a countable dense subset of Y, hence Y is separable as required.

(iii) This is straightforward.

(iv) This follows from (ii) and (iii). □

Theorem 3.2. *M(SSC,S) is provable in ZF.*

Proof. Let (X, d) be a SSC metric space and let T_d be the topology on X which is induced by d. Consider the base $\mathcal{B} = \{D(x, 1/n) : x \in X, n \in \mathbb{N}\}$ for T_d. Since X is SSC, there exists a subfamily $\mathcal{C} = \{C_n : n \in \mathbb{N}\}$ of $\mathcal{B}$ which is a base for T_d. Without loss of generality (wlog) assume that for each $n \in \mathbb{N}$, C_n does not have a unique center (i.e., if $C_n = D(x, 1/k)$ for some $x \in X$, $k \in \mathbb{N}$, then there exists $y \in X \backslash \{x\}$ such that $C_n = D(y, 1/k)$). For each $n \in \mathbb{N}$, let $D_n = \{C_n \backslash \{x\} : x \text{ is a center of } C_n\}$. Then $\mathcal{D} = \bigcup\{D_n : n \in \mathbb{N}\}$ is a base for T_d. Indeed, let $x \in X$ and $n \in \mathbb{N}$. Since $\mathcal{C}$ is a base, there exist $m, k \in \mathbb{N}$, and $y \in X$ such that $x \in C_m = D(y, 1/k) \subset D(x, 1/n)$. Wlog assume that $y \neq x$. Then $x \in C_m \backslash \{y\} \subset D(x, 1/n)$. Hence, $\mathcal{D}$ is a base for T_d and by our hypothesis on X, let $\mathcal{E} = \{E_n : n \in \mathbb{N}\} \subset \mathcal{D}$ be a base. For each $n \in \mathbb{N}$, let $k_n = \min\{k \in \mathbb{N} : \exists x \in X,\ E_n = C_k \backslash \{x\}\}$. It can be readily verified that for each $n \in \mathbb{N}$, there exists a unique element $x_{k_n} \in X$ such that $E_n = C_{k_n} \backslash \{x_{k_n}\}$ and that the set $F = \{x_{k_n} : n \in \mathbb{N}\}$ is dense in X. Thus, X is separable and the proof of the theorem is complete. □

Next we show that the statement *"every SSC metric space is hereditarily SSC"* is a theorem of ZF. First we need to establish the following Lemma.

Lemma 3.3. *The following statements are equivalent:*

(i) *CAC($\mathbb{R}$).*

(ii) *There exists a dense-in-itself SSC metric space.*

Proof. (i) ⇒ (ii) $\mathbb{R}$ with the standard metric $|\cdot|$ is a dense-in-itself metric space, and by Proposition 2.4, $\mathbb{R}$ is SSC.

(ii) ⇒ (i) Fix a dense-in-itself SSC metric space (X, d). Without loss of generality we may assume that X is infinite. Let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ be a disjoint family of non-empty subsets of $\mathcal{P}(\mathbb{N})$ (in ZF, $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$; see [5]). In view of Proposition 2.6 it suffices to show that $\mathcal{A}$ has a partial choice function. By Theorem 2, let $G = \{g_n : n \in \mathbb{N}\}$ be a dense subset of X. Let x be any element of X. Without loss of generality assume that for all $n \in \mathbb{N}$, $D(x, 1/n) \setminus B(x, 1/(n+1)) \neq \emptyset$. Via induction we construct a sequence $(x_m)_{m\in\mathbb{N}}$ such that for each $m \in \mathbb{N}$, $x_m \in G \cap (D(x, 1/m) \setminus B(x, 1/(m+1)))$ and a sequence $(\mathbf{x}_m)_{m\in\mathbb{N}}$ such that for each $m \in \mathbb{N}$, $\mathbf{x}_m$ is a sequence of elements of $G \cap (D(x, 1/m) \setminus B(x, 1/(m+1)))$ converging to the point x_m.

We start the induction by letting

$$n_1 = \min\{n \in \mathbb{N} : g_n \in D(x, 1) \setminus B(x, 1/2)\}$$

and $x_1 = g_{n_1}$. Then via a straightforward induction and using the fact that X is dense-in-itself, construct a sequence $\mathbf{x}_1 \subset G \cap (D(x, 1) \setminus B(x, 1/2))$ which is not eventually constant and which converges to x_1.

Assume that for each $i \leq k$ we have defined an element $x_i \in G \cap (D(x, 1/i) \setminus B(x, 1/(i+1)))$ and a sequence

$$\mathbf{x}_i \subset G \cap (D(x, 1/i) \setminus B(x, 1/(i+1)))$$

which converges to x_i.

Let $n_{k+1} = min\{n \in \mathbb{N} : g_n \in G \cap (D(x, 1/(k+1)) \setminus B(x, 1/(k+2)))\}$ and put $x_{k+1} = g_{n_{k+1}}$. Construct a sequence

$$\mathbf{x}_{k+1} \subset G \cap (D(x, 1/(k+1)) \setminus B(x, 1/(k+2)))$$

converging to x_{k+1}. The induction terminates.

For every $m \in \mathbb{N}$ and for every $Z \in A_m$, let

$$O_{mZ} = D(x, 1/m) \setminus (\{\mathbf{x}_m(i) : i \in Z\} \cup \{x_m\}).$$

It is straightforward to verify that O_{mZ} is an open set for each $m \in \mathbb{N}$ and for each $Z \in A_m$, and that the collection $\mathcal{B} = \{O_{mZ} : m \in \mathbb{N}, Z \in A_m\} \cup \{D(y, 1/n) : y \in X, n \in \mathbb{N}, x \notin D(y, 1/n)\}$ is a base for the metric topology on X. Since X is SSC, let $\{O_{n_i Z_{n_i}} : i \in \mathbb{N}\} \subset \mathcal{B}$ be a strictly decreasing neighbourhood base at x. Clearly, the function $f = \{(n_i, Z_{n_i}) : i \in \mathbb{N}\}$ is a partial choice function for the family $\mathcal{A}$. This completes the proof of (ii) $\Rightarrow$ (i) and of the Lemma. □

Remark 3.4. In contrast to the result of Lemma 3.3, the statement "*there exists a dense-in-itself uncountable hereditarily SS metric space*" does not imply the axiom CAC($\mathbb{R}$). Indeed, consider the basic Cohen model $\mathcal{M}1$ in [4]. It is well-known that CAC($\mathbb{R}$) fails in that model (see [4]). Now, the set $\mathbb{R}_{\mathcal{M}}$ of the reals in the countable transitive model $\mathcal{M}$ which serves as a ground model for the forcing construction of $\mathcal{M}_1$ is well-orderable in $\mathcal{M}$ as well as in $\mathcal{M}_1$. Hence, the aforementioned topological statement is obviously true of $\mathcal{M}_1$.

Theorem 3.5. *M(SSC,hSSC) is provable in ZF.*

Proof. Fix (X, d) a SSC metric space. Without loss of generality we may assume that X is infinite. If CAC($\mathbb{R}$) holds, then the conclusion follows from the fact that every second countable space is hereditarily second countable and from Proposition 2.4. So assume that CAC($\mathbb{R}$) fails. By Lemma 3.3 and Theorem 3.1 (i), it follows that X is a scattered space. Let $\alpha = |X|_{CB}$. If $\alpha = 1$ (i.e., $\mathrm{Iso}(X) = X$), then X is trivially hereditarily SSC. So, assume that $\alpha > 1$. We consider two cases:

Case 1. $\alpha = 2$. Fix a subspace Y of X. We will show that Y is SSC. If $Y \subset \mathrm{Iso}(X)$, then the conclusion is straightforward. So, assume that $Y \cap X_1 \neq \emptyset$. Let $\mathcal{B}$ be a base for (Y, d). Without loss of generality we may assume that $\forall O \in \mathcal{B}$, $|O \cap X_1| \leq 1$. We consider the folowing two cases:

(i) $\forall x \in (X_1 \cap Y), \forall^\infty n \in \mathbb{N}$ (i.e., for all but finitely many $n \in \mathbb{N}$), $|D(x, 1/n) \setminus D(x, 1/(n+1))| < \aleph_0$. For every $x \in (X_1 \cap Y)$ fix $n_x \in \mathbb{N}$ such that $|D(x, 1/n_x) \cap X_1| = 1$. Without loss of generality we may assume that:

(*) *For every* $x, y \in (X_1 \cap Y)$, $x \neq y$, $B(x, 1/n_x) \cap B(y, 1/n_y) = \emptyset$.

It is straightforward to verify that

$$\mathcal{B}' = \{\{x\} : x \in \mathrm{Iso}(X) \cap Y\} \cup \{O \in \mathcal{B} : \exists x \in X_1 \cap Y, x \in O \subset D(x, 1/n_x)\}$$

is a base for Y. We show that $\mathcal{B}'$ is countable. As X is SSC, it follows that $\mathrm{Iso}(X)$ is countable. Fix an enumeration of the set $[\mathrm{Iso}(X)]^{<\omega}$ of all finite subsets of $\mathrm{Iso}(X)$ (in ZF, $|[\omega]^{<\omega}| = |\omega|$).

Claim 1. For every $x \in (X_1 \cap Y)$ and every $O \in \mathcal{V}_x = \{W \in \mathcal{B} : x \in W \subset D(x, 1/n_x)\}$, $D(x, 1/n_x) \backslash O \in [\mathrm{Iso}(X)]^{<\omega}$.

Proof of claim 1. Fix $m > n_x$ such that $D(x, 1/m) \subset O$. Then, $D(x, 1/n_x) \backslash O \subseteq [D(x, 1/n_x) \backslash D(x, 1/(n_x + 1))] \cup [D(x, 1/(n_x + 1)) \backslash D(x, 1/(n_x + 2))] \cup \ldots \cup [D(x, 1/(m-1)) \backslash D(x, 1/m)]$. Since the set on the right hand side of the latter $\subseteq$ is finite (as a finite union of finite sets), it follows that $D(x, 1/n_x) \backslash O \in [\mathrm{Iso}(X)]^{<\omega}$ finishing the proof of claim 1.

Since for every $O, Q \in \mathcal{V}_x$, $O \neq Q$, $D(x, 1/n_x) \backslash O \neq D(x, 1/n_x) \backslash Q$, it follows from (*) and claim 1 that the function h which maps the element $Q \in \{O \in \mathcal{B} : \exists x \in X_1 \cap Y, x \in O \subset D(x, 1/n_x)\}$ to $D(x, 1/n_x) \backslash Q$ is $1:1$. Thus $\mathcal{B}'$ is countable and Y is SSC as required.

(ii) $\exists x \in (X_1 \cap Y), \exists^{\infty} n \in \mathbb{N}$ (there are infinitely many $n \in \mathbb{N}$) such that $|D(x, 1/n) \backslash D(x, 1/(n+1))| = \aleph_0$. Fix $x \in (X_1 \cap Y)$ satisfying the latter assumption. For our convenience we assume that:

$$\forall n \in \mathbb{N}, |D(x, 1/n) \backslash D(x, 1/(n+1))| = \aleph_0 \text{ and } B(x, 1) \cap X_1 = \{x\}.$$

We show that assumption (ii) cannot occur by establishing that:

Claim 2. $\mathrm{CAC}(\mathbb{R})$ holds.

Proof of claim 2. Let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ be a disjoint family of infinite subsets of $\mathcal{P}(\mathrm{Iso}(X))$. (In ZF, $|\mathcal{P}(\mathrm{Iso}(X))| = |\mathbb{R}|$). Without loss of generality we may assume that for every $n \in \mathbb{N}, \bigcup A_n \subset D(x, 1/n) \backslash D(x, 1/(n+1))$. It can be readily verified that

$$\mathcal{B} = \{\{y\} : y \in \mathrm{Iso}(X)\} \cup \{D(y, 1/(n+1)) : y \in X_1 \backslash \{x\}, n \in \mathbb{N}\} \cup \{D(x, 1/(n+1)) \cup O : O \in A_n, n \in \mathbb{N}\},$$

is a base for the topology T on X. Since X is SSC, it follows that there exists a family $\mathcal{C} = \{C_n : n \in \omega\} \subset \mathcal{B}$ such that $\mathcal{C}$ is a base for T.

Let $\mathcal{V}_x = \{D(x, 1/n_k) \cup On_k : k \in \mathbb{N}\} \subset \mathcal{C}$ be a neigbourhood base for x. Since $\mathcal{V}_x$ is infinite, it follows that the set $\{O_{n_k} : k \in \mathbb{N}\}$ is infinite. This immediately induces a partial choice function for the family $\mathcal{A}$ finishing the proof of claim 2 and case 1.

Case 2. $\alpha > 2$. Since X_2 is a non-empty closed subset of the scattered space X, it follows that $\text{Iso}(X_2) \neq \emptyset$. Fix $x \in \text{Iso}(X_2)$ and construct via induction a strictly increasing sequence of natural numbers $\{n_k : k \in \mathbb{N}\}$ such that: $\forall k \in \mathbb{N}, |D(x, 1/n_k) \backslash D(x, 1/n_{k+1})| = \aleph_0$. (Let $n_1 \in \mathbb{N}$ be such that $D(x, 1/n_1) \cap X_2 = \{x\}$. Since $x \notin \text{Iso}(X_1)$, $D(x, 1/n_1)$ contains infinitely many points of $\text{Iso}(X_1)$. Let $m \in \mathbb{N}$ and $y \in \text{Iso}(X_1)$ be such that $y \in V = D(x, 1/n_1) \backslash B(x, 1/m)$. Since $y \in \text{Iso}(X_1)$, let $W \subset V$ be an open neighbourhood of y such that $W \cap X_1 = \{y\}$. As $y \notin \text{Iso}(X)$, W contains infinitely many points of $\text{Iso}(X)$. Let $n_2 = \min\{n > n_1 : D(x, 1/n_1) \backslash D(x, 1/n)$ is an infinite subset of $\text{Iso}(X) \cup \text{Iso}(X_1)\}$. Since $\text{Iso}(X) \cup \text{Iso}(X_1)$ is countably infinite (due to the fact that X and X_1 are SSC), $D(x, 1/n_1) \backslash D(x, 1/n_2)$ is countably infinite. Continue this process by induction). Let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ be a disjoint family of infinite subsets of $\mathcal{P}(\text{Iso}(X) \cup \text{Iso}(X_1))$. Without loss of generality we may assume that for every $k \in \mathbb{N}$, $\bigcup A_{n_k} \subset D(x, 1/n_k) \backslash D(x, 1/n_{k+1})$. Clearly,

$$\mathcal{B} = \{O \in T_d : O \cap X_2 = \emptyset\} \cup \{D(y, 1/n_k) : k \in \mathbb{N}, y \in X_2 \backslash \{x\}\} \cup \{D(x, 1/n_{k+1}) \cup O : k \in \mathbb{N}, O \in A_{n_k}\}$$

is a base for T_d. We may continue now as in the proof of Claim 2 in order to verify that $\mathcal{A}$ has a partial choice function. Hence, CAC($\mathbb{R}$) holds and this contradicts our assumption (on the failure of CAC($\mathbb{R}$)). This completes the proof of Case 2 and of the theorem. □

Corollary 3.6. *The following statements are equivalent:*

(i) *CAC($\mathbb{R}$).*
(ii) *If X is a SS subspace of $\mathbb{R}$, then $\overline{X}$ is SS.*
(iii) *If X is a SSC subspace of $\mathbb{R}$, then $\overline{X}$ is SSC.*

Proof. (i) ⇒ (ii) This follows from the fact that CAC($\mathbb{R}$) holds iff $\mathbb{R}$ is hereditarily SS (see Proposition 2.5).

(ii) ⇒ (iii) Clearly, $\mathbb{Q}$ is SS in ZF, hence by (ii), $\mathbb{R} = \overline{\mathbb{Q}}$ is SS. By Proposition 2.3, CAC($\mathbb{R}$) holds, hence by Proposition 2.5, every subspace of $\mathbb{R}$ is SSC.

(iii) ⇒ (i) Let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ be a disjoint family of non-empty subsets of $\mathcal{P}(\mathbb{Q})$. For each $n \in \mathbb{N}$, pick an increasing sequence $(q_{n,m})_{m\in\mathbb{N}} \subset \mathbb{Q} \cap (1/(n+1), 1/n)$ converging to $1/n$. Let $X = \{q_{n,m} : n, m \in \mathbb{N}\}$. Clearly, X is a countable discrete subspace of $\mathbb{R}$, hence it is SSC (in ZF). By our hypothesis, $\overline{X}=X\cup\{0\}\cup\{1/n : n\in\mathbb{N}\}$ is a SSC space. Since the second Cantor-Bendixson derivative $\overline{X}_2 = \{0\}$, we may continue as in the proof of Case 2 in Theorem 3.5 in order to obtain a partial choice function for $\mathcal{A}$. □

We show next that every SSC metric space is SS in ZF, and that the reverse implication is not provable without some form of choice. Moreover, we establish that unlike the result of Theorem 3.5, subspaces of SS metric spaces need not be SS in ZF.

Theorem 3.7. (i) *M(SSC,SS) is provable in ZF.*
(ii) *CAC($\mathbb{R}$) iff M(SS,SSC) iff M(SS,hSS). Hence, M(SS,SSC) and M(SS,hSS) are not provable in ZF.*

Proof. (i) This follows from Theorems 3.2 and 3.5.

(ii) CAC($\mathbb{R}$) ⇒ M(SS,SSC) This follows from Proposition 2.4.

M(SS,SSC) ⇒ M(SS,hSS) This follows from Theorem 3.5 and from (i) of the present theorem.

M(SS,hSS) ⇒ CAC($\mathbb{R}$) Consider $X = (\mathbb{R} \times \{0\}) \cup (\mathbb{Q} \times \mathbb{Q})$ as a subspace of $\mathbb{R} \times \mathbb{R}$. Then X is a SS metric space. Indeed, let Y be a dense subset of X. Necessarily, we have that $Y^* = Y \cap (\mathbb{Q} \times \mathbb{Q})$ is countably infinite. We assert that Y^* is dense in Y. Let O be a non-empty open subset of X. Then $O \cap (\mathbb{Q} \times \mathbb{Q}) \neq \emptyset$ and since $\mathbb{R} \times \{0\}$ is closed in X, there exists an open set $P \subset O$ such that $P \cap (\mathbb{R} \times \{0\}) = \emptyset$. Since Y is dense in X, we must have that $P \cap Y^* \neq \emptyset$. Thus, Y is separable. By M(SS,hSS), the subspace $\mathbb{R}\times\{0\}$, hence $\mathbb{R}$, is SS, hence by Proposition 2.3, CAC($\mathbb{R}$) holds. □

Remark 3.8. (1) From the proof of Theorem 3.7 (ii) we infer that the statement "*every closed nowhere dense subset of a SS metric space is SS*" is equivalent to CAC($\mathbb{R}$), and consequently, it is not a theorem of ZF. (Note that in the proof of M(SS,hSS) ⇒ CAC($\mathbb{R}$) in Theorem 3.7, $\mathbb{R} \times \{0\}$ is a closed nowhere dense subset of the SS space X which fails to be SS in case CAC($\mathbb{R}$) fails).

(2) The statement "*there exists a continuum sized dense-in-itself SS metric space*" is provable in ZF. Indeed, consider the Cantor set C. Then $C \cup \mathbb{Q}$ is a dense-in-itself SS metric space of size $|\mathbb{R}|$.

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