RECOGNIZING INDECOMPOSABLE
SUBCONTINUA OF SURFACES FROM THEIR
COMPLEMENTS

by

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ABSTRACT. We prove two theorems which allow one to recognize indecomposable subcontinua of closed surfaces without boundary. If X is a subcontinuum of a closed surface S, we call the components of $S \setminus X$ the *complementary domains of* X. We prove that a continuum X is either indecomposable or the union of two indecomposable continua whenever it has a sequence $(U_n)_{n=1}^{\infty}$ of distinct complementary domains such that $\lim_{n \to \infty} \partial U_n = X$. We define a slightly stronger condition on the complementary domains of X, called the *double-pass condition*, which we conjecture is equivalent to indecomposability. We prove that this is so for continua which are not the boundary of one of their complementary domains.

1. INTRODUCTION

For us, a *continuum* is a compact connected metric space. A *closed surface* is a compact and connected, but not necessarily orientable, 2-manifold without boundary. We are interested in conditions which imply that a subcontinuum of a closed surface is topologically complicated. Further, we would like for these conditions to rely upon how the continuum is embedded in its ambient space, rather than to use internal characteristics of the continuum.

We use *indecomposability* as a criterion for topological complexity. A continuum X is *decomposable* if it can be written as the union of a pair of proper subcontinua A and B. The pair (A, B) is then called a *decomposition* of X. A continuum which is not decomposable is called *indecomposable*.

Non-degenerate indecomposable continua (i.e., indecomposable continua consisting of more than a point) are certainly quite complicated and have rich internal structure. For example, a non-degenerate continuum X is indecomposable if and only if, for all $p \in X$, the *composant* of p

$$C_p = \bigcup \{\text{proper subcontinua of } X \text{ containing } p\}$$

is a dense set of first category [10, Exercise 5.20a and Proposition 11.14], in which case there are uncountably many disjoint composants. In contrast, decomposable continua have either one or three composants [10, Theorem 11.15].

The rich structure of indecomposable continua provides many interesting internal methods of recognizing indecomposability. We are instead interested in recognizing indecomposability based on the continuum's interaction with the space in which it lies. This approach is in the spirit of some classical work of Kuratowski [7], Rutt [11], and Burgess [2].

In Section 2, we recall some earlier theorems which the current work extends. In Section 3, we extend a theorem of C. E. Burgess about planar continua. In Section 4 we extend the characterization of planar indecomposable continua in [3] to certain subcontinua of closed surfaces – those subcontinua which are not the boundary of any of their complementary domains. Finally, in Section 5 we state questions and conjectures.

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2. Prior Work and Notions

Let S be a closed surface, and $X \subset S$ be a continuum. A component of $S \setminus X$ is called a *complementary domain* of X. If X equals the boundary of one of its complementary domains, then X is called *unshielded*; otherwise, it is *shielded*.

In order to shorten some statements, we say that a continuum X is *2-indecomposable* if X is decomposable but cannot be written as the essential union of three proper subcontinua. By a theorem of Burgess [2, Theorem 1], a 2-indecomposable continuum is in some sense uniquely decomposable. Specifically, there exists a decomposition (A, B) of X where A and B are indecomposable continua. Further, if (C, D) is any other decomposition, then C and D each contain exactly one of A and B.

The first and most famous result in the vein of this work is by Kuratowski. It deals exclusively with continua which are unshielded.

Theorem 2.1 ([7, Theorem 11 of §62]). *Let X be a planar continuum. If X is the common boundary of three of its complementary domains, then X is either indecomposable or 2-indecomposable.*

C. E. Burgess generalized this and other related theorems. The following is a corollary to the main result in [2]. It complements Theorem 2.1 in that it can detect indecomposability in shielded continua.

Theorem 2.2 ([2, Corollary of Theorem 10]). *Let X be a planar continuum, and let $(U_n)_{n=1}^{\infty}$ be a sequence of distinct complementary domains of X. If $X = \lim_{n \to \infty} \partial U_n$, then X is either indecomposable or 2-indecomposable.*

Our Theorem 3.7 extends this theorem. It states that the above holds not only for continua in the plane, but for continua in all closed surfaces.

To state the next theorem, we need some terminology from [3]. We slightly extend the definitions to apply to an arbitrary closed surface, whereas the work from [3] applies only to $\mathbb{S}^2$.

Definition 2.3 (generalized crosscut). Let S be a closed surface and U a connected open subset with non-degenerate boundary. A *generalized crosscut* of U is a homeomorphic copy $A \subset U$ of the open interval $(0, 1)$ such that $\overline{A} \setminus A \subset \partial U$.

Generalized crosscuts are similar to crosscuts from prime end theory, except that the closure of a crosscut is by definition a compact arc. The closure of a generalized crosscut, however, need not be locally connected.

Definition 2.4 (shadow)**.** Let A be a generalized crosscut of U. A component V of $U \setminus A$ is called a *crosscut neighborhood* of A. The *shadow* of A corresponding to V is the set $\overline{V} \cap \partial U$.

Observe that a generalized crosscut has either one or two shadows. We can now state the *double-pass condition*.

Definition 2.5 (double-pass condition)**.** Let $X \subset S$ be a continuum, and let $(U_n)_{n=1}^{\infty}$ be a sequence of complementary domains of X. We say that $(U_n)_{n=1}^{\infty}$ satisfies the *double-pass condition* when, for any choice of generalized crosscuts K_n of U_n, there exists a sequence of shadows S_n of K_n so that $\lim_{n \to \infty} S_n = X$.

The main result of [3] is the following theorem.

Theorem 2.6 ([3, Theorem 1.4])**.** *A continuum $X \subset \mathbb{S}^2$ is indecomposable if and only if it has a sequence $(U_n)_{n=1}^{\infty}$ of complementary domains satisfying the double-pass condition.*

We extend this theorem in Section 4. Specifically, we prove that a *shielded* continuum in a surface is indecomposable if and only if it has a sequence of complementary domains satisfying the double-pass condition.

3. An Extension of a Theorem of Burgess

In this section, we extend Theorem 2.2 for planar continua to continua in arbitrary closed surfaces. Namely, we prove that a continuum in a surface which is the limit of boundaries of *distinct* complementary domains is either indecomposable or 2-indecomposable.

3.1. **Graphs and Cat's Cradles.** The following lemma gives us a convenient way to gain information about the genus of a particular surface.

Definition 3.1. The *complete bipartite graph* $K_{m,n}$ is the graph with $m + n$ vertices, $\{a_1, \ldots, a_m, b_1, \ldots, b_n\}$, and mn edges joining every vertex in $\{a_1, \ldots, a_m\}$ to every vertex in $\{b_1, \ldots, a_n\}$.

Lemma 3.2. *If S is a closed surface containing an embedding of the graph $K_{3,4g-1}$, then the genus of S is at least g.*

Proof. According to [4], the minimal genus g of any surface in which the graph $K_{m,n}$ can be embedded is $\left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$. (Here, $\lceil x \rceil$ denotes the smallest integer n such that $x \leq n$.) Substituting $m = 3$ and $n = 4g-1$ gives that the minimal genus for a surface to contain a copy of $K_{3,4g-1}$ is equal to $\left\lceil g - \frac{3}{4} \right\rceil = g$. $\qquad \square$

For the remainder of this section, let S denote a particular closed surface of genus g. Below we will use the following convenient notation: For an arc A, let $\operatorname{Int}(A)$ denote the open sub-arc of A joining the endpoints of A.

Definition 3.3 (cat's cradle). Let $d_1, d_3 \in S$ be distinct points. A collection of arcs $\{A_\alpha\}_{\alpha \in J}$ is called a *cat's cradle (between d_1 and d_3)* if

    (1) d_1 and d_3 are the endpoints of each A_α and
    (2) for distinct $\alpha, \beta \in J$, $\operatorname{Int}(A_\alpha) \cap \operatorname{Int}(A_\beta) = \emptyset$.

In addition, if each A_α intersects a set D_2 disjoint from $\{d_1, d_3\}$, then $\{A_\alpha\}_{\alpha \in J}$ is a cat's cradle *through D_2*.

We now prove a general fact about a cat's cradle through a closed disk in a surface. On the face of it, there are many ways in which the arcs of a cat's cradle could be arranged. It is certainly possible for the arcs to intersect the disk D_2 in parallel chords (in a way that one might call *linearly ordered*, defined precisely below). On the other hand, the arcs could be arranged in such a way that no chord in the intersection divides the disk between any other two chords in the intersection. The following lemma shows that this second possibility does not occur often in a surface of finite genus.

Lemma 3.4. *Let $d_1, d_3 \in S$, and let $D_2 \subset S$ be a closed disk. Suppose $(A_n)_{n=1}^{4g+3}$ is a cat's cradle between d_1 and d_3 through D_2, where g is the genus of S. Then no component of $D_2 \setminus \bigcup_{n=1}^{4g+3} A_n$ can have closure which meets every element of $(A_n)_{n=1}^{4g+3}$.*

Proof. Suppose some component V of $D_2 \setminus \bigcup_{n=1}^{4g+3} A_n$ has closure which meets every arc A_n. Designate a point $d_2 \in V$, and choose arcs $(A'_n)_{n=1}^{4g+3}$ satisfying the following for all different indices m and n:

    (1) A'_n joins d_2 to a point of A_n, with no proper sub-arc of A'_n doing so;

(2) $A'_n \cap A'_m = \{d_2\}$; and

(3) $\mathrm{Int}(A'_n) \subset V$ for each n.

The graph $\bigcup_{n=1}^{4g+3}(A_n \cup A'_n)$ is then homeomorphic to the graph $K_{3,4g+3}$, being the complete bipartite graph between the sets $\{d_1, d_2, d_3\}$ and $\bigcup_{n=1}^{4g+3}(A_n \cap A'_n)$. Lemma 3.2 indicates that the genus of S is at least $g+1$, contradicting the assumption that it is g. $\qquad\square$

Lemma 3.6 will be used in the proof of Theorem 3.7 to build interesting graphs in S. To state it, we need a definition.

Definition 3.5 (linearly ordered). Let $(A'_\alpha)_{\alpha \in J}$ be an ordered sequence of pairwise disjoint compact arcs in a closed disk D_2, each irreducible with respect to intersecting ∂D_2 twice. We say that $(A'_\alpha)_{\alpha \in J}$ is *linearly ordered* if, for $\alpha < \beta < \gamma$ in J, A'_α is separated from A'_γ in D_2 by A'_β. If J is finite, we call a component of $D_2 \setminus \bigcup_{n \in J} A'_n$ an *end component* if its closure meets only one element of $(A'_n)_{n \in J}$.

In what follows a *chord* of a disk D_2 is an arc in D_2 which intersects ∂D_2 exactly at its endpoints.

Lemma 3.6. *Let $(A_n)_{n=1}^\infty$ be a cat's cradle from d_1 to d_3 through a closed disk D_2, and suppose that $A_n \cap \partial D_2$ is finite for each n. Then there is a subsequence $(A_{n_i})_{i=1}^\infty$ and a collection of sub-arcs $(A'_{n_i})_{i=1}^\infty$, $A'_{n_i} \subset A_{n_i} \cap D_2$, so that $(A'_{n_i})_{i=1}^\infty$ is linearly ordered in D_2.*

Proof. First, using Lemma 3.4, notice that only finitely many components of $D_2 \cap \bigcup_{n=1}^\infty A_n$ do not meet D_2 in its interior. Hence, by passing to a subsequence, we may assume that $D_2 \setminus A_i$ is not connected for any $i \in \mathbb{N}$. In particular, $A_i \cap D_2$ contains a non-degenerate chord for every $i \in \mathbb{N}$ and therefore separates D_2.

There are two cases.

Case 1. One case is that, for any element A_α, there is an element A_β such that no element A_γ separates A_α from A_β in D_2. We will build by induction an increasing sequence $(n_k)_{k=1}^\infty$ and a corresponding sequence of chords $(A'_{n_k})_{k=1}^\infty$ with $A'_{n_k} \subset A_{n_k} \cap D_2$ such that, for all $k \geq 1$, the following two conditions are met:

(1) The chords $A'_{n_1}, \ldots, A'_{n_k}$ are linearly ordered; and

(2) some end component of $D_2 \setminus \bigcup_{i=1}^{k} A'_{n_i}$ bounded by A'_{n_k} intersects infinitely many elements of $(A_n)_{n>n_k}$. (Notice that there is a unique such end component if $k \neq 1$.)

For the base case of our induction, set $n_1 = 1$, and let $A'_{n_1} \subset A_{n_1}$ be any non-degenerate chord in D_2. There are then two components to $D_2 \setminus A'_{n_1}$, each an end component of $D_2 \setminus A'_{n_1}$. Infinitely many elements of the collection $(A_n)_{n \geq n_1}$ must intersect one of them. We have therefore satisfied the requirements for $k = 1$.

Now suppose that chords $A'_{n_1}, \ldots, A'_{n_k}$ have been found which satisfy conditions 1 and 2. Consider the collection $\{A_{N_1}, \ldots, A_{N_m}\}$ of arcs which are not separated from A'_{n_k} by any other A_n. (There may only be finitely many by Lemma 3.4.) Let V denote the end component of $D_2 \setminus \bigcup_{i=1}^{k} A'_{n_i}$ meeting A'_{n_k} Because each A_{N_i} intersects ∂D_2 in finitely many points, we see that $\bigcup_{i=1}^{m} A_{N_i}$ divides V into finitely many components. By choice of $N_1, \ldots, N_m$, no member of $(A_n)_{n>\max(N_1, \ldots, N_m)}$ intersects the component of $V \setminus \bigcup_{i=1}^{m} A_{N_i}$ which meets A'_{n_k}. Therefore, infinitely many members must intersect another component W of $V \setminus \bigcup_{i=1}^{m} A_{N_i}$. Let $A'_{n_{k+1}}$ be a minimal sub-arc of $\bigcup_{i=1}^{m} A_{N_i}$ which separates W from A'_k. Then W is contained in the end component of $D_2 \setminus \bigcup_{i=1}^{k+1} A'_{n_i}$ meeting $A'_{n_{k+1}}$. That end component meets infinitely many members of $(A_n)_{n>n_{k+1}}$, so we have extended our linearly ordered collection to $k+1$ elements.

By induction, we have a sequence of chords $(A'_{n_i})_{i=1}^{\infty}$. Since each finite subsequence $(A_{n_i})_{i=1}^{k}$ is linearly ordered, $(A_{n_i})_{i=1}^{\infty}$ is linearly ordered.

Case 2. There exists A_α such that any A_β is separated from A_α in D_2 by some A_γ.

Choose $n_1 > \alpha$, and let $A'_{n_1} \subset A_{n_1}$ be any non-degenerate chord. Find a sequence $(A_{n_i})_{i=1}^{\infty}$ such that A_{n_i} is separated from A'_α by $A_{n_{i+1}}$. Recall that D_2, as a two-dimensional disk, is *unicoherent*, which implies that any closed set which separates two points of D_2 also has a component which does so. Therefore, a chord $A'_{n_{i+1}} \subset A_{n_{i+1}}$ also separates A_{n_i} from A'_α. This process inductively yields the ordered sequence $(A_{n_i})_{i=2}^{\infty}$, which is evidently linearly ordered. $\square$

3.2. **Burgess's Theorem for Surfaces.** Now we move to the specific setting of a continuum with infinitely many complementary domains.

Theorem 3.7 (Burgess's Theorem for Surfaces). *Let X be a subcontinuum of a closed surface S, and let $(U_n)_{n=1}^{\infty}$ be a sequence of distinct complementary domains of X. If $X = \lim_{n\to\infty} \partial U_n$, then X is either indecomposable or 2-indecomposable.*

Proof. By [2, Theorem 1], we must only show that X is not the essential union of three proper subcontinua. By way of contradiction, suppose that $X = X_1 \cup X_2 \cup X_3$, where each X_i is a proper subcontinuum of X not contained in $\bigcup_{j\neq i} X_j$. We will first find appropriate closed disks D_1, D_2, and D_3. Designated points in D_1 and D_3 will function as the endpoints of a cat's cradle through D_2.

By the definitions of X_1, X_2, and X_3, there exist closed disks $D_1, D_2, D_3 \subset S$ such that

    (1) the interior of D_i intersects X_i, and
    (2) D_i is disjoint from $D_j \cup X_j$ when $i \neq j$.

Because of the increasing density of $(U_n)_{n=1}^{\infty}$, U_n intersects each D_i when n is large enough. By passing to a subsequence, assume that $U_n \cap D_i \neq \emptyset$ for each $n \in \mathbb{N}$ and $i \in \{1, 2, 3\}$. Then for each $n \in \mathbb{N}$ let $A_n \subset U_n$ be an arc such that, for each $i \in \{1, 2, 3\}$,

    (1) $A_n \cap D_i \neq \emptyset$,
    (2) no proper sub-arc of A_n intersects each D_i, and
    (3) $A_n \cap \partial D_i$ is finite.

Note that the last condition can be achieved since each U_n is open.

It is evident that $\text{Int}(A_n)$ intersects only one D_i for each n. By passing to a subsequence and relabeling the disks, we can assume without loss of generality that $\text{Int}(A_n) \cap D_2 \neq \emptyset$. Extend each A_n to an arc $\tilde{A}_n$ with arcs in D_1 and D_3 to obtain a cat's cradle from d_1 to d_3 through D_2, where d_i is a designated point of D_i. Notice that $\bigcup_{n=1}^{\infty} \tilde{A}_n$ is still disjoint from X_2, since $\tilde{A}_n \setminus (D_1 \cup D_3) \subset U_n$ is disjoint from X and $\tilde{A}_n \cap (D_1 \cup D_3) \subset D_1 \cup D_3$ is disjoint from X_2.

Let $(A'_{n_i})_{i=1}^{\infty}$ be the linearly ordered sequence of compact arcs in $\left(\bigcup_{n=1}^{\infty} \tilde{A}_n\right) \cap D_2$ guaranteed by Lemma 3.6. First, notice that X_2 separates $A_{n_i} \cap D_2$ from $A_{n_j} \cap D_2$ in D_2 when $i \neq j$. To see this, observe that $A_{n_i} \cap D_2 \subset U_{n_i}$, and analogously $A_{n_j} \subset U_{n_j}$. These are distinct complementary domains of X, so X separates A_{n_i} from A_{n_j} in S and thus in D_2. Since $X \cap D_2 \subset X_2$, we see

that X_2 separates A_{n_i} from A_{n_j} in D_2. Accordingly, choose points $x_1, \ldots, x_{4g+3} \in X_2$ so that

(1) each x_i lies between A'_{n_i} and $A'_{n_{i+1}}$ in D_2, and
(2) the set $\bigcup_{j \neq i} A_{n_j}$ does not separate x_i from A_{n_i} in D_2.

Since X_2 is connected and $X_2 \cap \bigcup_{j=1}^{4g+4} A_{n_j} = \emptyset$, there exists an arc $J \subset S \setminus \bigcup_{j=1}^{4g+4} A_{n_j}$ containing $\{x_i\}_{i=1}^{4g+3}$. By choice of x_i, there are disjoint arcs $(A''_{n_i})_{i=1}^{4g+3} \subset D_2$ which join J to A_{n_i} without intersecting any other A'_{n_j}. By collapsing the arc J to a point, the graph $K_{3,4g+3}$ is obtained in

$$G = J \cup \bigcup_{i=1}^{4g+3} (\tilde{A}_{n_i} \cup A''_{n_i}),$$

i.e., $K_{3,4g+3}$ is a minor of G. Lemma 3.2 concludes that the genus of S is at least $g + 1$, contradicting our assumption that the genus is g. $\qquad\square$

3.3. **A Partial Converse.** As can be expected, Theorem 3.7 has a partial converse: If X is indecomposable, then there exists a sequence $(U_n)_{n=1}^{\infty}$ of complementary domains of X, not necessarily distinct, such that $\lim_{n\to\infty} \partial U_n = X$. This fact will be used in the next section, so we prove it here. The proof of this property follows closely the outline of [3, Theorem 2.10], modified slightly to allow for a finite degree of *multicoherence*.

Definition 3.8 (multicoherent)**.** A connected topological space X is *multicoherent of degree k* if, for any pair of closed, connected subsets A and B such that $A \cup B = X$, the intersection $A \cap B$ consists of at most k components.

Closed surfaces and punctured closed surfaces are examples of finitely multicoherent spaces. Let S be a finitely multicoherent space, and suppose that $A \subset S$ is a closed set which separates p from q for points $p, q \in S$. Then, by [12, Theorem 1], there is a closed subset $B \subset A$ which has at most k components which also separates p from q. We use this property in the following proof. Here, if A and B are compact non-empty subsets of a metric space (X, d), $H_d(A, B)$ represents the Hausdorff distance between A and B in the hyperspace of non-empty compact subsets of X. See [10, Chapter 4] for details.

Theorem 3.9. *Let X be an indecomposable continuum in the closed surface S. Then there exists a sequence $(U_n)_{n=1}^{\infty}$ of (not necessarily distinct) complementary domains of X such that $\lim_{n\to\infty} \partial U_n = X$.*

Proof. This is clear if X is a point, so assume X is a non-degenerate indecomposable continuum. For the purposes of this proof, suppose that S is equipped with a metric d in which the set

$$B_\epsilon(p) = \{x \in S \mid d(p,x) < \epsilon\}$$

is simply connected when $\epsilon \in (0,1)$. Let p, q, and r lie in different composants of X. For each $n \in \mathbb{N}$, define

$$Q_n = \text{the component of } X \setminus B_{1/n}(p) \text{ containing } q, \text{ and}$$
$$R_n = \text{the component of } X \setminus B_{1/n}(p) \text{ containing } r.$$

Notice that $\lim_{n\to\infty} Q_n = \lim_{n\to\infty} R_n = X$, by density of composants. Since Q_n and R_n are different components of $X \setminus B_{1/n}(p)$, they are separated in $S \setminus B_{1/n}(p)$ by $S \setminus (B_{1/n}(p) \cup X)$. Thus, Q_n and R_n are closed and separated in the normal space $S \setminus B_{1/n}(p)$, so there is a subset K_n, closed in $S \setminus B_{1/n}(p)$, of $S \setminus (B_{1/n}(p) \cup X)$ which separates Q_n and R_n. Since $S \setminus B_{1/n}(p)$ is finitely multicoherent, say of degree k, then a subset

$$L_n^1 \cup \ldots \cup L_n^k$$

is a closed separator (in $S \setminus B_{1/n}(p)$) of Q_n and R_n, where the elements of the union are disjoint, closed, and connected (though perhaps some are empty). We may assume that they are ordered so that

$$H_d(L_n^1, X) \leq \ldots \leq H_d(L_n^k, X),$$

where $H_d(\emptyset, X)$ can be regarded as ∞. Moreover, since each L_n^i is disjoint from X and connected, there exists a complementary domain U_n^i such that

$$L_n^i \subset U_n^i.$$

(The set U_n^i can be any complementary domain of X if L_n^i is empty.)

The sequence $(U_n^1)_{n=1}^{\infty}$ formed in this way is the required sequence of complementary domains. We will show that any convergent subsequence of $(\partial U_n^1)_{n=1}^{\infty}$ converges to X, implying that the sequence itself converges to X.

First we demonstrate that $X \subset \liminf_{n\to\infty} \bigcup_{i=1}^{k} \partial U_n^i$. Choose $x \in X \setminus \{p\}$ and $0 < \epsilon < d(x,p)$. Let $N \in \mathbb{N}$ such that, for all $n \geq N$

(1) $Q_n \cap B_\epsilon(x) \neq \emptyset$,
(2) $R_n \cap B_\epsilon(x) \neq \emptyset$, and
(3) $B_{1/n}(p) \cap B_\epsilon(x) = \emptyset$.

For $n \geq N$, choose $q_n \in Q_n \cap B_\epsilon(x)$ and $r_n \in R_n \cap B_\epsilon(x)$. Let $A_n \subset B_\epsilon(x) \subset S \setminus B_{1/n}(p)$ be an arc joining q_n to r_n. Then

$$A_n \cap \bigcup_{i=1}^{k} L_n^i \neq \emptyset,$$

since $\bigcup_{i=1}^{k} L_n^i$ separates q_n from r_n in $S \setminus B_{1/n}(p)$. Since $L_n^i \subset U_n^i$ for each i and q_n, r_n are not in any U_n^i (they lie in X),

$$A_n \cap \bigcup_{i=1}^{k} \partial U_n^i \neq \emptyset,$$

implying that

$$\bigcup_{i=1}^{k} \partial U_n^i \cap B_\epsilon(x) \neq \emptyset.$$

This is true for all $n \geq N$, so $x \in \liminf_{n\to\infty} \bigcup_{i=1}^{k} \partial U_n^i$, and we have that $X = \lim_{n\to\infty} \bigcup_{i=1}^{k} \partial U_n^i$.

Now, let us consider the individual limits $\lim_{n\to\infty} \partial U_n^i$ for some fixed $i \leq k$. By passing to a subsequence, we may assume that the limit $X_i = \lim_{n\to\infty} \partial U_n^i$ exists for each $i \leq k$. Then, since $\lim_{n\to\infty} \bigcup_{i=1}^{k} \partial U_n^i = X$, we see that $\bigcup_{i=1}^{k} X_i = X$. However, this is a finite union of continua, and X is indecomposable, so at least one X_i is not a proper subcontinuum of X. By continuity, we see that $H_d(X_1, X) \leq H_d(X_i, X)$ for all $i \leq k$, so $X_1 = \lim_{n\to\infty} \partial U_n^1 = X$. $\square$

4. Characterization of Shielded Indecomposable Continua

A subcontinuum X of a closed surface S is called *shielded* if, for every complementary domain U of X, $\partial U \neq X$. We extend Theorem 2.6, which is a characterization of planar indecomposable continua, to shielded subcontinua of closed surfaces.

The assumption that a continuum is shielded already imparts some complexity. For instance, any sequence of complementary domains whose boundaries converge to the continuum must consist of infinitely many distinct elements. We use this property to bridge the gap left by the relatively weak separation properties of compact surfaces.

The proofs in this section will depend upon the existence of particularly well-behaved homeomorphisms of simply connected domains in a surface to the unit disk $\mathbb{D}$ in the plane. Specifically, we wish for a null sequence of crosscuts in a domain to correspond to a null sequence of crosscuts in $\mathbb{D}$. We will prove that conformal isomorphisms have this property, so for the remainder we will assume that the closed surface S is endowed with a conformal structure and that d is the corresponding metric.

Lemma 4.1. *If X is a non-degenerate continuum in S, then all of its simply connected complementary domains are conformally isomorphic to the unit disk.*

Remark. Note that, in contrast to the planar case, some complementary domains may not be simply connected.

Proof. Let U be a simply connected complementary domain of X. Let P be the universal covering space of S, with corresponding conformal covering map $\pi : P \to S$. Recall that, by the Uniformization Theorem [9, Theorem 1.1], P is conformally isomorphic to a simply connected subset of the Riemann sphere. If $\hat{U}$ is a component of $\pi^{-1}(U)$, then $\hat{U}$ is simply connected (since U is) and $\pi|_{\hat{U}}$ is a covering map. In fact, $\pi|_{\hat{U}}$ is a conformal isomorphism since its trivial fundamental group is isomorphic to the group of deck transformations for $\pi|_{\hat{U}}$. However, $\hat{U}$ misses $\pi^{-1}(X)$, so $\hat{U}$ is conformally isomorphic to the unit disk by the classical Riemann mapping theorem. $\qquad\qquad\square$

The proof of the following lemma is identical to the proof of [3, Lemma 3.4], and is included here for completeness.

Lemma 4.2. *Let U be a simply connected open subset of S with non-degenerate boundary. Let $\phi : U \to \mathbb{D}$ be a conformal isomorphism. Then the image of a null sequence $(K_n)_{n=1}^{\infty}$ of crosscuts of U is a null sequence of crosscuts in $\mathbb{D}$.*

Proof. Let $(K_n)_{n=1}^\infty$ be a null sequence of crosscuts of U with image sequence $(A_n)_{n=1}^\infty = (\phi(K_n))_{n=1}^\infty$. Without loss of generality, assume that $(K_n)_{n=1}^\infty$ converges to a point $x \in \partial U$. By passing to a subsequence, we may assume that the image sequence converges to a continuum $L \subset \overline{\mathbb{D}}$. Since $(K_n)_{n=1}^\infty$ does not accumulate on a subset of U, we see that $L \subset \partial \mathbb{D}$.

Let $t \in L$. There exists a chain of crosscuts $(A_n')_{n=1}^\infty$ of $\mathbb{D}$ converging to t which maps to a null sequence $(K_n')_{n=1}^\infty$ of crosscuts of U by ϕ^{-1}. (See [9, Lemma 17.9]; the proof does not rely on the planarity of U.) We may assume that $(K_n')_{n=1}^\infty$ converges to a point of ∂U by passing to a subsequence. For each $m \in \mathbb{N}$, that $(A_n)_{n=1}^\infty$ accumulates on t implies that all but finitely many A_n intersect the crosscut neighborhood of A_m' corresponding to t. Also, since $(K_n)_{n=1}^\infty$ forms a null sequence in U and converges to x, we see that all but finitely many K_n (thus A_n) lie entirely within the crosscut neighborhood of K_m' (thus A_m') corresponding to t. However, the crosscut neighborhoods of A_m' form a null sequence as $m \to \infty$, so $(A_n)_{n=1}^\infty$ form a null sequence. $\qquad\square$

Lemma 4.3. *Suppose $U \subset S$ is open, connected, and simply connected with non-degenerate boundary, and let $\phi : U \to \mathbb{D}$ be a conformal isomorphism. Let B_1 and B_2 be disjoint closed disks meeting ∂U, and let $E_i \subset \partial \mathbb{D}$ denote the set of endpoints of the crosscuts of $\mathbb{D}$ which form $\phi((\partial B_i) \cap U)$. If, for different i and j, E_i lies in a component of $\partial \mathbb{D} \setminus E_j$, then there is a generalized crosscut K of U which separates $B_1 \cap U$ from $B_2 \cap U$ in U. Moreover, if ∂U is locally connected, then K is a crosscut of U.*

Proof. Identical to proof of Lemma 3.5 in [3]. $\qquad\square$

Theorem 4.4. *A shielded subcontinuum X of a closed surface S is indecomposable if and only if it has a sequence of complementary domains which satisfies the double-pass condition.*

Proof. Suppose first that X is indecomposable. By Theorem 3.9, there is a sequence $(U_n)_{n=1}^\infty$ of complementary domains of X such that $\lim_{n\to\infty} \partial U_n = X$. For each $n \in \mathbb{N}$, let K_n be a generalized crosscut of U_n. Let A_n and B_n be shadows of K_n so that $A_n \cup B_n = \partial U_n$, with $H_d(A_n, X) \le H_d(B_n, X)$.

Then A_n and B_n are subcontinua of X. There is a subsequence $(U_{n_i})_{i=1}^{\infty}$ so that $(A_{n_i})_{i=1}^{\infty}$ and $(B_{n_i})_{i=1}^{\infty}$ converge to subcontinua A and B of X. We see that $A \cup B = X$, since $A_{n_i} \cup B_{n_i} = \partial U_{n_i}$. Since X is indecomposable, either A or B is not a proper subcontinuum of X. We have $H_d(X, A_{n_i}) \leq H_d(X, B_{n_i})$, so $A = X$. This is true for all choices of subsequences $(U_{n_i})_{i=1}^{\infty}$ where $(A_{n_i})_{i=1}^{\infty}$ and $(B_{n_i})_{i=1}^{\infty}$ both converge, so $(U_n)_{n=1}^{\infty}$ satisfies the double-pass condition.

Now, assume that a continuum $X \subset S$ has a sequence $(U_n)_{n=1}^{\infty}$ of complementary domains satisfying the double-pass condition. By way of contradiction, suppose that (X_1, X_2) is a decomposition of X. There are disjoint closed disks $D_1, D_2 \subset S$ so that, for $i \in \{1, 2\}$,

(1) the interior of D_i intersects X_i, and

(2) if $i \neq j$, D_i is disjoint from $X_j \cup D_j$.

Since X satisfies the double-pass condition, there exists $N \geq 1$ such that, for all $n \geq N$, no generalized crosscut of U_n separates $D_1 \cap U$ from $D_2 \cap U$ in U_n. Without loss of generality, $N = 1$.

Now we make use of the assumption that X is shielded. Since $\partial U_n \neq X$ for each n and $\lim_{n \to \infty} \partial U_n = X$, no element of $(U_n)_{n=1}^{\infty}$ appears infinitely often in the sequence. Therefore, we can assume by passing to a subsequence that $(U_n)_{n=1}^{\infty}$ consists of different simply connected complementary domains of X.

Let $\phi_n : U_n \to \mathbb{D}$ be a conformal isomorphism. According to Lemma 4.3, the sets $E_{n,1}$ and $E_{n,2}$, comprised of the endpoints of the crosscuts constituting $\phi_n((\partial D_1) \cap U_n)$ and $\phi_n((\partial D_2) \cap U_n)$ separate each other in $\partial \mathbb{D}$. It is evident that $D_1 \cap U_n$ and $D_2 \cap U_n$ may or may not separate the other in U_n. By passing to a subsequence, we can assume that, for every $n \in \mathbb{N}$, either

(1) $D_1 \cap U_n$ separates $D_2 \cap U_n$ in U_n, or

(2) neither separates the other in U_n.

We will find a crosscut $F_n \subset \mathbb{D} \setminus \phi(D_2)$ of $\mathbb{D}$ which joins points of $E_{n,1}$ which separate $E_{n,2}$ in $\partial \mathbb{D}$. In the case that $D_1 \cap U_n$ separates $D_2 \cap U_n$, a component of $(\partial D_1) \cap U_n$ also does, so we can define F_n as a component of $\phi((\partial D_1) \cap U_n)$ which separates $\phi_n(D_2 \cap U_n)$ in $\mathbb{D}$. In the second case, there are components K_1 and K_2 of $\phi_n((\partial D_1) \cap U_n)$ whose endpoints separate $E_{2,n}$ in $\partial \mathbb{D}$. Let F_n be the union of an arc joining $k_1 \in \mathrm{Int}(K_1)$ to $k_2 \in \mathrm{Int}(K_2)$ with one component each of $K_1 \setminus \{k_1\}$ and $K_2 \setminus \{k_2\}$. Notice that $\phi_n^{-1}(F_n)$ is in fact a crosscut of U_n in either case, since points of F_n close to ∂U_n are in ∂D_1.

Let A_n be an arc in U_n joining components of $D_2 \cap U_n$ which are separated by $\phi_n^{-1}(F_n)$, intersecting $\phi_n^{-1}(F_n)$ transversely exactly once. Let $C_n \subset D_2 \cup A_n$ be a simple closed curve containing A_n, formed by joining the endpoints of A_n to a designated point $a \in D_2$ with line segments otherwise disjoint from $C_1 \cup \ldots C_{n-1}$.

Let $Y = C_1 \cup C_2 \cup \ldots \cup C_{2g+1}$, where g is the genus of S. This is the one-point union of $2g + 1$ simple closed curves, so $S \setminus Y$ is not connected (see for instance [1, 5.14]). Since the arcs $F_1, \ldots, F_{2g+1}$ intersect Y exactly once transversely and have their endpoints in X_2, we conclude that each component of $S \setminus Y$ contains points of X_1. However, since Y is the union of arcs in $S \setminus X$ and B_2, we see that Y is disjoint from X_1. This contradicts the assumption that X_1 is connected. □

Remark. Suppose that the continuum in question has locally connected boundary components. Then Lemma 4.3 provides a crosscut (rather than a generalized crosscut) to show that X fails the double-pass condition. Hence, we can observe that a continuum with locally connected boundary components is indecomposable if and only if it satisfies the double-pass condition with crosscuts.

5. Conclusion

Though the results presented here are advances, the corresponding theorems for planar continua are much stronger. Here we discuss stronger generalizations which may hold.

5.1. **Burgess's Theorem.** Theorem 3.7 is a direct translation of the corollary to [2, Theorem 10]. In turn, that theorem follows from a fundamental theorem of C. E. Burgess, reproduced below.

Theorem ([2, Theorem 9]). *Let $H \subset \mathbb{S}^2$ be closed, $M \subset \mathbb{S}^2$ a continuum, and M_1, M_2, and M_3 subcontinua of M. Suppose that K_1, K_2, and K_3 are closed disks, disjoint from each other and H, and K_i intersects M_j if and only if $i = j$. Then there do not exist three complementary domains of $M \cup H$ all of which intersect K_1, K_2, and K_3.*

Question 5.1. Is there an extension of this theorem to closed surfaces?

Using Burgess's theorem, one can prove that a planar continuum which is the common boundary of three of its complementary domains [7] or is the impression of one of its prime ends [11] is either indecomposable or 2-indecomposable. Also, a planar continuum which is the limit of a disjoint sequence of shadows is indecomposable [3]. Theorem 3.7 was proven here without proving the analog of Burgess's theorem, but interesting theorems would surely follow if Question 5.1 had a positive answer.

5.2. **Kuratowski's Theorem.** In particular, the requirement of Theorem 3.7 that X have infinitely many distinct domains whose boundaries limit to the continuum is probably stronger than necessary for the conclusion. This motivates the following question.

Question 5.2. Is there a finite version of Theorem 3.7 like Kuratowski's theorem? Specifically, if a continuum X in a closed surface of genus g is the common boundary of three complementary domains, what is the maximal $n(g)$ so that X is the union of $n(g)$ indecomposable continua?

A partial answer is illustrated by an example. Let $C_1, C_2, C_3 \subset \mathbb{C}_\infty \setminus \{\infty\}$ be homeomorphic copies of the pseudocircle such that, for distinct i, j, k,

(1) C_i is contained in the closure of the component of $\mathbb{S}^2 \setminus C_j$ containing ∞,
(2) $C_i \cap C_j$ is a single point not in C_k, and
(3) $C_i \cap C_k$ is in a different composant of C_k than $C_j \cap C_k$.

This is a continuum with 5 complementary domains. Two complementary domains, U and V, meet each element of $\{C_1, C_2, C_3\}$. There are three complementary domains, W_1, W_2, and W_3, where W_i is the bounded complementary domain of C_i.

No proper subcontinuum of $C_1 \cup C_2 \cup C_3$ can separate U from V because of condition 3, so $\partial U = \partial V = C_1 \cup C_2 \cup C_3$. Remove disks $D_1, D_1' \subset W_1$, $D_2 \subset W_2$, and $D_3 \subset W_3$, and paste cylinders joining ∂D_1 to ∂D_2 and $\partial D_1'$ to ∂D_3 in a way resulting in an orientable surface S of genus 2. Then $C_1 \cup C_2 \cup C_3$ is the common boundary of three domains, though it is not the union of just two indecomposable continua. This indicates that an extension of Kuratowski's theorem along these lines must take the genus into account.

5.3. **Characterization Theorem.** Theorem 4.4 is significantly weaker than Theorem 2.6, since Theorem 2.6 holds for continua which are boundaries of connected open sets in $\mathbb{S}^2$. The author suspects the statement of Theorem 2.6 holds true in surfaces, with some qualification. For instance, an essentially embedded simple closed curve in a torus satisfies the double-pass condition as it is stated here, but artificially so.

One can define a *generalized crosscut with curves* of a domain U as the disjoint union of a generalized crosscut and a finite number of simple closed curves. By defining shadows in analogy to shadows of generalized crosscuts, one obtains an equivalent notion in simply connected domains – adding disjoint simple closed curves to a generalized crosscut does not change the shadows. However, generalized crosscuts with curves allows stronger separation in domains which are not simply connected.

Question 5.3. Let X be a continuum in a closed surface S. If for every sequence $(C_i)_{i=1}^{\infty}$ of generalized crosscuts with curves there exists a choice of shadows $(S_i)_{i=1}^{\infty}$, where S_i a shadow of C_i, which converge to X, is X indecomposable?

5.4. **Higher Dimensions.** One may ask how useful recognition from the complement might be in more general spaces. Specifically, in $\mathbb{R}^3$, this approach does not look useful. Formulations of prime end theory in $\mathbb{R}^3$ have had limited applicability. Further, M. Lubański constructed in [8] a family of absolute neighborhood retracts which can be the common boundary of any finite number of domains in $\mathbb{R}^3$. Kuratowski [7, p. 560] noted that this can be extended to an infinite number of complementary domains.

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(Clinton P. Curry) Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294-1170

E-mail address, Clinton P. Curry: `clintonc@uab.edu`