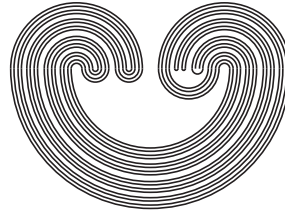

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ON POSITIONAL DIMENSION-LIKE FUNCTIONS

by

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ON POSITIONAL DIMENSION-LIKE FUNCTIONS

D. N. GEORGIU, S. D. ILIADIS, AND A. C. MEGARITIS

ABSTRACT. In [5] the so called positional dimension-like functions of the type ind were introduced. These functions were studied only with respect to the property of universality. Here, we first give some relations between positional dimension-like functions of the type ind and then study these functions with respect to other standard properties of dimension theory.

1. PRELIMINARIES

The realm of spaces is the class of all T_0 -spaces. The class of all ordinals is denoted by \mathcal{O} and the first infinite cardinal is denoted by ω . In the class \mathcal{O} we denote by $(+)$ the natural sum of Hessenberg (see, for example, [6]). We note the following properties of the natural sum:

- (1) $\alpha(+)\beta = \beta(+)\alpha$,
- (2) if $\alpha_1 < \alpha_2$, then $\alpha_1(+)\beta < \alpha_2(+)\beta$, and
- (3) $\alpha(+n) = \alpha + n$ for $n < \omega$.

We also consider two symbols, “ -1 ” and “ ∞ ”. It is assumed that $-1 < \alpha < \infty$ for every $\alpha \in \mathcal{O}$ and $-1(+)\alpha = \alpha(+)(-1) = \alpha$, $\infty(+)\alpha = \alpha(+)\infty = \infty$ for every $\alpha \in \mathcal{O} \cup \{-1, \infty\}$.

Let Q be a subset of a space X . We denote by $\text{Int}_X(Q)$, $\text{Cl}_X(Q)$, and $\text{Bd}_X(Q)$ the interior, the closure, and the boundary of Q in X , respectively.

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We recall (see [5]) that a family B of open subsets of X (containing X and the empty set) is said to be a *p-base for Q in X* if the set $\{Q \cap U : U \in B\}$ is a base for the subspace Q . A p-base B for Q in X is said to be a *pos-base* if for every $x \in Q$ and an open neighbourhood U of x in X there exists an element V of B such that $x \in V \subseteq U$. A p-base B for Q in X is said to be a *ps-base* if B is a base for the space X .

Definition 1.1. We denote by $p_0\text{-ind}$ the *dimension-like function* whose domain is the class of all pairs (Q, X) , where Q is a subset of a space X , and whose range is the class $\mathcal{O} \cup \{-1, \infty\}$ satisfying the following conditions:

- (i) $p_0\text{-ind}(Q, X) = -1$ if and only if $Q = X = \emptyset$.
- (ii) $p_0\text{-ind}(Q, X) \leq \alpha$, where $\alpha \in \mathcal{O}$, if and only if there exists a p-base B for Q in X such that for every $U \in B$ we have

$$p_0\text{-ind}(Q \cap \text{Bd}_X(U), \text{Bd}_X(U)) < \alpha.$$

Definition 1.2. We denote by $p_1\text{-ind}$ the *dimension-like function* whose domain is the class of all pairs (Q, X) , where Q is a subset of a space X , and whose range is the class $\mathcal{O} \cup \{-1, \infty\}$ satisfying the following conditions:

- (i) $p_1\text{-ind}(Q, X) = -1$ if and only if $Q = \emptyset$.
- (ii) $p_1\text{-ind}(Q, X) \leq \alpha$, where $\alpha \in \mathcal{O}$, if and only if there exists a p-base B for Q in X such that for every $U \in B$ we have

$$p_1\text{-ind}(Q \cap \text{Bd}_X(U), X) < \alpha.$$

Remark 1. If in Definitions 1.1 and 1.2 instead of the p-base B we consider a pos-base (respectively, a ps-base), then the dimension-like function $p_i\text{-ind}$, $i \in \{0, 1\}$, will be denoted by $\text{pos}_i\text{-ind}$ (respectively, by $\text{ps}_i\text{-ind}$). Note that the dimension-like function $\text{pos}_1\text{-ind}$ is the transfinite extension of the relative small inductive dimension given in [7] and [8] (see also [4]).

Remark 2. It is known (see [5]) that for every pair (Q, X) we have $p_i\text{-ind}(Q, X) \in \{-1, \infty\} \cup \tau^+$, $\text{pos}_i\text{-ind}(Q, X) \in \{-1, \infty\} \cup \tau^+$, and $\text{ps}_i\text{-ind}(Q, X) \in \{-1, \infty\} \cup \tau^+$, $i = 0, 1$, where τ is the weight of X and τ^+ the least cardinal greater than τ . Also, if df denotes one of the above dimension-like functions, then $df(Q, X) = \infty$ if and only if the inequality $df(Q, X) \leq \alpha$ is not true for any $\alpha \in \tau^+ \cup \{-1\}$.

The above dimension like-functions were introduced in [5] under the name “positional dimension like-functions of the type ind”. These functions were studied only with respect to the property of universality, that is if df is one of the above dimension-like functions and $\alpha \in \tau^+ \cup \{-1\}$, then in the class \mathbb{P} of all pairs (Q^X, X) , where Q^X is a subset of a space X such that $df(Q^X, X) \leq \alpha$, there exists a universal element. (An element (Q^T, T) of \mathbb{P} is said to be universal in \mathbb{P} if for every $(Q^X, X) \in \mathbb{P}$ there exists a homeomorphism i_T^X of X into T such that $i_T^X(Q^X) \subseteq Q^T$.) About some other “positional dimension-like functions” under the name “relative dimensions” see for example [1] and [2].

In section 2 of this paper we give some relations between of the defined dimension-like functions. In section 3 we consider subspace theorems and in section 4 sum theorems. Finally, in section 5 we give some results connected with the mapping of spaces.

2. THE RELATION BETWEEN POSITIONAL DIMENSION-LIKE FUNCTIONS OF THE TYPE ind

Proposition 2.1. *For every subset Q of a space X we have*

$$\text{ind}(Q) \leq \text{p}_i\text{-ind}(Q, X), \quad i \in \{0, 1\}.$$

Proof. We prove that

$$\text{ind}(Q) \leq \text{p}_0\text{-ind}(Q, X). \tag{1}$$

The case $i = 1$ is similar. Let $\text{p}_0\text{-ind}(Q, X) = \alpha \in \mathcal{O} \cup \{-1, \infty\}$. The relation (1) is clear if $\alpha = -1$ or $\alpha = \infty$. Suppose that $\alpha \in \mathcal{O}$ and that (1) is true for every pair (Q^Y, Y) with $\text{p}_0\text{-ind}(Q^Y, Y) < \alpha$. Since $\text{p}_0\text{-ind}(Q, X) = \alpha$, there exists a p-base B for Q in X such that for every $U \in B$ we have

$$\text{p}_0\text{-ind}(Q \cap \text{Bd}_X(U), \text{Bd}_X(U)) < \alpha.$$

To prove that $\text{ind}(Q) \leq \alpha$ it suffices to show that

$$\text{ind}(\text{Bd}_Q(Q \cap U)) < \alpha$$

for every $U \in B$. Since

$$\text{Bd}_Q(Q \cap U) \subseteq Q \cap \text{Bd}_X(U),$$

by inductive assumption we have

$$\begin{aligned} \text{ind}(\text{Bd}_Q(Q \cap U)) &\leq \text{ind}(Q \cap \text{Bd}_X(U)) \\ &\leq \text{p}_0\text{-ind}(Q \cap \text{Bd}_X(U), \text{Bd}_X(U)) < \alpha. \quad \square \end{aligned}$$

Proposition 2.2. *For every subset Q of a space X we have*

$$\text{p}_i\text{-ind}(Q, X) \leq \text{pos}_i\text{-ind}(Q, X) \leq \text{ps}_i\text{-ind}(Q, X), \quad i \in \{0, 1\}.$$

Proof. Suggestively we prove by induction only the inequality

$$\text{p}_0\text{-ind}(Q, X) \leq \text{pos}_0\text{-ind}(Q, X), \quad (2)$$

where $\text{pos}_0\text{-ind}(Q, X) = \alpha \in \mathcal{O} \cup \{-1\}$. The relation (2) is clear for $\alpha = -1$. There exists a pos-base B for Q in X such that for every $U \in B$ we have $\text{pos}_0\text{-ind}(Q \cap \text{Bd}_X(U), \text{Bd}_X(U)) < \alpha$. By induction,

$$\begin{aligned} \text{p}_0\text{-ind}(Q \cap \text{Bd}_X(U), \text{Bd}_X(U)) &\leq \text{pos}_0\text{-ind}(Q \cap \text{Bd}_X(U), \text{Bd}_X(U)) \\ \text{and since } B \text{ is also a p-base for } Q \text{ in } X, &\text{p}_0\text{-ind}(Q, X) \leq \alpha. \quad \square \end{aligned}$$

By induction it can be proved also the following proposition.

Proposition 2.3. *For every subset Q of a space X we have*

$$\text{ps}_0\text{-ind}(Q, X) = \text{ind}(X).$$

Propositions 2.1, 2.2, and 2.3 imply the following corollary.

Corollary 2.4. *For every space X we have*

$$\text{p}_0\text{-ind}(X, X) = \text{pos}_0\text{-ind}(X, X) = \text{ps}_0\text{-ind}(X, X) = \text{ind}(X).$$

The following examples show that in the realm of T_0 -spaces the inequalities in Propositions 2.1 and 2.2 cannot be replaced by equalities.

Example 2.5. Let $X = \{a, b\}$, $Q = \{a\}$, $K = \{b\}$, and

$$\tau = \{\emptyset, \{b\}, X\}$$

be a topology on X . Then, it can be proved that

$$\begin{aligned} \text{ind}(X) &= 1, \\ \text{p}_0\text{-ind}(Q, X) &= \text{pos}_0\text{-ind}(Q, X) = 0, \\ \text{p}_0\text{-ind}(K, X) &= 0, \quad \text{pos}_0\text{-ind}(K, X) = 1, \quad \text{and} \\ \text{ps}_1\text{-ind}(Q, X) &= 0. \end{aligned}$$

Example 2.6. Let $X = \{a, b, c\}$, $Q = \{a, b\}$, and

$$\tau = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$$

be a topology on X . It can be proved that

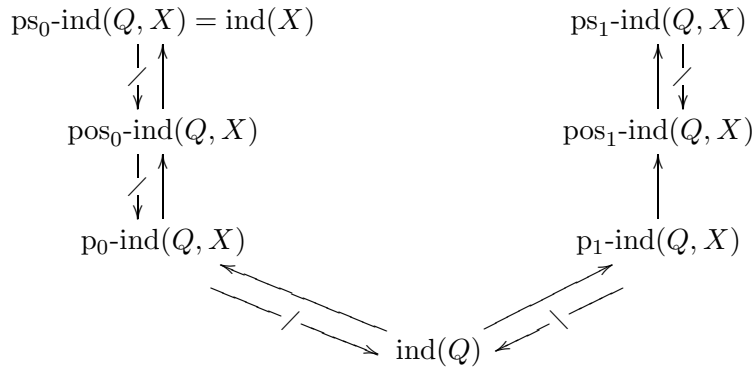
$$\text{ind}(Q) = 0, \text{ind}(X) = 1,$$

$$\text{p}_1\text{-ind}(X, X) = \text{pos}_1\text{-ind}(X, X) = 2,$$

$$\text{p}_i\text{-ind}(Q, X) = \text{pos}_i\text{-ind}(Q, X) = 1, \quad i \in \{0, 1\}, \text{ and}$$

$$\text{ps}_1\text{-ind}(K, X) = \infty.$$

The relations between the considered positional dimension-like functions of the type ind are summarized in the following diagram, where “ \rightarrow ” means “ \leq ” and “ \nrightarrow ” means that “in general $\not\leq$ ”.



Remark. According to Examples 2.5 and 2.6 it is possible that $\text{ps}_1\text{-ind}(K, Y) < \text{ind}(Y)$ and $\text{ind}(X) < \text{ps}_1\text{-ind}(Q, X)$ for some spaces X and Y and $Q \subseteq X$, $K \subseteq Y$.

Question. Find a space X and a subset Q of X such that

$$\text{p}_1\text{-ind}(Q, X) < \text{pos}_1\text{-ind}(Q, X).$$

It can be proved by induction that Proposition 1 of [7] (given for finite dimensions) can be extended to transfinite dimensions. So, we have the following proposition.

Proposition 2.7. *Let X be a hereditarily normal space (that is every subspace of X is normal) and $Q \subseteq X$. Then,*

$$\text{ind}(Q) = \text{pos}_1\text{-ind}(Q, X).$$

Propositions 2.1, 2.2, and 2.7 imply the following corollary.

Corollary 2.8. *For every subset Q of a hereditarily normal space X we have $\text{ind}(Q) = \text{p}_1\text{-ind}(Q, X) = \text{pos}_1\text{-ind}(Q, X)$.*

By Proposition 2.2 and Theorem 1 of [8] we have the following proposition.

Proposition 2.9. *Let Q be a subset of a Tychonoff space X . If $\text{ind}(Q) < \omega$, then*

$$\text{p}_1\text{-ind}(Q, X) \leq \text{pos}_1\text{-ind}(Q, X) \leq \text{ind}(Q) + 1.$$

It is shown in [8] that if X is a normal space and Q is a closed subset of X such that $\text{ind}(Q) = 0$, then $\text{pos}_1\text{-ind}(Q, X) = 0$. Therefore, by Proposition 2.2 we have the following proposition.

Proposition 2.10. *Let X be a normal space and Q a closed subset of X . If $\text{ind}(Q) = 0$, then*

$$\text{p}_1\text{-ind}(Q, X) = \text{pos}_1\text{-ind}(Q, X) = 0.$$

3. SUBSPACE THEOREMS

Proposition 3.1. *Let $i \in \{0, 1\}$ and Q, K be two subsets of a space X with $K \subseteq Q$. Then,*

- (a) $\text{p}_i\text{-ind}(K, X) \leq \text{p}_i\text{-ind}(Q, X)$,
- (b) $\text{pos}_i\text{-ind}(K, X) \leq \text{pos}_i\text{-ind}(Q, X)$, and
- (c) $\text{ps}_i\text{-ind}(K, X) \leq \text{ps}_i\text{-ind}(Q, X)$.

Proof. We prove the inequality

$$\text{p}_0\text{-ind}(K, X) \leq \text{p}_0\text{-ind}(Q, X). \quad (3)$$

The proofs of all other inequalities are similar. Let $\text{p}_0\text{-ind}(Q, X) = \alpha \in \mathcal{O} \cup \{-1, \infty\}$. The relation (3) is clear if $\alpha = -1$ or $\alpha = \infty$. Let $\alpha \in \mathcal{O}$ and suppose that (3) is true for any $K \subseteq Q \subseteq X$ with $\text{p}_0\text{-ind}(Q, X) < \alpha$. There exists a p-base B for Q in X such that for every $U \in B$ we have

$$\text{p}_0\text{-ind}(Q \cap \text{Bd}_X(U), \text{Bd}_X(U)) < \alpha.$$

Since $K \cap \text{Bd}_X(U) \subseteq Q \cap \text{Bd}_X(U)$, by inductive assumption,

$$\begin{aligned} \text{p}_0\text{-ind}(K \cap \text{Bd}_X(U), \text{Bd}_X(U)) &\leq \\ \text{p}_0\text{-ind}(Q \cap \text{Bd}_X(U), \text{Bd}_X(U)) &< \alpha \end{aligned}$$

and since B is also a p-base for K in X , $\text{p}_0\text{-ind}(K, X) \leq \alpha$. \square

Proposition 3.2. *Let $i \in \{0, 1\}$, Y be a subspace of a space X , and $Q \subseteq Y$. Then,*

- (a) $\text{p}_i\text{-ind}(Q, Y) \leq \text{p}_i\text{-ind}(Q, X)$,
- (b) $\text{pos}_i\text{-ind}(Q, Y) \leq \text{pos}_i\text{-ind}(Q, X)$, and
- (c) $\text{ps}_i\text{-ind}(Q, Y) \leq \text{ps}_i\text{-ind}(Q, X)$.

Proof. Suggestively we prove the inequality

$$\text{ps}_1\text{-ind}(Q, Y) \leq \text{ps}_1\text{-ind}(Q, X). \quad (4)$$

Let $\text{ps}_1\text{-ind}(Q, X) = \alpha \in \mathcal{O} \cup \{-1, \infty\}$. The relation (4) is clear if $\alpha = -1$ or $\alpha = \infty$. Let $\alpha \in \mathcal{O}$ and suppose that (4) is true for any $Q \subseteq Y \subseteq X$ with $\text{ps}_1\text{-ind}(Q, X) < \alpha$. There exists a ps-base B for Q in X such that for every $U \in B$ we have

$$\text{ps}_1\text{-ind}(Q \cap \text{Bd}_X(U), X) < \alpha.$$

Since

$$\text{Bd}_Y(U \cap Y) \subseteq Y \cap \text{Bd}_X(U) \subseteq \text{Bd}_X(U),$$

by Proposition 3.1,

$$\begin{aligned} \text{ps}_1\text{-ind}(Q \cap \text{Bd}_Y(U \cap Y), X) &\leq \\ \text{ps}_1\text{-ind}(Q \cap \text{Bd}_X(U), X) &< \alpha. \end{aligned}$$

Also, by inductive assumption,

$$\begin{aligned} \text{ps}_1\text{-ind}(Q \cap \text{Bd}_Y(U \cap Y), Y) &\leq \\ \text{ps}_1\text{-ind}(Q \cap \text{Bd}_Y(U \cap Y), X) &< \alpha \end{aligned}$$

and since the set $\{U \cap Y : U \in B\}$ is a ps-base for Q in Y , $\text{ps}_1\text{-ind}(Q, Y) \leq \alpha$. \square

Proposition 3.3. *Let Y be a dense subspace of a space X and $Q \subseteq Y$. Then, $\text{p}_1\text{-ind}(Q, Y) = \text{p}_1\text{-ind}(Q, X)$.*

Proof. By Proposition 3.2 it suffices to prove that

$$\text{p}_1\text{-ind}(Q, X) \leq \text{p}_1\text{-ind}(Q, Y). \quad (5)$$

Let $\text{p}_1\text{-ind}(Q, Y) = \alpha \in \mathcal{O} \cup \{-1, \infty\}$. The relation (5) is clear if $\alpha = -1$ or $\alpha = \infty$. Let $\alpha \in \mathcal{O}$ and suppose that (5) is true for any $Q \subseteq Y \subseteq X$ with $\text{p}_1\text{-ind}(Q, Y) < \alpha$, where Y is dense in X .

There exists a p-base B for Q in Y such that for every $U \in B$ we have $\text{p}_1\text{-ind}(Q \cap \text{Bd}_Y(U), Y) < \alpha$.

The set of all open subsets V of X such that $V \cap Y \in B$ is a p-base for Q in X . Since Y is dense in X , for every open subset V of X we have

$$\text{Bd}_Y(V \cap Y) = Y \cap \text{Bd}_X(V)$$

and, therefore,

$$Q \cap \text{Bd}_Y(V \cap Y) = Q \cap \text{Bd}_X(V).$$

By inductive assumption, if $V \cap Y \in B$ we have

$$\begin{aligned} \text{p}_1\text{-ind}(Q \cap \text{Bd}_X(V), X) &= \text{p}_1\text{-ind}(Q \cap \text{Bd}_Y(V \cap Y), X) \\ &\leq \text{p}_1\text{-ind}(Q \cap \text{Bd}_Y(V \cap Y), Y) < \alpha. \end{aligned}$$

Thus, $\text{p}_1\text{-ind}(Q, X) \leq \alpha$. \square

It can be proved by induction that Lemma 2 of [8] (given for finite dimensions) can be extended to transfinite dimensions. Hence, we obtain next proposition.

Proposition 3.4. *Let Y be a dense subspace of a space X and $Q \subseteq Y$. Then, $\text{pos}_1\text{-ind}(Q, Y) = \text{pos}_1\text{-ind}(Q, X)$.*

4. SUM THEOREMS

Proposition 4.1. *Let Q_1 and Q_2 be two subsets of a space X . Then,*

$$\text{pos}_0\text{-ind}(Q_1 \cup Q_2, X) \leq \text{pos}_0\text{-ind}(Q_1, X)(+)\text{pos}_0\text{-ind}(Q_2, X) \quad (6).$$

Proof. We prove relation (6) by induction on α , where

$$\alpha = \text{pos}_0\text{-ind}(Q_1, X)(+)\text{pos}_0\text{-ind}(Q_2, X).$$

If $\alpha = -1$, then $\text{pos}_0\text{-ind}(Q_1, X) = \text{pos}_0\text{-ind}(Q_2, X) = -1$ which means that $Q_1 \cup Q_2 = X = \emptyset$ and, therefore, (6) is true.

Suppose that for any space X and its subsets Q_1, Q_2 relation (6) is true if

$$\text{pos}_0\text{-ind}(Q_1, X)(+)\text{pos}_0\text{-ind}(Q_2, X) < \alpha,$$

where α is a fixed ordinal. We shall prove (6) for the case

$$\text{pos}_0\text{-ind}(Q_1, X)(+)\text{pos}_0\text{-ind}(Q_2, X) = \alpha.$$

Let

$$\text{pos}_0\text{-ind}(Q_1, X) = \alpha_1 \quad \text{and} \quad \text{pos}_0\text{-ind}(Q_2, X) = \alpha_2,$$

where $\alpha_1, \alpha_2 \in \mathcal{O} \cup \{-1\}$. If one of the elements α_1, α_2 is equal to -1 , then the other is also equal to -1 and, therefore, $\alpha = -1$ is not ordinal. Hence, we can suppose that $\alpha_1, \alpha_2 \in \mathcal{O}$.

There exists a pos-base B_1 for Q_1 in X and a pos-base B_2 for Q_2 in X such that

$$\text{pos}_0\text{-ind}(Q_1 \cap \text{Bd}_X(U_1), \text{Bd}_X(U_1)) < \alpha_1$$

and

$$\text{pos}_0\text{-ind}(Q_2 \cap \text{Bd}_X(U_2), \text{Bd}_X(U_2)) < \alpha_2$$

for every $U_1 \in B_1$ and $U_2 \in B_2$.

The set $B = B_1 \cup B_2$ is a pos-base for $Q_1 \cup Q_2$ in X . Let $U \in B$, for example, $U \in B_1$. Then,

$$\text{pos}_0\text{-ind}(Q_1 \cap \text{Bd}_X(U), \text{Bd}_X(U)) < \alpha_1$$

and, by Propositions 3.1 and 3.2,

$$\text{pos}_0\text{-ind}(Q_2 \cap \text{Bd}_X(U), \text{Bd}_X(U)) \leq \text{pos}_0\text{-ind}(Q_2, X) = \alpha_2.$$

By inductive assumption we have

$$\begin{aligned} & \text{pos}_0\text{-ind}((Q_1 \cup Q_2) \cap \text{Bd}_X(U), \text{Bd}_X(U)) = \\ & \text{pos}_0\text{-ind}((Q_1 \cap \text{Bd}_X(U)) \cup (Q_2 \cap \text{Bd}_X(U)), \text{Bd}_X(U)) \leq \\ & \text{pos}_0\text{-ind}(Q_1 \cap \text{Bd}_X(U), \text{Bd}_X(U))(+) \\ & \text{pos}_0\text{-ind}(Q_2 \cap \text{Bd}_X(U), \text{Bd}_X(U)) < \alpha_1(+) \alpha_2 = \alpha. \end{aligned}$$

Thus, $\text{pos}_0\text{-ind}(Q_1 \cup Q_2, X) \leq \alpha$. □

Proposition 4.2. *Let Q_1 and Q_2 be two subsets of a space X . Then,*

$$\text{pos}_1\text{-ind}(Q_1 \cup Q_2, X) \leq \text{pos}_1\text{-ind}(Q_1, X)(+) \text{pos}_1\text{-ind}(Q_2, X) + 1 \quad (7)$$

and

$$\text{ps}_1\text{-ind}(Q_1 \cup Q_2, X) \leq \text{ps}_1\text{-ind}(Q_1, X)(+) \text{ps}_1\text{-ind}(Q_2, X) + 1. \quad (8)$$

Proof. We prove relation (8) by induction on α , where

$$\alpha = \text{ps}_1\text{-ind}(Q_1, X)(+) \text{ps}_1\text{-ind}(Q_2, X).$$

If $\alpha = -1$, then $\text{ps}_1\text{-ind}(Q_1, X) = \text{ps}_1\text{-ind}(Q_2, X) = -1$ which means that $Q_1 \cup Q_2 = \emptyset$ and, therefore, (8) is true.

Suppose that for any space X and its subsets Q_1, Q_2 relation (8) is true if

$$\text{ps}_1\text{-ind}(Q_1, X)(+)\text{ps}_1\text{-ind}(Q_2, X) < \alpha,$$

where α is a fixed ordinal. We shall prove (8) for the case

$$\text{ps}_1\text{-ind}(Q_1, X)(+)\text{ps}_1\text{-ind}(Q_2, X) = \alpha.$$

Let

$$\text{ps}_1\text{-ind}(Q_1, X) = \alpha_1 \quad \text{and} \quad \text{ps}_1\text{-ind}(Q_2, X) = \alpha_2,$$

where $\alpha_1, \alpha_2 \in \mathcal{O} \cup \{-1\}$. If $\alpha_1 = -1$ or $\alpha_2 = -1$, then $Q_1 = \emptyset$ or $Q_2 = \emptyset$, respectively and the relation (8) is true.

There exists a ps-base B_1 for Q_1 in X and a ps-base B_2 for Q_2 in X such that

$$\text{ps}_1\text{-ind}(Q_1 \cap \text{Bd}_X(U_1), X) < \alpha_1$$

and

$$\text{ps}_1\text{-ind}(Q_2 \cap \text{Bd}_X(U_2), X) < \alpha_2$$

for every $U_1 \in B_1$ and $U_2 \in B_2$. The set $B = B_1 \cup B_2$ is a ps-base for $Q_1 \cup Q_2$ in X . Let $U \in B$, for example, $U \in B_1$. Then,

$$\text{ps}_1\text{-ind}(Q_1 \cap \text{Bd}_X(U), X) < \alpha_1$$

and, by Proposition 3.1,

$$\text{ps}_1\text{-ind}(Q_2 \cap \text{Bd}_X(U), X) \leq \text{ps}_1\text{-ind}(Q_2, X) = \alpha_2.$$

By inductive assumption we have

$$\begin{aligned} \text{ps}_1\text{-ind}((Q_1 \cup Q_2) \cap \text{Bd}_X(U), X) &= \\ \text{ps}_1\text{-ind}((Q_1 \cap \text{Bd}_X(U)) \cup (Q_2 \cap \text{Bd}_X(U)), X) &\leq \\ \text{ps}_1\text{-ind}(Q_1 \cap \text{Bd}_X(U), X)(+) & \\ \text{ps}_1\text{-ind}(Q_2 \cap \text{Bd}_X(U), X) + 1 &< \alpha_1(+) \alpha_2 + 1 = \alpha + 1. \end{aligned}$$

Thus, $\text{ps}_1\text{-ind}(Q_1 \cup Q_2, X) \leq \alpha + 1$.

The proof of the relation (7) is similar. \square

Remark. The relation (7) is an extension of relation (5) of [7] (see page 25) to transfinite dimension in the realm of T_0 -spaces.

5. SOME OTHER RESULTS

Proposition 5.1. *Let $f : X \rightarrow Y$ be a continuous map and $Q \subseteq X$. If the restriction $f|_Q$ of the map f to Q is a homeomorphism, then*

$$p_1\text{-ind}(Q, X) \leq p_1\text{-ind}(f(Q), Y). \tag{9}$$

Proof. We prove relation (9) by induction on the element

$$p_1\text{-ind}(f(Q), Y) \in \mathcal{O} \cup \{-1, \infty\}.$$

This relation is true if $p_1\text{-ind}(f(Q), Y) = -1$ or $p_1\text{-ind}(f(Q), Y) = \infty$. Suppose that the relation (9) is true if $p_1\text{-ind}(f(Q), Y) < \alpha \in \mathcal{O}$ and prove it in the case where $p_1\text{-ind}(f(Q), Y) = \alpha$.

There exists a p-base B for $f(Q)$ in Y such that for every $W \in B$ we have

$$p_1\text{-ind}(f(Q) \cap \text{Bd}_Y(W), Y) < \alpha.$$

The set $\{f^{-1}(W) : W \in B\}$ is a p-base for Q in X . We prove that $p_1\text{-ind}(Q \cap \text{Bd}_X(f^{-1}(W)), X) < \alpha$ for every $W \in B$. Since the map f is continuous, we have $f(\text{Bd}_X(f^{-1}(W))) \subseteq \text{Bd}_Y(W)$. By Proposition 3.1,

$$\begin{aligned} p_1\text{-ind}(f(Q \cap \text{Bd}_X(f^{-1}(W))), Y) &= \\ p_1\text{-ind}(f(Q) \cap f(\text{Bd}_X(f^{-1}(W))), Y) &\leq \\ p_1\text{-ind}(f(Q) \cap \text{Bd}_Y(W), Y) &< \alpha. \end{aligned}$$

Thus, by inductive assumption,

$$p_1\text{-ind}(Q \cap \text{Bd}_X(f^{-1}(W)), X) \leq p_1\text{-ind}(f(Q) \cap \text{Bd}_Y(W), Y) < \alpha$$

and, therefore, $p_1\text{-ind}(Q, X) \leq \alpha$. □

Similarly, we have the following proposition.

Proposition 5.2. *Let $f : X \rightarrow Y$ be a continuous map and $Q \subseteq X$. If the restriction $f|_Q$ of the map f to Q is a homeomorphism, then*

$$\text{pos}_1\text{-ind}(Q, X) \leq \text{pos}_1\text{-ind}(f(Q), Y).$$

Remark. The above proposition is a generalization of Lemma 1 of [8] for transfinite dimension in the realm of T_0 -spaces.

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