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COREFLECTIVE SUBCONSTRUCTS OF THE CONSTRUCTS OF AFFINE SETS

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Abstract. For a topological construct, we give necessary and sufficient conditions to be isomorphic to a coreflective subconstruct of a category of affine sets. This means that the objects can be described isomorphically as sets structured by a collection of functions. We also characterize the hereditary coreflective subconstructs of the categories of affine sets and the subcategories constructed from an algebra stucture. We prove that these two types of subconstructs do not coincide. As an application of these results we find a relation between the affine sets over $[0, \infty]$ and the metrically generated categories. Finally, we will give some examples of $(\mathbb{T}, \mathsf{V})$-categories which can be embedded in the category of affine sets over V.

1. Introduction

Recently, categories of affine sets, metrically generated topological categories, and categories of $(\mathbb{T}, \mathsf{V})$-algebras have been introduced as unifications of the different isomorphic descriptions of topological spaces.

The first unification that will be considered in this paper originated from ideas of Diers [20, 21, 22, 23]. Diers aimed at obtaining a classification of concrete geometrical categories. The objects of these categories are sets equipped with a geometrical structure.

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This type of topological category became known as categories KSet consisting of affine sets over K, the precise definition of which is explained in section 2. Strongly related to Diers' point of view are the categories that became known as K-Chu spaces [3, 7], as used in theoretical computer science [34]. The precise relation to categories of affine sets is dealt with in [30]. In several papers [26, 28, 27], Giuli has convincingly shown that the setting of affine sets benefits from the existence of the Zariski closure operator, which allows for nice topological results on separation, completeness and compactness. Another issue in the work of Giuli concerns the characterization of hereditary coreflective subcategories of KSet, leaving the following questions as open problems:

(1) Can every hereditary coreflective subcategory of KSet be obtained by putting a suitable algebraic theory on K?
(2) For each regular cardinal α the category $\mathsf{Tight}(\alpha)$, consisting of all topological spaces for which every point in the closure of a subset is also in the closure of a smaller subset of cardinality less than α, is a hereditary coreflective subcategory of Top [29, 1]. Are all these subcategories $\mathsf{Tight}(\alpha)$ of Top obtained by using a suitable algebraic theory?

The second unification we will encounter is the setting of metrically generated categories. The focus here is on categories in which "metrizable objects" generate the whole construct, as will be explained in section 3. Metrically generated theories were introduced in [19] and have proven to be a very suitable setting too for developing separation, completeness and compactness results [9, 17, 25]. We will be dealing mostly with the so called local theories introduced in [38].

There is a third and entirely new and important approach to categorical topology, which we should mention here, that started in [15] and [16]. It is based on earlier papers of Barr [2] and Lawvere [32]. Barr described topological spaces as relational Eilenberg-Moore Algebras with respect to the ultrafilter monad on Set. Lawvere presented generalized metric spaces (using the identity monad) as V-categories, where V is the halfline $\bar{\mathsf{R}}_+$. By replacing the Boolean two element lattice used by Barr and the lattice $\bar{\mathsf{R}}_+$ used by Lawvere, by some arbitrary quantale V, and by replacing the ultrafilter monad and the identity monad by some arbitrary Set-monad, the authors of [15] and [16], form the category $(\mathbb{T}, \mathsf{V})$-Cat of $(\mathbb{T}, \mathsf{V})$-algebras in a natural way. More details are to be found in section 4. Meanwhile this type of topological categories has been shown to be extremely suitable for the intrinsic approach of topology since the natural classes of open or of proper maps available in $(\mathbb{T}, \mathsf{V})$-Cat fulfill all the properties needed for such an approach [35].

The main issue of our paper is to contribute to the comparison of the various types of topological categories mentioned above. This comparison is successful for those categories having an initially dense object. As a first theorem we prove that topological categories with an initially dense object are exactly those embeddable as a coreflective subcategory of some KSet. Our second theorem states that in order to be embeddable as a *hereditary* coreflective subcategory of some KSet, the topological category has to posses an initially dense *injective* object. Moreover, we are able to completely characterize those subcategories of KSet that can be described via a suitable algebraic theory on K. Using these characterizing theorems we solve both open questions cited above. Question (1) is solved in the negative by producing a counterexample, question (2) is answered affirmatively.

As corollaries of our main theorems, we conclude that all local metrically generated theories are indeed embeddable as coreflective subcategories of $[0, \infty]\mathsf{Set}$. Others, like Unif do not have an initially dense object, and are therefore not coreflectively embeddable.

Our main theorems also imply that among categories of type $(\mathbb{T}, \mathsf{V})$-Cat, those for which there is an initially dense object with underlying set V are embeddable as coreflective subcategories of VSet. To the knowledge of the author the question which categories $(\mathbb{T}, \mathsf{V})$-Cat have an initially dense object is not settled yet. Some results in this direction showing that V can be endowed with a $(\mathbb{T}, \mathsf{V})$-categorical structure can be found in [12]. We will mention some monads for which V endowed with this structure is indeed an initially dense object.

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2. Coreflective subconstructs of KSet

Recall that, for a set K, an *affine set* over K is a pair $(X, \mathcal{A})$ with X a set and $\mathcal{A}$ a subset of the power set K^X. An *affine map* from $(X, \mathcal{A})$ to $(Y, \mathcal{B})$ is a function $f\colon X \to Y$ such that $\beta \circ f \in \mathcal{A}$ for all $\beta \in \mathcal{B}$. The construct, with objects the affine sets over K and morphisms the affine maps, will be denoted by KSet.

For every set K, the construct KSet is a topological construct [20]. If X is a set, $(X_i, \mathcal{A}_i)_{i\in I}$ a family of affine sets over K, then $\mathcal{A} = \{\alpha_i \circ f_i \,|\, i \in I, \alpha_i \in \mathcal{A}_i\}$ is the unique initial structure on X with respect to the source $(f_i\colon X \to X_i)_{i\in I}$.

Compactness and completeness for categories of affine sets are studied in [26, 28, 27, 10]. A lot of the theorems about compactness and completeness are inherited by the hereditary coreflective subconstructs of KSet.

In this section, we will characterize the coreflective and hereditary coreflective subconstructs of the construct KSet for a set K. Initially dense objects will play an important role in the characterization of these constructs. Recall that an object Y of the category X is *initially dense* in X if for every X-object X, there exists an initial source $(f_i\colon X \to Y)_{i\in I}$ with codomain Y.

It is clear that the affine set $\mathsf{K} = (K, \{id_K\})$ over K is an initially dense object of KSet. Since initial sources are preserved by a coreflector, it immediately follows that in a coreflective subconstruct X of KSet with coreflector $\mathbf{c}\colon \mathsf{KSet} \to \mathsf{X}$, $\mathbf{c}(\mathsf{K})$ is initially dense.

On the other hand, let X be a topological construct with an initially dense object $(K, \mathcal{U}_K)$. With every X-object $(X, \mathcal{U})$, we can associate the affine set over K: $(X, \mathrm{Hom}((X, \mathcal{U}), (K, \mathcal{U}_K)))$.

Proposition 2.1. *If X is a topological construct with initially dense object $(K, \mathcal{U}_K)$ and $(X, \mathcal{U}), (X, \mathcal{V})$ are X-objects such that $\mathrm{Hom}((X, \mathcal{U}), (K, \mathcal{U}_K)) = \mathrm{Hom}((X, \mathcal{V}), (K, \mathcal{U}_K))$, then $\mathcal{U} = \mathcal{V}$.*

Proof. Since $(K, \mathcal{U}_K)$ is initially dense in X, there exists an initial source $(\alpha_i\colon (X, \mathcal{U}) \to (K, \mathcal{U}_K))_{i\in I}$. Since $\mathrm{Hom}((X, \mathcal{U}), (K, \mathcal{U}_K)) = \mathrm{Hom}((X, \mathcal{V}), (K, \mathcal{U}_K))$, this implies that $\alpha_i \circ id_X\colon (X, \mathcal{V}) \to (K, \mathcal{U}_K)$ is a X-morphism for each $i \in I$. By initiality of the source

$$(\alpha_i\colon (X, \mathcal{U}) \to (K, \mathcal{U}_K))_{i\in I},$$

it follows that $id_X\colon (X, \mathcal{V}) \to (X, \mathcal{U})$ is a X-morphism.

In a similar way, one can prove that $id_X\colon (X,\mathcal{U}) \to (X,\mathcal{V})$ is a X-morphism. We can conclude that the X-strucures $\mathcal{U}$ and $\mathcal{V}$ on X coincide. $\square$

Proposition 2.2. *Let* X *be a topological construct with initially dense object* $(K,\mathcal{U}_K)$. *If* $(X,\mathcal{U}), (Y,\mathcal{V})$ *are* X*-objects and* $f\colon X \to Y$ *is a function, then the following are equivalent:*

(1) $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ *is a* X*-morphism.*
(2) $f\colon (X, \mathrm{Hom}((X,\mathcal{U}),(K,\mathcal{U}_K))) \to (Y, \mathrm{Hom}((X,\mathcal{V}),(K,\mathcal{U}_K)))$ *is an affine map.*

Proof. Let

$$f\colon (X, \mathrm{Hom}((X,\mathcal{U}),(K,\mathcal{U}_K))) \to (Y, \mathrm{Hom}((Y,\mathcal{V}),(K,\mathcal{U}_K)))$$

be an affine map. Since $(K,\mathcal{U}_K)$ is initially dense in X, there exists an initial source $(\alpha_i\colon (Y,\mathcal{V}) \to (K,\mathcal{U}_K))_{i\in I}$. Since f is an affine map and $\alpha_i \in \mathrm{Hom}((Y,\mathcal{V}),(K,\mathcal{U}_K))$, we have that $\alpha_i \circ f$ is a X-morphism for every $i \in I$. By initiality of the source

$$(\alpha_i\colon (Y,\mathcal{V}) \to (K,\mathcal{U}_K))_{i\in I},$$

it follows that $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ is a X-morphism.

The other implication immediately follows from the fact that the composition of X-morphisms is a X-morphism. $\square$

For a topological construct with initially dense object $(K,\mathcal{U}_K)$, let $\mathsf{KSet}_{\mathsf{X}}$ be the full subconstruct of KSet consisting of all affine sets $(X,\mathcal{A})$ over K such that there exists an X-object $(X,\mathcal{U})$ with $\mathcal{A} = \mathrm{Hom}((X,\mathcal{U}),(K,\mathcal{U}_K))$. The following proposition immediately follows from the previous two propositions.

Proposition 2.3. *If* X *is a topological construct with initially dense object* $(K,\mathcal{U}_K)$*, then* X *is isomorphic to* $\mathsf{KSet}_{\mathsf{X}}$. $\square$

Proposition 2.4. *For a topological construct* X *with initially dense object* $(K,\mathcal{U}_K)$*,* $\mathsf{KSet}_{\mathsf{X}}$ *is a concretely coreflective subconstruct of* KSet.

Proof. For an affine set $(X,\mathcal{A})$, let $\mathcal{U}$ be the initial X-structure on X for the source $(\alpha\colon X \to (K,\mathcal{U}_K))_{\alpha\in\mathcal{A}}$. It is clear that

$$id_X\colon (X, \mathrm{Hom}((X,\mathcal{U}),(K,\mathcal{U}_K))) \to (X,\mathcal{A})$$

is an affine map.

Now suppose that $f\colon (X', \mathrm{Hom}((X',\mathcal{U}'), (K,\mathcal{U}_K)) \to (X, \mathcal{A})$ is an affine map with $(X',\mathcal{U}')$ an X-object. Then, for all $\alpha \in \mathcal{A}$, we have that all compositions $\alpha \circ f\colon (X',\mathcal{U}') \to (K,\mathcal{U}_K)$ are X-morphisms. Since $(\alpha\colon (X,\mathcal{U}) \to (K,\mathcal{U}_K))_{\alpha\in\mathcal{A}}$ is an initial source, it follows that $f\colon (X',\mathcal{U}') \to (X,\mathcal{U})$ is a X-morphism. By 2.2 it now follows that $f\colon (X', \mathrm{Hom}((X',\mathcal{U}'), (K,\mathcal{U}_K))) \to (X, \mathrm{Hom}((X,\mathcal{U}), (K,\mathcal{U}_K)))$ is an affine map. We can conclude that $(X, \mathrm{Hom}((X,\mathcal{U}), (K,\mathcal{U}_K)))$ is the coreflection of $(X, \mathcal{A})$. □

Theorem 2.5. *A topological construct* X *is isomorphic to a concretely coreflective subconstruct of* KSet *if and only if* X *has an initially dense object with underlying set* K. □

Let M be a class of morphisms in a category X. Recall that an object X is called *M-injective* provided that for every morphism $m\colon Y \to Z$ in M and every morphism $f\colon Y \to X$, there exists a morphism $g\colon Z \to X$ with $f = g \circ m$. If M is the class of all embeddings, the M-injective objects are called injective objects. If M consists of all initial morphisms, the M-injective objects are called initially injective objects.

A subcategory Y of X is said to be closed under initial morphisms in X provided that whenever $m\colon X \to Y$ is an initial morphism and Y belongs to Y, then X belongs to Y. If this property only holds for the initial monomorphisms (embeddings), then the subcategory Y is called a hereditary subcategory of X.

Theorem 2.6. *A topological construct* X *is isomorphic to a hereditary concretely coreflective subconstruct of* KSet *if and only if* X *has an initially dense injective object with underlying set* K.

Proof. We first prove that for a hereditary coreflective subconstruct X of KSet, the coreflection $\mathbf{c}(\mathsf{K})$ of the affine set $\mathsf{K} = (K, \{id_K\})$ is an injective object. Let $(X, \mathcal{A})$ be an X-object, let $(A, \mathcal{A}|_A = \{\alpha|_A|\alpha \in \mathcal{A}\})$ be a subspace of $(X, \mathcal{A})$ and let $f\colon (A, \mathcal{A}|_A) \to \mathbf{c}(\mathsf{K})$ be an affine map. This implies that $id_K \circ f \in \mathcal{A}|_A$ and hence there exists $\alpha \in \mathcal{A}$ such that $\alpha|_A = f$. Since $\alpha\colon (X, \mathcal{A}) \to \mathsf{K}$ is an affine map and $(X, \mathcal{A})$ is an X-object, we have that $\alpha\colon (X, \mathcal{A}) \to \mathbf{c}(\mathsf{K})$ is also an affine map. It now follows that $\mathbf{c}(\mathsf{K})$ is an initially dense injective object of X.

If X is a topological construct with initially dense injective object $(K,\mathcal{U}_K)$, we know that X is isomorphic to the coreflective subconstruct $\mathsf{KSet}_{\mathsf{X}}$ of KSet. Let $(X,\mathcal{U})$ be an X-object and $(A,\mathcal{U}|_A)$ a subspace of $(X,\mathcal{U})$. Since $(K,\mathcal{U}_K)$ is an injective object, every morphism $\alpha\colon (A,\mathcal{U}|_A) \to (K,\mathcal{U}_K)$ can be extended to a X-morphism $\bar{\alpha}\colon (X,\mathcal{U}) \to (K,\mathcal{U}_K)$. Hence,

$$\mathrm{Hom}((X,\mathcal{U}),(K,\mathcal{U}_K))|_A = \mathrm{Hom}((A,\mathcal{U}|_A),(K,\mathcal{U}_K))$$

and we can conclude that $\mathsf{KSet}_{\mathsf{X}}$ is a hereditary subconstruct of KSet. □

We will now recall a general method to construct hereditary coreflective subcategories of KSet [20, 21, 28].

In order to define a subconstruct of KSet, we put an algebra structure on K. Recall that an *algebra structure* on the set K is a class of operations

$$\Omega = \{\omega_i\colon K^{n_i} \to K \mid i \in I\}$$

of arbitrary arities. Hence the n_i are arbitrary cardinal numbers, and there is no condition on the size of the indexing system I. For every set X, by point-wise extension, the powerset K^X carries an algebra structure. We denote by $\mathsf{KSet}(\Omega)$ the subconstruct of KSet consisting of those affine sets $(X,\mathcal{A})$ for which $\mathcal{A}$ is an Ω-subalgebra of the function algebra K^X. The objects in $\mathsf{KSet}(\Omega)$ are called *affine sets over the algebra* (K,Ω).

In [22], Y. Diers proved the following theorem.

Theorem 2.7. *A topological construct* X *is isomorphic to* $\mathsf{KSet}(\Omega)$ *for some algebraic theory* (K,Ω) *if and only if* X *has an initially dense object with underlying set* K *which is initially injective.*

Using this theorem, we can prove the following characterization of the categories of the form $\mathsf{KSet}(\Omega)$.

Proposition 2.8. X *is isomorphic to* $\mathsf{KSet}(\Omega)$ *iff* X *is a concretely coreflective subconstruct of* KSet *which is closed under initial morphisms in* KSet.

Proof. We first prove that $\mathsf{KSet}(\Omega)$ is closed under initial morphisms. Let $(Y,\mathcal{B}) \in \mathsf{KSet}(\Omega)$ and $f\colon (X,\mathcal{A}) \to (Y,\mathcal{B})$ be an initial affine map. For $\omega_T\colon K^T \to K \in \Omega$ and $(\beta_t)_{t\in T} \in \mathcal{B}^T$, we have $\omega_T((\beta_t \circ f)_{t\in T}) = \omega_T((\beta_t)_{t\in T}) \circ f$. This implies that $\mathcal{A} = \{\beta \circ f \mid \beta \in \mathcal{B}\}$ is a Ω-subalgebra of K^X.

For a concretely coreflective subconstruct X of KSet, which is closed under initial morphisms in KSet, we already proved that $\mathbf{c}(\mathsf{K})$ is an initially dense injective object. We now prove that $\mathbf{c}(\mathsf{K})$ is also an initially injective object. Let $m\colon (X,\mathcal{A}) \to (Y,\mathcal{B})$ be an initial X-morphism and let $f\colon (X,\mathcal{A}) \to \mathbf{c}(\mathsf{K})$ be an affine map. This implies that $id_K \circ f \in \mathcal{A}$ and hence there exists $\beta \in \mathcal{B}$ such that $\beta \circ m = f$. Since $\beta\colon (Y,\mathcal{B}) \to \mathsf{K}$ is an affine map and $(Y,\mathcal{B})$ is an X-object, we have that $\beta\colon (Y,\mathcal{B}) \to \mathbf{c}(\mathsf{K})$ is also an affine map. It now follows from the previous theorem that there exists an algebraic theory Ω on K such that X is isomorphic to $\mathsf{KSet}(\Omega)$. □

This answers Giuli's first question in [26], where he asks whether every hereditary coreflective subcategory of KSet is of the form $\mathsf{KSet}(\Omega)$. It seems that for a coreflective subconstruct of KSet being closed under embeddings is not sufficient to be of the form $\mathsf{KSet}(\Omega)$. The categories of the form $\mathsf{KSet}(\Omega)$ satisfy the stronger condition of being closed under initial morphisms. We now give an example of a hereditary coreflective subconstruct of $\mathsf{3Set}$ which is not of the form $\mathsf{3Set}(\Omega)$.

Recall that a *pretopological space* is a structured set $(X,\mathcal{V})$ where the structure $\mathcal{V}$ is a function assigning a neighbourhood filter $\mathcal{V}(x)$ to each point $x \in X$ and $\mathcal{V}(x)$ satisfies the condition $\mathcal{V}(x) \subseteq \dot{x}$. A function $f\colon (X,\mathcal{V}) \to (Y,\mathcal{W})$ between pretopological spaces is continuous if $\mathcal{W}(f(x)) \subseteq f(\mathcal{V}(x))$ for each $x \in X$. The category of pretopological spaces and continuous maps is denoted by PrTop.

In [11], neigbourhoodfilters are replaced by neighbourhoodstacks in order to obtain the extensional hull of the category Cl of closure spaces. A preclosure space is a structured set $(X,\mathcal{V})$, where the structure $\mathcal{V}$ is a function assigning to each point $x \in X$ a neighbourhood stack $\mathcal{V}(x) \subset P(X)$ such that:

(1) $X \in \mathcal{V}(x)$
(2) $\forall V \in \mathcal{V}(x) : x \in V$
(3) $\forall V \in \mathcal{V}(x) : V \subset W \Rightarrow W \in \mathcal{V}(x)$

The construct with objects the preclosure spaces and morphisms the continuous maps is denoted by PrCls.

The pretopological space $\mathbf{3}$ with underlying set $\{0,1,2\}$ and neighbourhoodfilters

$$\mathcal{V}(0) = \mathcal{V}(2) = \{0,1,2\} \text{ and } \mathcal{V}(1) = \{\{1,2\},\{0,1,2\}\}$$

is an initially dense object of the category PrTop of pretopological spaces [5]. This pretopological space is also an initially dense object of the category PrCls of preclosure spaces [11]. Hence, 2.5 implies that both categories are coreflective subconstructs of the category 3Set of affine sets over $\{0, 1, 2\}$.

Proposition 2.9. PrTop *and* PrCls *are hereditary subconstructs of* 3Set.

Proof. Let $(X, \mathcal{V})$ be a pretopological space and A a subset of X. We will prove that $\mathrm{Hom}((X, \mathcal{V}), \mathbf{3})|_A$ consisting of the restrictions to A of the continuous functions from $(X, \mathcal{V})$ to $\mathbf{3}$ coincide with $\mathrm{Hom}((A, \mathcal{V}|_A), \mathbf{3})$. If $\alpha\colon (X, \mathcal{V}) \to \mathbf{3}$ is a continuous function, then it is clear that the restriction $\alpha|_A\colon (A, \mathcal{V}|_A) \to \mathbf{3}$ is also continuous. On the other hand, let $\alpha\colon (A, \mathcal{V}|_A) \to \mathbf{3}$ be a continuous function. Consider the function $\bar{\alpha}\colon X \to \{0, 1, 2\}$, defined by $\bar{\alpha}(x) = \alpha(x)$ for $x \in A$ and $\bar{\alpha}(x) = 2$ for $x \notin A$. It is clear that $\bar{\alpha}\colon (X, \mathcal{V}) \to \mathbf{3}$ is a continuous extension of α. □

Remark that PrTop and PrCls are not closed under initial morphisms in 3Set, because the pretopological space $\mathbf{3}$ is not an initially injective object of 3Set. Let I_2 be the indiscrete pretopological space with underlying set $\{0, 1\}$ and let I_1 be the indiscrete pretopological space with underlying set $\{0\}$. Let $f\colon \mathsf{I}_2 \to \mathbf{3}$ be the continuous function defined by $f(0) = 0$ and $f(1) = 2$. The constant function $c_0\colon \mathsf{I}_2 \to \mathsf{I}_1$ is an initial morphism in PrTop, but there is no continuous function $g\colon \mathsf{I}_1 \to \mathbf{3}$ such that $g \circ c_0 = f$. Hence, PrTop and PrCls are hereditary coreflective subconstructs of 3Set which can not be constructed from an algebra structure on $\{0, 1, 2\}$.

We now look at the second open problem stated by Giuli in [26]. The notion of tightness was introduced by Arhangel'skii in [1]. Let α be a regular cardinal. A topological space (X, τ) is α-tight if it satisfies the following condition:

$$\text{if } x \in cl_\tau B, \text{ then } x \in cl_\tau A \text{ for some } A \subset B, |A| < \alpha.$$

The category $\mathsf{Tight}(\alpha)$ consisting of the α-tight topological spaces and continuous maps is a concretely coreflective subconstruct of Top. The coreflection was explicitly described by Arhangel'skii [1]. Further details of this coreflection can also be found in [6]. In [29], Giuli and Husek proved that $\mathsf{Tight}(\alpha)$ is also a hereditary subconstruct of Top. Since Top is a hereditary concretely coreflective subconstruct of the category SSet of affine sets over $S = \{0,1\}$, it immediately follows that $\mathsf{Tight}(\alpha)$ is also a hereditary concretely coreflective subconstruct of SSet.

Proposition 2.10. $\mathsf{Tight}(\alpha)$ *is closed under initial morphisms in* SSet.

Proof. Top is isomorphic to $\mathsf{SSet}(\Omega)$ with Ω the algebraic structure on S containing the constant operations, the operations $\omega_i(a_t) = \max\limits_{t\in n_i} a_t$ for arbitrary cardinals n_i and $\omega_i'(a_t) = \min\limits_{t\in n_i} a_t$ for every finite cardinal n_i. From 2.8, it follows that Top is closed under initial morphisms in SSet. Hence, it is sufficient to prove that $\mathsf{Tight}(\alpha)$ is closed under initial morphisms in Top. Let $f\colon (X,\tau_X) \to (Y,\tau_Y)$ be an initial morphism in Top, with (Y,τ_Y) a $\mathsf{Tight}(\alpha)$-object. For $B \subseteq X$ and $x \in clB$, we have $f(x) \in cl(f(B))$. Since (Y,τ_Y) is a $\mathsf{Tight}(\alpha)$-object, there exists a subset C of $f(B)$ with cardinality less than α such that $f(x) \in cl(C)$. For every $c \in C$, choose $x_c \in f^{-1}(\{c\})$ and let $A = \{x_c | c \in C\}$. A is a set with cardinality less than α such that $f(A) = C$. Since f is an initial morphism in Top, we have $clA = f^{-1}(cl(f(A))) = f^{-1}(cl(C))$. This implies that $x \in cl(A)$. We can conclude that (X,τ_X) is a $\mathsf{Tight}(\alpha)$-object. $\square$

It now follows from 2.8 that all categories $\mathsf{Tight}(\alpha)$ are of the form $\mathsf{SSet}(\Omega)$ for some algebraic theory Ω on S. This answers the second question of Giuli in [26].

3. Metrically generated categories

Another general type of topological categories are the metrically generated categories, which were introduced by R. Lowen and E. Colebunders in [19]. In this section, we will use the results of the previous section in order to compare the categories of affine sets with the metrically generated theories. First, we gather all the preliminary material from [19] that is needed to introduce the metrically generated constructs.

A function $d\colon X \times X \to [0,\infty]$ is called a quasi-pre-metric if it is zero on the diagonal. We will drop "pre" if d satisfies the triangle inequality and we will drop "quasi" if d is symmetric. For a quasi-pre-metric d, we will denote by d^{-1} the quasi-pre-metric defined by $d^{-1}(x,y) = d(y,x)$ for all $x,y \in X$. We will consider the pointwise order on the set of quasi-pre-metrics on X (i.e. $d \leqslant e \Leftrightarrow \forall x, y \in X : d(x,y) \leqslant e(x,y)$). Denote by Met the construct of quasi-pre-metrics and contractions (a map $f\colon (X,d) \to (X',d')$ is a contraction if $d' \circ f \times f \leq d$) and by $\mathsf{Met(X)}$ the fiber of Met-structures on X. Met is a topological construct and the indiscrete objects are precisely those for which every pair of points has distance 0.

A *base category* $\mathcal{C}$ is a full and isomorphism-closed concrete subconstruct of Met which is closed for initial morphisms and contains all Met-indiscrete spaces. In this paper we will consider the base category $\mathcal{C}^{\Delta}$ consisting of all quasi-metric spaces, $\mathcal{C}^{\Delta s}$ the construct of metric spaces, $\mathcal{C}^{\Delta s\theta}$ the construct of totally bounded metric spaces and $\mathcal{C}^{\Delta\theta}$ the construct of totally bounded quasi-metric spaces (a quasi-metric d on X is totally bounded if $d \vee d^{-1}$ is a totally bounded metric on X).

Given a base category $\mathcal{C}$, a topological construct X is called *$\mathcal{C}$-metrically generated* if there exists a concrete functor $K\colon \mathcal{C} \to \mathsf{X}$ such that K preserves initial morphisms and $K(\mathcal{C})$ is initially dense in X.

We now recall that there exists a model category for all $\mathcal{C}$-metrically generated constructs. For any collection $\mathcal{B}$ of quasi-pre-metrics on X, we put $\mathcal{B}{\downarrow} := \{e \in \mathsf{Met(X)} \mid \exists d \in \mathcal{B} : e \leq d\}$. We say that $\mathcal{B}$ is a basis for $\mathcal{D}$ if $\mathcal{D} = \mathcal{B}{\downarrow}$.

$\mathsf{M}^{\mathcal{C}}(\mathsf{X})$ is the construct with objects, pairs $(X,\mathcal{D})$ where X is a set and $\mathcal{D}$ is a collection of quasi-pre-metrics on X with basis consisting of $\mathcal{C}$-metrics.
$\mathcal{D}$ is called a *$\mathcal{C}$-meter* and $(X,\mathcal{D})$ a *$\mathcal{C}$-metered space*. If $(X,\mathcal{D})$ and $(X',\mathcal{D}')$ are $\mathcal{C}$-metered spaces and $f\colon (X,\mathcal{D}) \to (X',\mathcal{D}')$, then we say that f is a contraction if $d' \circ f \times f \in \mathcal{D}$ for all $d' \in \mathcal{D}'$.

ξ is called an *expander* on $\mathsf{M}^{\mathcal{C}}$ if for any set X, ξ provides us with a function

$$\xi\colon \mathsf{M}^{\mathcal{C}}(X) \to \mathsf{M}^{\mathcal{C}}(X) : \mathcal{D} \to \xi(\mathcal{D})$$

such that the following properties are fulfilled:

E1. $\mathcal{D} \subseteq \xi(\mathcal{D})$
E2. $\mathcal{D} \subseteq \mathcal{D}' \Rightarrow \xi(\mathcal{D}) \subseteq \xi(\mathcal{D}')$
E3. $\xi(\xi(\mathcal{D})) = \xi(\mathcal{D})$
E4. $f\colon Y \to X$ and $\mathcal{D} \in \mathsf{M}^{\mathcal{C}}(X) \Rightarrow \xi(\mathcal{D}) \circ f \times f \subseteq \xi(\mathcal{D} \circ f \times f \downarrow)$

Given an expander ξ on $\mathsf{M}^{\mathcal{C}}$, $\mathsf{M}^{\mathcal{C}}_{\xi}$ is the full coreflective subconstruct of $\mathsf{M}^{\mathcal{C}}$ with objects, those $\mathcal{C}$-metered spaces $(X,\mathcal{D})$ for which $\xi(\mathcal{D}) = \mathcal{D}$.

The main result of [19] states that a topological construct is $\mathcal{C}$-metrically generated if and only if X is concretely isomorphic to $\mathsf{M}^{\mathcal{C}}_{\xi}$ for some expander ξ on $\mathsf{M}^{\mathcal{C}}$.

For any meter $\mathcal{D}$, we put $[\mathcal{D}]^{\mathcal{C}} = \{d \in \mathcal{D} \mid d\ \mathcal{C}\text{-meter}\}$. For any expander ξ on $\mathsf{M}^{\mathsf{Met}}$, there is an adapted version on $\mathsf{M}^{\mathcal{C}}$ defined by $\xi^{\mathcal{C}}(\mathcal{D}) = [\xi(\mathcal{D})]^{\mathcal{C}}\downarrow$. Throughout this paper we will consider local expanders. An expander ξ on $\mathsf{M}^{\mathcal{C}}$ is called *local* if $loc^{\mathcal{C}} \leqslant \xi$ where *loc* is the following expander on $\mathsf{M}^{\mathsf{Met}}$.

$$loc(\mathcal{D}) = \{e \,|\, \forall x \in X,\, \exists d \in \mathcal{D} : e(x,y) \leqslant d(x,y)\}.$$

Local expanders are studied in detail in [38]. In this paper, we will consider the following examples of local expanders on $\mathsf{M}^{\mathsf{Met}}$. For a meter $\mathcal{D}$ on a set X:

- $\xi_T(\mathcal{D}) = \{e \,|\, \forall \epsilon > 0, \forall x \in X, \exists d_1,..,d_n \in \mathcal{D}, \exists \delta > 0 : \sup_{i=1}^{n} d_i(x,y) < \delta \Rightarrow e(x,y) < \epsilon\}$
- $\xi_A(\mathcal{D}) = \{e \,|\, \forall \epsilon > 0, \forall x \in X, \forall \omega < \infty\, \exists d_1,..,d_n \in \mathcal{D} : e(x,y) \wedge \omega \leqslant \sup_{i=1}^{n} d_i(x,y) + \epsilon\}$
- $\eta(\mathcal{D}) = \{e \,|\, \forall x \in X, \forall \omega < \infty\,, \exists d \in \mathcal{D} : e(x,y) \wedge \omega \leqslant d(x,y)\}$

For the expander ξ_T, the construct $\mathsf{M}^{\mathcal{C}}_{\xi_T^{\mathcal{C}}}$ is isomorphic to Top in case $\mathcal{C}$ equals $\mathcal{C}^{\Delta}$ or $\mathcal{C}^{\Delta\theta}$, to the construct Creg of completely regular spaces in case $\mathcal{C}$ equals $\mathcal{C}^{\Delta s}$ or $\mathcal{C}^{\Delta s\theta}$. For the expander ξ_A, the construct $\mathsf{M}^{\mathcal{C}}_{\xi_A^{\mathcal{C}}}$ is isomorphic to Ap in case $\mathcal{C}$ equals $\mathcal{C}^{\Delta}$ or $\mathcal{C}^{\Delta\theta}$, to the construct Uap of uniform approach spaces in case $\mathcal{C}$ equals $\mathcal{C}^{\Delta s}$ or $\mathcal{C}^{\Delta s\theta}$.

In [8], initially dense objects are determined for the local metrically generated categories. There, we proved that for $\mathcal{C}$ any of the base categories $\mathcal{C}^{\Delta}, \mathcal{C}^{\Delta s}, \mathcal{C}^{\Delta\theta}, \mathcal{C}^{\Delta s\theta}$ and ξ a local expander on $\mathsf{M}^{\mathcal{C}}$, $\mathsf{M}^{\mathcal{C}}_{\xi}$ has an initially dense object with underlying set $[0,\infty]$. Combining this with the results of the previous section, we have the following relation between local metrically generated categories and categories of affine sets.

Proposition 3.1. *For $\mathcal{C}$ any of the base categories $\mathcal{C}^{\Delta}, \mathcal{C}^{\Delta s}, \mathcal{C}^{\Delta\theta}, \mathcal{C}^{\Delta s\theta}$ and ξ a local expander on $\mathsf{M}^{\mathcal{C}}$, $\mathsf{M}^{\mathcal{C}}_{\xi}$ is isomorphic to a coreflective subconstruct of* $[0,\infty]\mathsf{Set}$.

We now describe the coreflective subconstruct of $[0,\infty]\mathsf{Set}$, which is isomorphic to $\mathsf{M}_{loc}^{\mathcal{C}^\Delta}$. All the local metrically generated constructs with $\mathcal{C} \subset \mathcal{C}^\Delta$ can be embedded in the following subconstruct of the construct of affine sets.

Proposition 3.2. *$\mathsf{M}_{loc}^{\mathcal{C}^\Delta}$ is isomorphic to the full subconstruct of $[0,\infty]\mathsf{Set}$ consisting of the affine sets $(X,\mathcal{A})$ over $[0,\infty]$ satisfying the following four conditions:*

(1) $c_a \in \mathcal{A}$ *for all* $a \in [0,\infty]$
(2) $a \in [0,\infty], \alpha \in \mathcal{A} \Rightarrow A_a(\alpha) = \alpha + c_a \in \mathcal{A}$
(3) $a \in [0,\infty], \alpha \in \mathcal{A} \Rightarrow S_a(\alpha) = (\alpha - c_a) \vee 0 \in \mathcal{A}$
(4) *$\mathcal{A}$ is closed under infima of the collections $(\alpha_i)_{i\in I} \in \mathcal{A}$ such that $\forall x \in X$, there exists $j \in I : \inf\limits_{i\in I} \alpha_i(x) = \alpha_j(x)$.*

Proof. In [8], it was proved that $([0,\infty], loc^{\mathcal{C}^\Delta}(d_{\mathbb{P}}^{-1}\downarrow))$, with

$$d_{\mathbb{P}}\colon [0,\infty]\times[0,\infty] \to [0,\infty] \colon (x,y) \to (x-y)\vee 0$$

is initially dense in $\mathsf{M}_{loc}^{\mathcal{C}^\Delta}$. It then follows from 2.3 that $\mathsf{M}_{loc}^{\mathcal{C}^\Delta}$ is isomorphic to $[0,\infty]\mathsf{Set}_{\mathsf{M}_{\mathsf{loc}}^{\mathcal{C}^\Delta}}$ and the isomorphism on the objects is given by $F(X,\mathcal{D}) = (X, \mathcal{A}_{\mathcal{D}} = \{\alpha\colon X \to [0,\infty] \,|\, d_\alpha^{-1} \in \mathcal{D}\})$ with $d_\alpha^{-1}(x,y) = (\alpha(y) - \alpha(x)) \vee 0$. So, we only have to prove that $[0,\infty]\mathsf{Set}_{\mathsf{M}_{\mathsf{loc}}^{\mathcal{C}^\Delta}}$ consists of the affine sets over $[0,\infty]$ satisfying the four conditions above. We first prove that the affine sets $(X,\mathcal{A}_{\mathcal{D}})$ satisfy the four conditions:

(1) $d_{c_a}^{-1}(x,y) = (a-a)\vee 0 = 0 \in \mathcal{D}$
(2) $$d_{A_a(\alpha)}^{-1}(x,y) = ((\alpha(y)+a) - (\alpha(x)+a)) \vee 0 = (\alpha(y)-\alpha(x))\vee 0 = d_\alpha^{-1}(x,y)$$
(3) $$\begin{aligned} d_{S_a(\alpha)}^{-1}(x,y) &= (((\alpha(y)-a)\vee 0) - ((\alpha(x)-a)\vee 0))\vee 0 \\ &\leqslant \alpha(y)-\alpha(x)\vee 0 = d_\alpha^{-1}(x,y) \end{aligned}$$
(4) Let $(\alpha_i)_{i\in I} \in \mathcal{A}_{\mathcal{D}}$ and for all $x \in X$, there exists $j \in I$ such that $\inf\limits_{i\in I} \alpha_i(x) = \alpha_j(x)$.
$$\begin{aligned} d_{\inf \alpha_i}^{-1}(x,y) &= (\inf_{i\in I} \alpha_i(y) - \inf_{i\in I} \alpha_i(x)) \vee 0 \\ &= (\inf_{i\in I} \alpha_i(y) - \alpha_j(x)) \vee 0 \\ &\leqslant (\alpha_j(y) - \alpha_j(x)) \vee 0 = d_{\alpha_j}^{-1}(x,y). \end{aligned}$$
This implies that $d_{\inf \alpha_i}^{-1} \in loc^{\mathcal{C}^\Delta}(\mathcal{D}) = \mathcal{D}$.

On the other hand, let $(X, \mathcal{A})$ be an affine set such that $\mathcal{A}$ satisfies the four conditions. Let $\mathcal{D} = loc^{\mathcal{C}^\Delta}\{d_\alpha^{-1} \mid \alpha \in \mathcal{A}\}$. By definition of $\mathcal{D}$ and $\mathcal{A}_\mathcal{D}$ follows immediately that $(X, \mathcal{D})$ is a $\mathsf{M}_{loc}^{\mathcal{C}^\Delta}$-object and $\mathcal{A} \subseteq \mathcal{A}_\mathcal{D}$.

If $\alpha \in \mathcal{A}_\mathcal{D}$, then $d_\alpha^{-1} \in \mathcal{D}$. Hence, for all $x \in X$, there exists $\beta_x \in \mathcal{A}$ such that $d_\alpha^{-1}(x,y) \leqslant d_{\beta_x}^{-1}(x,y)$. This means that

$$(\alpha(y) - \alpha(x)) \vee 0 \leqslant (\beta_x(y) - \beta_x(x)) \vee 0.$$

Which implies $\alpha(y) \leqslant ((\beta_x(y) - \beta_x(x)) \vee 0) + \alpha(x)$. Since $\mathcal{A}$ satisfies condition 2 and 3, we have that the functions $\alpha_x \colon X \to [0,\infty]$ defined by

$$\alpha_x(y) = ((\beta_x(y) - \beta_x(x)) \vee 0) + \alpha(x)$$

belong to $\mathcal{A}$ for all $x \in X$. For all $x \in X$, we have that $\alpha \leqslant \alpha_x$ and $\alpha(x) = \alpha_x(x)$. This implies that $\alpha = \inf_{x \in X} \alpha_x$ and $\inf_{x\in X} \alpha_x(y) = \alpha_y(y)$ for all $y \in X$. By condition 4, we can conclude that $\alpha \in \mathcal{A}$. $\square$

In [8], it was proved that if an expander ξ on $\mathsf{M}^{\mathcal{C}^\Delta}$ satisfies the slightly stronger condition $\eta \leqslant \xi$, then $([0,\infty], \xi(d_\mathbb{P}\downarrow))$ is also an initially dense object of $\mathsf{M}_\xi^{\mathcal{C}^\Delta}$. Hence, for these expanders, $\mathsf{M}_\xi^{\mathcal{C}^\Delta}$ is isomorphic to a second subconstruct of $[0,\infty]\mathsf{Set}$. We now describe the largest coreflective subconstruct of $[0,\infty]\mathsf{Set}$ obtained this way.

Proposition 3.3. $\mathsf{M}_\eta^{\mathcal{C}^\Delta}$ *is isomorphic to the full subconstruct of* $[0,\infty]\mathsf{Set}$ *consisting of the affine sets* $(X, \mathcal{A})$ *over* $[0,\infty]$ *satisfying the following five conditions:*

(1) $c_a \in \mathcal{A}$ *for all* $a \in [0,\infty]$
(2) $a \in [0,\infty], \alpha \in \mathcal{A} \Rightarrow A_a(\alpha) = \alpha + c_a \in \mathcal{A}$
(3) $a \in [0,\infty], \alpha \in \mathcal{A} \Rightarrow S_a(\alpha) = (\alpha - c_a) \vee 0 \in \mathcal{A}$
(4) $a \in [0,\infty], \alpha \in \mathcal{A} \Rightarrow \alpha \wedge c_a \in \mathcal{A}$
(5) $\mathcal{A}$ *is closed under suprema of the collections* $(\alpha_i)_{i\in I}$ *that satisfy the following condition:* $\forall x \in X$:
If $\sup_{i\in I} \alpha_i(x) \neq \infty$, then there exists $j \in I : \sup_{i \in I} \alpha_i(x) = \alpha_j(x)$
If $\sup_{i\in I} \alpha_i(x) = \infty$, then $\forall \omega < \infty, \exists j \in I$ such that :
$\omega \leqslant (\alpha_j(x) - \alpha_j(y)) \wedge 0$ for all y with $\sup_{i\in I} \alpha_i(y) \neq \infty$.

Proof. Since $([0,\infty], \eta^{\mathcal{C}^\Delta}(d_{\mathbb{P}}\downarrow))$ is initially dense in $\mathsf{M}_\eta^{\mathcal{C}^\Delta}$, it follows from 2.3 that $\mathsf{M}_\eta^{\mathcal{C}^\Delta}$ is isomorphic to $[0,\infty]\mathsf{Set}_{\mathsf{M}_\eta^{\mathcal{C}^\Delta}}$ and the isomorphism on the objects is given by

$$F(X,\mathcal{D}) = (X, \mathcal{A}_\mathcal{D} = \{\alpha\colon X \to [0,\infty] \,|\, d_\alpha \in \mathcal{D}\})$$

with $d_\alpha(x,y) = (\alpha(x) - \alpha(y)) \vee 0$. Analogously to the previous proof, one can prove that $[0,\infty]\mathsf{Set}_{\mathsf{M}_\eta^{\mathcal{C}^\Delta}}$ consists of the affine sets over $[0,\infty]$ satisfying the conditions above. $\square$

If we apply the isomorphism to the category $\mathsf{M}_{\xi_T}^{\mathcal{C}^\Delta}$, we get the following isomorphic description for the construct Top of topological spaces and continuous maps.

Proposition 3.4. Top *is isomorphic to the category of affine sets* $(X,\mathcal{A})$ *over* $[0,\infty]$ *satisfying the following conditions:*

(1) $c_a \in \mathcal{A}$ *for all* $a \in [0,\infty]$
(2) $a \in [0,\infty], \alpha \in \mathcal{A} \Rightarrow A_a(\alpha) = \alpha + c_a \in \mathcal{A}$
(3) $a \in [0,\infty], \alpha \in \mathcal{A} \Rightarrow S_a(\alpha) = (\alpha - c_a) \vee 0 \in \mathcal{A}$
(4) $\alpha, \beta \in \mathcal{A} \Rightarrow \alpha \wedge \beta \in \mathcal{A}$
(5) $(\alpha_i)_{i\in I} \in \mathcal{A} \Rightarrow \sup\limits_{i\in I} \alpha_i \in \mathcal{A}$
(6) $\alpha \in \mathcal{A} \Rightarrow \mu_\infty \circ \alpha \in \mathcal{A}$, *with* $\mu_\infty\colon [0,\infty] \to [0,\infty], \mu_\infty(0) = 0$ *and* $\mu_\infty(x) = \infty$ *for* $x \neq 0$.

Remark that this description was also obtained in [18]. If we apply the isomorphism to the category $\mathsf{M}_{\xi_A}^{\mathcal{C}^\Delta}$, we get the description of approach spaces using regular function frames.

4. $(\mathbb{T},\mathsf{V})$-categories

In this section, we explain how the theorems of section 2 can contribute to a comparison of the categories of affine sets and $(\mathbb{T},\mathsf{V})$-categories or lax algebras. Throughout this section V will denote a (commutative and unital) quantale. Hence, V is a complete lattice which carries a commutative and associative operation $\otimes$ with a neutral element k, such that

$$u \otimes \bigvee_{i\in I} v_i = \bigvee_{i\in I} (u \otimes v_i)$$

for all $u, v_i \in V$. Since V is a complete lattice, the preservation of suprema by $u \otimes -\colon \mathsf{V} \to \mathsf{V}$ is equivalent to the existence of a right adjoint $\hom(u,-)\colon \mathsf{V} \to \mathsf{V}$ to $u \otimes -$. Therefore, we have a map $\hom\colon \mathsf{V}\times\mathsf{V} \to \mathsf{V}$ such that for all $u, v, w \in \mathsf{V}$,

$$u \otimes v \leqslant w \Leftrightarrow v \leqslant \hom(u, w)$$

For example, the two-element chain $2 = \{\bot, \top\}$ with $\otimes = \wedge$ and $k = \top$ is a quantale. The category V-Mat of V-matrices [4, 16] has sets as objects, and a morphism $r\colon X \nrightarrow Y$ in V-Mat is a map $r\colon X \times Y \to \mathsf{V}$. Composition of V-matrices $r\colon X \nrightarrow Y$ and $s\colon Y \nrightarrow Z$ is defined as matrix multiplication:

$$s \cdot r(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z)$$

For example, 2-Mat is isomorphic to the category Rel with sets as objects and relations as morphisms. The identity $1_X\colon X \nrightarrow X$ in V-Mat is the V-matrix which sends all diagonal elements (x, x) to k and all other elements to the bottom element $\bot$ of V. Each Set-map $f\colon X \to Y$ can be interpreted as the V-matrix $f\colon X \nrightarrow Y$ given by

$$f(x, y) = \begin{cases} k & \text{if } f(x) = y \\ \bot & \text{otherwise} \end{cases}$$

To keep notations simple, we will write $f\colon X \to Y$ instead of $f\colon X \nrightarrow Y$ to designate a V-matrix induced by a map. The complete order on V induces a complete order on V-$\mathsf{Mat}(X, Y) = \mathsf{V}^{X\times Y}$: for V-matrices $r, r'\colon X \nrightarrow Y$ we define

$$r \leqslant r' :\Leftrightarrow \forall x \in X,\ \forall y \in Y :\ r(x, y) \leqslant r'(x, y)$$

The *transpose* $r^\circ\colon Y \nrightarrow X$ of a V-matrix $r\colon X \nrightarrow Y$ is defined by $r^\circ(y, x) = r(x, y)$.

Recall that a *monad* $\mathbb{T} = (T, e, m)$ on Set consists of a functor $T\colon \mathsf{Set} \to \mathsf{Set}$ together with natural transformations $e\colon Id \to T$ and $m\colon TT \to T$ such that

$$m \cdot Te = 1_T = m \cdot e_T \quad \text{and} \quad m \cdot Tm = m \cdot m_T$$

A *lax extension* of a monad $\mathbb{T} = (T, e, m)$ on Set is a map

$$\begin{array}{rcl} \bar{T}\colon \mathsf{V}\text{-}\mathsf{Mat} & \to & \mathsf{V}\text{-}\mathsf{Mat} \\ (r\colon X \nrightarrow Y) & \mapsto & (\bar{T}r\colon TX \nrightarrow TY) \end{array}$$

satisfying for all $r\colon X \nrightarrow Y, s\colon Y \nrightarrow Z$ and $f\colon X \to Y$ the conditions:

(1) $s \leqslant r \Rightarrow \bar{T}s \leqslant \bar{T}r$

(2) $\bar{T}s \cdot \bar{T}r \leqslant \bar{T}(s \cdot r)$
(3) $Tf \leqslant \bar{T}f$ and $(Tf)^\circ \leqslant \bar{T}f^\circ$
(4) $e_Y \cdot r \leqslant \bar{T}r \cdot e_X$
(5) $m_Y \cdot \bar{T}\bar{T}r \leqslant \bar{T}r \cdot m_X$

In [2] Barr shows how to extend a *monad* $\mathbb{T} = (T, e, m)$ on Set to $\mathsf{2\text{-}Mat} = \mathsf{Rel}$. For each relation $r\colon X \nrightarrow Y$, we let G_r denote its graph $G_r \subset X \times Y$. With $p\colon G_r \to X$ and $q\colon G_r \to Y$ being the respective projection maps, we have $r = q \cdot p^\circ$. The *Barr extension* T_B of $\mathbb{T}$ to Rel is defined by $T_B X = TX$ and $T_B r = Tq \cdot T_B p^\circ$, where $T_B p^\circ = (Tp)^\circ$. If the functor T satisfies the Beck-Chevalley Condition, then the Barr extension is a lax extension.

In [14], Clementino and Hofmann describe the construction of a lax $\mathsf{V\text{-}Mat}$-extension out of a Rel-extension. For a V-matrix $r\colon X \nrightarrow Y$ and $v \in \mathsf{V}$, they first define relations $r_v\colon X \nrightarrow Y$

$$r_v(x, y) = \top \Leftrightarrow v \leqslant r(x, y)$$

Given a lax extension $\bar{T}\colon \mathsf{Rel} \to \mathsf{Rel}$ of a Set-monad $\mathbb{T} = (T, e, m)$ and a constructively completely distributive lattice V with $k = \top$ or $T\emptyset = \emptyset$, a lax extension $T_{\mathsf{V}}\colon \mathsf{V\text{-}Mat} \to \mathsf{V\text{-}Mat}$ is defined by

$$T_{\mathsf{V}} r(\mathfrak{x}, \mathfrak{n}) = \bigvee \{v \in \mathsf{V} \mid \bar{T} r_v(\mathfrak{x}, \mathfrak{n}) = \top\},$$

for any V-matrix $r\colon X \nrightarrow Y$ and $\mathfrak{x} \in TX, \mathfrak{n} \in TY$. This extension was called the strata extension in [36]. We refer to [14, 35] for further details.

For a Set-monad $\mathbb{T} = (T, e, m)$ equipped with a lax extension $\bar{T}$ of T, the category $(\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat}$ of $(\mathbb{T}, \mathsf{V})$-*categories*, or *lax algebras*, has as objects pairs (X, a) with X a set and its structure $a\colon TX \nrightarrow X$ is a reflexive and transitive V-matrix:

$$1_X \leqslant a \cdot e_X \quad \text{and} \quad a \cdot \bar{T}a \leqslant a \cdot m_X$$

Morphisms $f\colon (X, a) \to (Y, b)$ are Set-maps $f\colon X \to Y$ satisfying $f \cdot a \leqslant b \cdot Tf$, and composing as in Set. $(\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat}$ is a topological construct [13] and the intial structures are formed as follows. Given $(f_i\colon X \to (Y_i, b_i))_{i \in I}$, then the initial structure $a\colon TX \nrightarrow X$ is given by $a = \bigwedge_{i \in I} f_i^\circ \cdot b_i \cdot Tf_i$.

In order to apply the theorems of section 2 to $(\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat}$, the category $(\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat}$ needs an initially dense object. The question which categories $(\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat}$ have an initially dense object is still open. A candidate for an initially dense object can be found in [12].

In that paper, Clementino and Hofmann consider the structure $hom_\xi = hom \cdot \xi\colon TV \nrightarrow V$ with $\xi\colon TV \to Y$ defined by $\xi(\mathfrak{r}) = \bigvee\{v \in V \mid \mathfrak{r} \in T(\{u \in V \mid v \leqslant u\})\}$. For a lax extension constructed as described above (the strata extension applied to the Barr extension), they prove that (V, hom_ξ) is a $(\mathbb{T},\mathsf{V})$-category.

The following theorem is an immediate consequence of theorem 2.5.

Theorem 4.1. *If* (V, hom_ξ) *is an initially dense object of* $(\mathbb{T},\mathsf{V})$-Cat*, then* $(\mathbb{T},\mathsf{V})$-Cat *is isomorphic to a coreflective subconstruct of* VSet*.*

This theorem can be applied to the following two examples of monads.

Examples.

1. Identity monad $\mathbb{I}$. The identity monad $\mathbb{I}$ is the triple $(Id, 1, 1)$ and the strata extension of the identity functor $Id\colon \mathsf{Set} \to \mathsf{Set}$ is given by the identity $Id\colon \mathsf{V}$-$\mathsf{Mat} \to \mathsf{V}$-$\mathsf{Mat}$. The category $(\mathbb{I},\mathsf{V})$-Cat is the category V-Cat of V-enriched categories and V-functors. V-enriched categories (or simply V-categories) were introduced and studied in [24, 31] in the more general context of symmetric monoidal-closed categories. For a nice presentation of this theory, we refer to [32]. A V-category is a set X together with a V-matrix $a\colon X \times X \to V$ satisfying

$$k \leqslant a(x,x) \quad (R) \quad \text{and} \quad a(x,y) \otimes a(y,z) \leqslant a(x,z) \quad (T)$$

for all $x, y, z \in X$. Given V-categories (X,a) and (Y,b), a V-functor $f\colon (X,a) \to (Y,b)$ is a map $f\colon X \to Y$ such that, for each $x, x' \in X$, $a(x,x') \leqslant b(f(x), f(x'))$.

Proposition 4.2. $(\mathsf{V}, \mathrm{hom})$ *is an initially dense object of* V-Cat*.*

Proof. For every V-category $(X, a\colon X \times X \to V)$, we will prove that the source

$$(a(x,-)\colon (X,a) \to (\mathsf{V}, \mathrm{hom}))_{x \in X},$$

where $a(x,-)(y) = a(x,y)$ for all $y \in X$, is initial in V-Cat. Since the V-matrix a satisfies condition (T), it follows that

$$a(y,z) \leqslant hom(a(x,y), a(x,z))$$

for all $x, y, z \in X$. Since a also satisfies condition (R), we have

$$hom(a(y,y), a(y,z) \geqslant hom(k, a(y,z)) = a(y,z)$$

for all $y, z \in X$. We can conclude that

$$a(y,z) = \bigwedge_{x \in X} hom(a(x,y), a(x,z))$$

for all $y, z \in X$. □

Combining this with theorem 2.5, we can conclude:

Corollary 4.3. V-Cat *is isomorphic to a coreflective subconstruct of* VSet.

2. Powerset monad $\mathbb{P} = (P, e, m)$. The powerset functor P sends a set X to the set PX of subsets of X and sends a map $f\colon X \to Y$ to $Pf\colon PX \to PY$ defined by $Pf(A) = \{f(x) \mid x \in A\}$ where $A \subseteq X$. For $x \in X$ and $\mathcal{A} \in PPX$, the maps e_X and m_X are given by

$$e_X(x) = \{x\} \qquad \text{and} \qquad m_X(\mathcal{A}) = \cup\mathcal{A}.$$

We consider the extension of the powerset functor defined by

$$P_{\mathsf{V}} r(A, B) = \bigwedge_{y \in B} \bigvee_{x \in A} r(x, y),$$

where $A \in PX$ and $B \in PY$. Seal [37] proved that the category $(\mathbb{P}, \mathsf{V})$-Cat is isomorphic to the category V-Cls of V-closure spaces and continuous maps.

Definition 4.4. A V-*closure space* is a pair (X, c), where X is a set and $c\colon PX \times X \to \mathsf{V}$ satisfies:

(C1) $\forall x \in X, \forall A \subseteq X : a \in A \Rightarrow c(A, a) \geq k$;
(C2) $\forall x \in X, \forall A \subseteq B \subseteq X : c(A, x) \leq c(B, x)$;
(C3) $\forall x \in X, \forall A, B \subseteq X : \bigwedge_{y \in B} c(A, y) \otimes c(B, x) \leq c(A, x)$.

A map $f\colon (X, c) \to (X', c')$ is *continuous* if

$$\forall A \subseteq X, \forall x \in X : c(A, x) \leq c'(f(A), f(x)).$$

The 2-closure spaces are exactly the closure spaces. In case $\mathsf{V} = ([0, \infty]^{\mathrm{op}}, +, 0)$, V-Cls coincides with the non-additive approach spaces. On V, we consider the V-closure space (V, c_V) where

$$c_V\colon P\mathsf{V} \times \mathsf{V} \to \mathsf{V}\colon (A, x) \mapsto \hom(\bigwedge A, x)$$

In an analogous way to the proof for approach spaces in [33], we can prove the following theorem.

Theorem 4.5. (V, c_V) *is an initially dense object of* V-Cls.

Corollary 4.6. V-Cls *is isomorphic to a coreflective subconstruct of* VSet.

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