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REMARKS ON THE GENERIC EXISTENCE OF ULTRAFILTERS ON ω

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Abstract. The purpose of this note is to contrast the generic existence of certain kinds of ultrafilters on ω with the existence of $2^{\mathfrak{c}}$-many of them. First, we prove that it is consistent with ZFC that there are $2^{\mathfrak{c}}$-many Q-points but Q-points do not exist generically. This answers in the negative a question by R. Michael Canjar. Then we define the strong generic existence of a class of ultrafilters and show that the strong generic existence of selective ultrafilters is equivalent to the their generic existence. However, we prove a result that implies that for several classes of ultrafilters, including P-points and nowhere dense ultrafilters, the strong generic existence of P-points is not equivalent to their generic existence.

1. Preliminaries

We use standard set theoretic notation. We say that $\mathcal{A} \subseteq [\omega]^\omega$ has the *strong finite intersection property* (SFIP) provided that the intersection of any finite subfamily is infinite. The filter generated by $\mathcal{A}$ is denoted $\langle \mathcal{A} \rangle$. The letter $\mathcal{F}$ will always denote a filter on ω containing the cofinite filter. A *basis* for $\mathcal{F}$ is a family $\mathcal{B} \subseteq \mathcal{F}$ such that for every $F \in \mathcal{F}$ there exists a $B \in \mathcal{B}$ such that $B \subseteq F$. Given any filter $\mathcal{F}$ let $\chi(\mathcal{F})$ be the minimum cardinality of a basis of $\mathcal{F}$. This $\chi(\mathcal{F})$ is called the *character of* $\mathcal{F}$. We say that $\mathcal{F}$

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is κ-*generated* provided that $\chi(\mathcal{F}) = \kappa$ and $\mathcal{F}$ is $< \kappa$-*generated* provided that $\chi(\mathcal{F}) < \kappa$. The letters $\mathcal{U}$ and $\mathcal{V}$ will always denote nonprincipal ultrafilters on ω. The cardinal $\mathfrak{u}$ is defined as

$$\mathfrak{u} = \min\{|\mathcal{B}|\colon \mathcal{B} \text{ is basis of an ultrafilter on } \omega\}.$$

Two filters $\mathcal{F}_0$ and $\mathcal{F}_1$ are *orthogonal* provided that there exists an $X \in [\omega]^\omega$ such that $X \in \mathcal{F}_0$ and $\omega \setminus X \in \mathcal{F}_1$. This is denoted $\mathcal{F}_0 \perp \mathcal{F}_1$. An ultrafilter $\mathcal{U}$ is a *Q-point* provided that for every finite-to-one $f\colon \omega \to \omega$ there exists a $U \in \mathcal{U}$ such that $f{\restriction}U$ is one-to-one. On the other hand, $\mathcal{U}$ is *rapid* provided that for every $f\colon \omega \to \omega$ there is a $U \in \mathcal{U}$ such that $|U \cap f(n)| \le n$ for every $n < \omega$. Every Q-point is rapid but not every rapid ultrafilter is a Q-point. An ultrafilter $\mathcal{U}$ is a *P-point* provided that for every partition $\mathcal{P}$ of ω either $\mathcal{P} \cap \mathcal{U} \neq \emptyset$ or there exists a $U \in \mathcal{U}$ such that $|U \cap P| < \omega$ for every $P \in \mathcal{P}$. We will call such a $U \in \mathcal{U}$ a *partial pseudo-selector of* $\mathcal{P}$. If in this definition we require instead that there exists a $U \in \mathcal{U}$ such that $|U \cap P| \le 1$ for every $P \in \mathcal{P}$, we obtain the definition of a *selective* or *Ramsey ultrafilter*, and we will call such a $U \in \mathcal{U}$ a *partial selector of* $\mathcal{P}$. It is well known that an ultrafilter is selective if and only if it is both a P-point and a Q-point. An ultrafilter which is both a P-point and a rapid ultrafilter is called *semiselective*. If $f, g \in \omega^\omega$, we declare $f \le^* g$ when $|\{n < \omega\colon f(n) > g(n)\}| < \omega$. A family $\mathcal{G} \subseteq \omega^\omega$ is *dominating* provided that for every $f \in \omega^\omega$ there is a $g \in \mathcal{G}$ such that $f \le^* g$, and it is *unbounded* provided that there is no single $f \in \omega^\omega$ such that $g \le^* f$ for every $g \in \mathcal{G}$. The cardinals $\mathfrak{d}$, $\mathfrak{b}$, $\mathsf{cov}(\mathcal{M})$, and $\mathsf{non}(\mathcal{N})$ denote the minimum cardinality of a dominating family, an unbounded family, a family of meager sets whose union covers $\mathbb{R}$, and a non-measure zero subset of $\mathbb{R}$, respectively. These cardinals are related as shown in Figure 1, where $\kappa \to \lambda$ means $\kappa \le \lambda$.

$$\begin{tikzcd}
\mathfrak{b} \arrow[r] & \mathfrak{d} \arrow[r] & \mathfrak{c} \\
\omega_1 \arrow[r] \arrow[u] & \mathsf{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathsf{non}(\mathcal{N}) \arrow[u]
\end{tikzcd}$$

FIGURE 1. A fragment of Cichon's diagram.

2. Generic and strong generic existence of ultrafilters

Definition 2.1. Let $\mathscr{C}$ be a class of ultrafilters on ω and let κ be an uncountable cardinal. We abbreviate by $GE(\mathscr{C}, \kappa)$ the statement "every $< \kappa$-generated filter can be extended to an ultrafilter in $\mathscr{C}$." Here, GE stands for "generic existence."

In what follows, we will denote by P, Q, R, S, and SS the classes of P-points, Q-points, rapid, selective, and semi-selective ultrafilters, respectively.

The next three propositions characterize the generic existence of these ultrafilters in terms of $\mathsf{cov}(\mathcal{M})$ and $\mathfrak{d}$.

Proposition 2.2 (Ketonen [13])**.**

$$GE(P, \mathfrak{c}) \Leftrightarrow \mathfrak{d} = \mathfrak{c}.$$

Proposition 2.3 (Canjar [12])**.**

$$GE(Q, \mathfrak{d}) \Leftrightarrow \mathsf{cov}(\mathcal{M}) = \mathfrak{d} \Leftrightarrow GE(R, \mathfrak{d}).$$

Proposition 2.4 (Canjar [12]; Bartoszynski and Judah [10])**.**

$$GE(S, \mathfrak{c}) \Leftrightarrow \mathsf{cov}(\mathcal{M}) = \mathfrak{c} \Leftrightarrow GE(SS, \mathfrak{c}).$$

In [12, p. 240], R. Michael Canjar asked, Assuming that $\mathfrak{c}$ is regular, does the existence of $2^{\mathfrak{c}}$-many selective ultrafilters imply $GE(S, \mathfrak{c})$? We answered this negatively in [17] by constructing a model of ZFC where $\mathfrak{c} = \omega_2$, and there are $2^{\mathfrak{c}}$-many selective ultrafilters but $\mathsf{cov}(\mathcal{M}) < \mathfrak{c}$. The same question for $\mathfrak{c}$ singular is an unpublished result by James E. Baumgartner who noticed that in the Bell-Kunen model described in [6], $\mathfrak{c} = \omega_{\omega_1}$, $\mathsf{cov}(\mathcal{M}) = \omega_1$, and there are $2^{\mathfrak{c}}$-many selective ultrafilters on ω.

Definition 2.5. Let M be a model of ZFC. A forcing notion $\mathbb{P}$ is ω^ω-bounding provided that for every $\mathbb{P}$-generic filter G over M and for every $f \in \omega^\omega \cap M[G]$, there exists a $g \in \omega^\omega \cap M$ such that $\forall n < \omega\ f(n) < g(n)$.

Proposition 2.6 (Millán [17])**.** *There is a model N of* ZFC *such that*

$$N \models \text{“}\mathfrak{c} = \omega_2 + |S| = 2^{\mathfrak{c}} + \neg GE(S, \mathfrak{c})\text{.”}$$

Proof: Let M be such that $M \models$ "$\mathsf{ZFC} + \mathsf{CH} + 2^{\omega_1} = 2^{\omega_2} = \omega_3$." If $\mathbb{P} \in M$ is the partial order to add ω_2-many Sacks reals iteratively with countable supports and G is $\mathbb{P}$-generic over M, then

$$M[G] \models \text{“}\mathsf{ZFC} + \mathfrak{c} = \omega_2 + 2^{\omega_1} = 2^{\omega_2} = \omega_3.\text{”}$$

Now, CH in M implies that $\mathfrak{d}^M = \omega_1^M$. Since $\mathbb{P}$ is ω^ω-bounding and proper, we have that $\omega_1^{M[G]} = \omega_1^M = \mathfrak{d}^M = \mathfrak{d}^{M[G]}$. In particular, $M[G] \models$ "$\mathsf{cov}(\mathcal{M}) < \mathfrak{c}$." By Proposition 2.4, $M[G] \models$ "$\neg GE(S, \mathfrak{c})$." To see that $M[G] \models$ "$|S| = 2^{\mathfrak{c}}$," use CH to construct in M, 2^{ω_1}-many selective ultrafilters. Then invoke a theorem by Baumgartner and Richard Laver [4, Theorem 4.5] to extend these to 2^{ω_1}-many selective ultrafilters in $M[G]$. Therefore,

$$M[G] \models \text{“}\mathfrak{c} \text{ is regular } + |S| = 2^{\mathfrak{c}} + \neg GE(S, \mathfrak{c}).\text{”}$$

Hence, $N = M[G]$ works. $\square$

Canjar also asked [12], Does the existence of $2^{\mathfrak{d}}$-many Q-points imply $GE(Q, \mathfrak{d})$?

We will answer this question negatively by constructing a model of "ZFC+$|Q| = 2^{\mathfrak{c}} + \neg GE(Q, \mathfrak{d})$."

Definition 2.7. Let $f\colon \omega \to \omega$ be any function. We say that $U \subseteq \omega$ is *f-rare* if $f(m) < n$ for every $m, n \in U$ with $m < n$.

Definition 2.8. A family $\mathcal{U} \subseteq [\omega]^\omega$ is *rare* if for every $f\colon \omega \to \omega$ there exists a $U_f \in \mathcal{U}$ which is f-rare.

Proposition 2.9 (Mathias [16]; Taylor, unpublished. See also Blass [8])**.** *An ultrafilter $\mathcal{U}$ on ω is a Q-point if and only if $\mathcal{U}$ is a rare family.*

Let $\mathcal{U}$ and $\mathcal{V}$ be two families of subsets of ω and let $\Psi(\mathcal{U}, \mathcal{V})$ be an abbreviation of the statement, "$\mathcal{U} \neq \mathcal{V}$ and both $\mathcal{U}$ and $\mathcal{V}$ are Q-points."

Lemma 2.10. *Let M be a transitive model of* ZFC*, let $\mathcal{U}, \mathcal{V} \in M$ such that $M \models$ "$\Psi(\mathcal{U}, \mathcal{V})$," and let $\mathbb{P} \in M$ be an ω^ω-bounding forcing notion. Then, for any $\mathbb{P}$-generic filter G over M,*

$$M[G] \models \text{“}\mathsf{ZFC} + \exists\, \bar{\mathcal{U}}\ \exists\, \bar{\mathcal{V}}\ (\mathcal{U} \subseteq \bar{\mathcal{U}}\ \wedge\ \mathcal{V} \subseteq \bar{\mathcal{V}}\ \wedge\ \Psi(\bar{\mathcal{U}}, \bar{\mathcal{V}})).\text{”}$$

Proof: Let $\mathcal{U}$ be a Q-point in M, let G be a $\mathbb{P}$-generic filter over M, and let $f \in \omega^\omega \cap M[G]$. Then there exists a $g \in \omega^\omega \cap M$ such that $f(n) < g(n)$ for every $n < \omega$. Since $M \models$ "$\mathcal{U}$ is a Q-point," we can find a $U_g \in \mathcal{U}$ which is g-rare. Since f is dominated by g, we conclude that U_g is also f-rare. Therefore, $M[G] \models$ "$\mathcal{U}$ is rare." If $M \models$ "$\mathcal{U} \neq \mathcal{V}$," then there exists a $U \in (\mathcal{U} \setminus \mathcal{V}) \cap M \subseteq (\mathcal{U} \setminus \mathcal{V}) \cap M[G]$. This implies that $U \in \mathcal{U}$ and $\omega \setminus U \in \mathcal{V}$. If $\bar{\mathcal{U}}$ and $\bar{\mathcal{V}}$ are ultrafilters in $M[G]$, extending $\mathcal{U}$ and $\mathcal{V}$, respectively, then $\bar{\mathcal{U}}$ and $\bar{\mathcal{V}}$ are distinct Q-points. Hence, $M[G] \models$ "$\Psi(\bar{\mathcal{U}}, \bar{\mathcal{V}})$." □

Corollary 2.11. *If an ω^ω-bounding forcing notion preserves P-points, then it preserves selective ultrafilters.*

Theorem 2.12. *There are models N_i of* ZFC *for $i = 0, 1$ such that*

(a) $N_0 \models$ "$\mathsf{ZFC} + \mathfrak{c} = \omega_2 + |Q| = 2^{\mathfrak{c}} + \neg GE(Q, \mathfrak{d})$," *and*
(b) $N_1 \models$ "$\mathsf{ZFC} + \mathfrak{c} = \omega_{\omega_1} + |Q| = 2^{\mathfrak{c}} + \neg GE(Q, \mathfrak{d})$."

Proof: Suppose that $\kappa \in \{\omega_2, \omega_{\omega_1}\}$. If $M \models$ "$\mathsf{ZFC} + \mathsf{GCH}$," let $\mathbb{P} = \mathbf{C}(\kappa) \in M$ be the notion of forcing for adding κ-many Cohen reals and let G be a $\mathbb{P}$-generic filter over M; then we have that $M[G] \models$ "$\kappa = \mathfrak{c} = \mathsf{cov}(\mathcal{M}) = \mathfrak{d}$." By Proposition 3.2, it follows that $M[G] \models$ "$\mathfrak{c} = \kappa + |Q| = 2^{\mathfrak{c}}$." Let $\mathbb{Q} = \mathbf{B}(\kappa) \in M[G]$ be the measure algebra for adding κ-many random reals. Let H be a $\mathbb{Q}$-generic filter over $M[G]$. Then $\mathbb{Q}$ is ω^ω-bounding and $M[G][H] \models$ "$\mathfrak{c} = \kappa = \mathfrak{d}$." By Lemma 2.10, we can extend each of the Q-points existing in $M[G]$ to at least one Q-point in $M[G][H]$ obtaining $2^{\mathfrak{c}}$-many Q-points altogether. Hence, $M[G][H] \models$ "$|Q| = 2^{\mathfrak{c}}$." Let $S \in M[G][H]$ be the set formed by the first ω_1-many random reals added. Since S is a Sierpinski set in $M[G][H]$, it is non-measurable, so $M[G][H] \models$ "$\mathsf{non}(\mathcal{N}) = \omega_1$." On the other hand, $\mathsf{cov}(\mathcal{M}) \leq \mathsf{non}(\mathcal{N})$; hence, $M[G][H] \models$ "$\mathsf{cov}(\mathcal{M}) < \mathfrak{d}$." By Proposition 2.3, $M[G][H] \models$ "$|Q| = 2^{\mathfrak{c}} \wedge \neg GE(Q, \mathfrak{d})$." Therefore, models $N_0 = M[G][H]$, when $\kappa = \omega_2$, and $N_1 = M[G][H]$, when $\kappa = \omega_{\omega_1}$, satisfy the conclusion of the theorem. □

3. Generic versus strong generic existence of ultrafilters on ω

In this section we show that for some classes of ultrafilters $\mathscr{C}$, $GE(\mathscr{C}, \mathfrak{c})$ fails to be a good indicator of the abundance of ultrafilters from $\mathscr{C}$. As an alternative, we propose $SGE(\mathscr{C}, \mathfrak{c})$ instead.

Definition 3.1. $SGE_\lambda(\mathscr{C}, \kappa)$ abbreviates the statement, "every $<\kappa$-generated filter can be extended to 2^λ-many ultrafilters in $\mathscr{C}$," where SGE stands for *"strong generic existence."* When $\lambda = \mathfrak{c}$, we drop the subindex and write $SGE(\mathscr{C},\kappa)$. We will use $SGE(\mathscr{C},\mathfrak{c})$ to abbreviate *"the strong generic existence of ultrafilters in $\mathscr{C}$."*

Proposition 3.2 (Millán [17])**.**

$$SGE(Q,\mathfrak{d}) \Leftrightarrow \mathsf{cov}(\mathcal{M}) = \mathfrak{d}.$$

Actually, we proved in [17] that the identity $\mathsf{cov}(\mathcal{M}) = \mathfrak{d}$ implies that every $< \mathfrak{d}$-generated filter can be extended to $2^{\mathfrak{c}}$-many $\mathfrak{c}$-generated Q-points. The other direction follows from Proposition 2.3 and Lemma 3.5(b) below.

As an immediate consequence of Proposition 3.2, we have the following dichotomy result.

Corollary 3.3. *Suppose that $\mathfrak{c} \leq \mathsf{cov}(\mathcal{M})^+$, then either there is a $\mathfrak{c}$-generated Q-point or there are no Q-points at all.*

Proof: If $\mathfrak{c} \leq \mathsf{cov}(\mathcal{M})^+$, then $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{d} \leq \mathfrak{c} \leq \mathsf{cov}(\mathcal{M})^+$. Suppose that $\mathsf{cov}(\mathcal{M}) = \mathfrak{d}$. Then we are done by Proposition 3.2 and the remark below it. If $\mathsf{cov}(\mathcal{M}) < \mathfrak{d}$, then $\mathfrak{d} = \mathfrak{c}$. Since, by Proposition 2.9, every Q-point has character $\geq \mathfrak{d}$ and either there is a Q-point (in which case it has character $\mathfrak{c}$) or there are no Q-points at all, this completes the argument. □

Notice that propositions 2.3 and 3.2 can be combined to obtain the following.

Proposition 3.4. $SGE(Q,\mathfrak{d}) \Leftrightarrow GE(Q,\mathfrak{d})$.

Lemma 3.5. *Let $\mathscr{C}$ be a class of ultrafilters and let κ, λ, and μ be cardinals. Then*

(a) $SGE_0(\mathscr{C},\kappa) \Leftrightarrow GE(\mathscr{C},\kappa)$,
(b) $\lambda \leq \mu \Rightarrow SGE_\mu(\mathscr{C},\kappa) \Rightarrow SGE_\lambda(\mathscr{C},\kappa)$, *and*
(c) $SGE_1(\mathscr{C},\kappa) \Leftrightarrow (GE(\mathscr{C},\kappa) \wedge \kappa \leq \mathfrak{u})$.

Proof: Parts (a) and (b) are obvious. For part (c), one implication follows from parts (a) and (b) and the fact that $\mathfrak{u} < \kappa \Rightarrow \neg SGE_1(\mathscr{C},\kappa)$. For the other implication, let $\mathcal{F}$ be a filter with $\chi(\mathcal{F}) < \kappa$. Since $\kappa \leq \mathfrak{u}$, $\mathcal{F}$ cannot be an ultrafilter and there exists an $X \in [\omega]^\omega$ such that $\mathcal{F} \cup \{X\}$ and $\mathcal{F} \cup \{\omega \setminus X\}$ both have the SFIP. Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be the filters generated by $\mathcal{F} \cup \{X\}$ and $\mathcal{F} \cup \{\omega \setminus X\}$, respectively. Then $\chi(\mathcal{F}_0) = \chi(\mathcal{F}_1) = \chi(\mathcal{F}) < \kappa$. So we can use $GE(\mathscr{C},\kappa)$ to extend $\mathcal{F}_0$ and $\mathcal{F}_1$ to ultrafilters $\mathcal{U}_0$ and $\mathcal{U}_1$ in $\mathscr{C}$. By our choice of X, these ultrafilters are distinct. □

Definition 3.6. We call a class $\mathscr{C}$ of ultrafilters on ω to be κ-inductive provided that there exist formulas $\langle \phi_\xi(Y) \colon \xi < \kappa \rangle$ such that for every $\mathcal{U}$

$$\mathcal{U} \in \mathscr{C} \Leftrightarrow \forall \xi < \kappa \, \exists U \in \mathcal{U} \; \phi_\xi(U).$$

Lemma 3.7. *The classes P, S, and Q are $\mathfrak{c}$-inductive.*

Proof: To see that P is $\mathfrak{c}$-inductive, let $\langle \mathcal{P}_\xi \colon \xi < \mathfrak{c} \rangle$ be a listing of the partitions of ω into infinitely many pieces and consider for every $\xi < \mathfrak{c}$ the formula

$$\phi_\xi(Y) \Leftrightarrow [(\exists \mathcal{G} \in [\mathcal{P}_\xi]^{<\omega})(Y \subseteq \bigcup \mathcal{G}) \vee (\forall P \in \mathcal{P}_\xi)(|Y \cap P| < \omega)].$$

For S, replace $|Y \cap P| < \omega$ by $|Y \cap P| \leq 1$ in the formulas above. For Q, let $\langle \mathcal{P}_\xi \colon \xi < \mathfrak{c} \rangle$ be a listing of the partitions of ω into finite pieces and consider $\phi_\xi(Y) \Leftrightarrow (\forall P \in \mathcal{P}_\xi)(|Y \cap P| \leq 1)$ for every $\xi < \mathfrak{c}$. $\square$

Lemma 3.8. *The class Q is $\mathfrak{d}$-inductive.*

Proof: Let $\langle f_\xi \in \omega^\omega \colon \xi < \mathfrak{d} \rangle$ be a dominating family and consider for each $\xi < \mathfrak{d}$ the formula

$$\phi_\xi(Y) \Leftrightarrow (\forall m < \omega)(\forall n < \omega) \; ((m, n \in Y \wedge m < n) \Rightarrow f_\xi(m) < n).$$

Then the $\mathfrak{d}$-inductivity of Q follows from Proposition 2.9. $\square$

Theorem 3.9. *If $\kappa \geq 1$ and $\mathscr{C}$ is a κ-inductive class of ultrafilters, then*

$$SGE_\kappa(\mathscr{C}, \kappa) \Leftrightarrow (GE(\mathscr{C}, \kappa) \wedge \kappa \leq \mathfrak{u}).$$

Proof: It is obvious that $SGE_\kappa(\mathscr{C}, \kappa)$ implies both $GE(\mathscr{C}, \kappa)$ and $\kappa \leq \mathfrak{u}$. So suppose that $G(\mathscr{C}, \kappa)$ and $\kappa \leq \mathfrak{u}$ hold and that $\mathcal{F}$ is a $< \kappa$-generated filter. We will construct inductively a tree $\langle \mathcal{F}_s \colon s \in 2^{<\kappa} \rangle$ of filters satisfying the following requirements for every $\xi < \kappa$ and $s \in 2^\xi$.

(1) $\mathcal{F}_\emptyset = \mathcal{F}$;
(2) $\mathcal{F}_{s \restriction \gamma} \subseteq \mathcal{F}_s$ for every $\gamma < \xi$;
(3) $\chi(\mathcal{F}_s) \leq \max\{\chi(\mathcal{F}), |\xi|\}$;
(4) $\mathcal{F}_{s^\frown\langle 0 \rangle} \perp \mathcal{F}_{s^\frown\langle 1 \rangle}$;
(5) $\mathcal{F}_s = \bigcup\{\mathcal{F}_{s \restriction \gamma} \colon \gamma < \xi\}$ if ξ is limit; and
(6) there exists $X^i_\xi \in \mathcal{F}_{s^\frown\langle i \rangle}$ such that $\phi_\xi(X^i_\xi)$ hold for $i = 0, 1$.

If this construction can be completed, then for every $g \in 2^\kappa$, let $\mathcal{U}_g$ be an ultrafilter extending the filter $\mathcal{F}_g = \bigcup\{\mathcal{F}_{g\restriction\xi} \colon \xi < \kappa\}$. By conditions (1), (4), and (6), the ultrafilters $\mathcal{U}_g$ extend $\mathcal{F}$, are pairwise distinct, and are all in $\mathscr{C}$. To see that this construction can be completed, we need only to check the inductive hypothesis for the successor ordinal case. Suppose that $s \in 2^\xi$ and that $\mathcal{F}_s$ has been defined. We want to define $\mathcal{F}_{s^\frown\langle 0\rangle}$ and $\mathcal{F}_{s^\frown\langle 1\rangle}$. By the induction hypothesis, $\chi(\mathcal{F}_s) < \kappa \leq \mathfrak{u}$, so we can find a $Y \in [\omega]^\omega$ such that $\mathcal{F}_s \cup \{Y\}$ and $\mathcal{F}_s \cup \{\omega \setminus Y\}$ have both SFIP. Since $\mathcal{F}_0^* = \langle \mathcal{F}_s \cup \{Y\}\rangle$ and $\mathcal{F}_1^* = \langle \mathcal{F}_s \cup \{\omega \setminus Y\}\rangle$, we have that $\chi(\mathcal{F}_0^*) = \chi(\mathcal{F}_1^*) = \chi(\mathcal{F}_s) < \kappa$. Also, since $GE(\mathscr{C}, \kappa)$ holds, there exist $\mathcal{U}_i \in \mathscr{C}$ extending $\mathcal{F}_i^*$ for $i = 0, 1$. Thus, it is possible to pick $X_\xi^i \in \mathcal{U}_i$ such that $\phi_\xi(X_\xi^i)$ for $i = 0, 1$. Put $\mathcal{F}_{s^\frown\langle 0\rangle} = \langle \mathcal{F}_s \cup \{Y, X_\xi^0\}\rangle$ and $\mathcal{F}_{s^\frown\langle 1\rangle} = \langle \mathcal{F}_s \cup \{\omega \setminus Y, X_\xi^1\}\rangle$. Then these filters satisfy the requirements. □

Corollary 3.10. *If $\mathscr{C}$ is $\mathfrak{c}$-inductive like P, Q, or S, then*

$$SGE(\mathscr{C}, \mathfrak{c}) \Leftrightarrow SGE_1(\mathscr{C}, \mathfrak{c}).$$

Proof: This follows from Lemma 3.5(c), Lemma 3.7, and Theorem 3.9. □

Corollary 3.11. $SGE(S, \mathfrak{c}) \Leftrightarrow GE(S, \mathfrak{c})$.

Proof: This follows from Theorem 3.9 and Proposition 2.4. □

Corollary 3.12. $SGE(P, \mathfrak{c}) \Leftrightarrow min\{\mathfrak{u}, \mathfrak{d}\} = \mathfrak{c}$.

Proof: This follows from Theorem 3.9 and Proposition 2.2. □

By a theorem of Jason Aubrey [1], $\min\{\mathfrak{u}, \mathfrak{d}\} = \min\{\mathfrak{r}, \mathfrak{d}\}$. Here, $\mathfrak{r}$ is the *refinement* or *reaping number*. (See [7].) Therefore, we can rephrase Corollary 3.12 as

Corollary 3.13. $SGE(P, \mathfrak{c}) \Leftrightarrow min\{\mathfrak{r}, \mathfrak{d}\} = \mathfrak{c}$.

Theorem 3.14. *There is a model N of* ZFC *such that*

$$N \models \text{“}GE(P, \mathfrak{c}) \wedge \neg SGE(P, \mathfrak{c})\text{.”}$$

Proof: Let M be such that $M \models$ “$\mathsf{ZFC} + \mathsf{GCH}$,” and consider in M a countable support forcing iteration $\langle\langle \mathbb{P}_\alpha, \dot{Q}_\alpha\rangle \colon \alpha < \omega_2\rangle$ such that

$$\forall \alpha < \omega_2 \quad \Vdash_\alpha \text{“}\dot{Q}_\alpha \simeq \text{rational perfect set forcing”}$$

(see [19]; [10, p. 360]; and [9]), and let G be a $\mathbb{P}_{\omega_2}$-generic filter over M. Then P-points in M generate P-points in $M[G]$ and

$$M[G] \models \text{“Every } P\text{-point is } \omega_1\text{-generated} + \mathfrak{d} = \mathfrak{c} = \omega_2 = 2^{\omega_1}.\text{”}$$

(See [9].) Therefore,

$$M[G] \models \text{“}GE(P, \mathfrak{c}) \wedge |P| = \mathfrak{c}.\text{”}$$

Hence, model $N = M[G]$ works. □

Corollary 3.15. $\mathrm{Con}(\mathsf{ZFC}) \Longrightarrow \mathrm{Con}(\mathsf{ZFC}{+}GE(P,\mathfrak{c}){+}\neg SGE(P,\mathfrak{c}))$.

4. Other classes of ultrafilters

Theorem 3.9 can be applied to get a similar characterization for $SGE(\mathscr{C}, \mathfrak{c})$ as in Corollary 3.13 for other classes of ultrafilters as well. These depend, of course, on the characterization of $GE(\mathscr{C}, \mathfrak{c})$ in terms of cardinal invariants.

Definition 4.1 (Baumgartner [3])**.** Let X be a non-empty set and let $\mathcal{I} \subseteq \mathcal{P}(X)$ be a set containing the singletons and closed under subsets. An ultrafilter $\mathcal{U}$ on ω is an $\mathcal{I}$-ultrafilter provided that for every $f\colon \omega \to X$ there exists a $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}$.

If $X = 2^\omega$ and $\mathcal{I}$ is the ideal of countable closed, nowhere-dense, measure-zero subsets of 2^ω, then $\mathcal{I}$-ultrafilters are called *countable closed, nowhere-dense, and measure-zero ultrafilters*, respectively. If $X = \omega^\omega$ and $\mathcal{I}$ is the ideal of σ-compact subsets of ω^ω, then the $\mathcal{I}$-ultrafilters are called *σ-compact ultrafilters*. If $\alpha < \omega_1$, put $\mathcal{I}_\alpha = \{A \subseteq \omega_1 \colon \mathrm{o.t}(A) \le \alpha\}$ and $\mathcal{J}_\alpha = \{A \subseteq \omega_1 \colon \mathrm{o.t}(A) < \alpha\}$. If $\mathcal{I} = \mathcal{I}_\alpha$ or $\mathcal{I} = \mathcal{J}_\alpha$ for some $\alpha < \omega_1$, then $\mathcal{I}$-ultrafilters are called *ordinal ultrafilters*. Let O, CC, ND, MZ, and K_σ denote the classes of ordinal, countable closed, nowhere-dense, measure-zero, and σ-compact ultrafilters, respectively.

Lemma 4.2. *If $\mathscr{C} \in \{O, CC, ND, MZ, K_\sigma\}$, then $\mathscr{C}$ is $\mathfrak{c}$-inductive.*

Proof: Let $\langle f_\xi \colon \xi < \mathfrak{c}\rangle$ be a listing of X^ω and consider the family of formulas $\langle \phi_\xi(Y) \colon \xi < \mathfrak{c}\rangle$ where $\phi_\xi(Y) \Leftrightarrow f_\xi[Y] \in \mathcal{I}$. □

We refer the reader to [5] and [11] for proofs of the following propositions.

Proposition 4.3 (Brendle [11])**.** *If $\mathscr{C} \in \{O, CC\}$, then*

$$GE(\mathscr{C}, \mathfrak{c}) \Leftrightarrow \mathfrak{d} = \mathfrak{c}.$$

Proposition 4.4 (Brendle [11]). $GE(ND, \mathfrak{c}) \Leftrightarrow \mathsf{cof}(\mathcal{M}) = \mathfrak{c}$.

Proposition 4.5 (Brendle [11]). $GE(MZ, \mathfrak{c}) \Leftrightarrow \mathsf{cof}(\mathcal{E}, \mathcal{M}) = \mathfrak{c}$.

Proposition 4.6 (Barney [5]). $GE(K_\sigma, \mathfrak{c}) \Leftrightarrow \mathfrak{d} = \mathfrak{c}$.

Theorem 4.7. *If* $\mathscr{C} \in \{O, CC, K_\sigma\}$, *then*

(a) $SGE(\mathscr{C}, \mathfrak{c}) \Leftrightarrow min\{\mathfrak{u}, \mathfrak{d}\} = \mathfrak{c}$;
(b) $SGE(ND, \mathfrak{c}) \Leftrightarrow min\{\mathfrak{u}, \mathsf{cof}(\mathcal{M})\} = \mathfrak{c}$;
(c) $SGE(MZ, \mathfrak{c}) \Leftrightarrow min\{\mathfrak{u}, \mathsf{cof}(\mathcal{E}, \mathcal{M})\} = \mathfrak{c}$.

Proof: Apply Theorem 3.9 and propositions 4.1 and 4.4 for (a), Proposition 4.2 for (b), and Proposition 4.3 for (c). □

Theorem 4.8. *There is a model* N *of* ZFC *such that if* $\mathscr{C} \in \{O, CC, ND, MZ, K_\sigma\}$, *then*

$$N \models \text{“}GE(\mathscr{C}, \mathfrak{c}) \wedge \neg SGE(\mathscr{C}, \mathfrak{c})\text{.”}$$

Proof: The model from Theorem 3.14 works since it is known that in this model, $\mathfrak{d} = \mathsf{cof}(\mathcal{M}) = \mathsf{cof}(\mathcal{E}, \mathcal{M}) = \mathfrak{c}$. □

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