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A CORRECTION TO "THE CONNECTIVITY STRUCTURE OF THE HYPERSPACES $C_\epsilon(X)$"

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Abstract. We demonstrate that Proposition 3.1 of [Eric L. McDowell and B. E. Wilder, *The connectivity structure of the hyperspaces* $C_\epsilon(X)$, Topology Proc. **27** (2003), no. 1, 223–232] is false by constructing a locally connected metric continuum which admits a non-locally connected small-point hyperspace.

Let X be a continuum with metric d. For any $\epsilon > 0$ the set $C_{d,\epsilon}(X) = \{A \in C(X) : \text{diam}_d(A) \le \epsilon\}$ is called a *small-point hyperspace* of X. The notation $C_\epsilon(X)$ is used when the metric on X is understood.

Proposition 3.1 of [2] asserts that X is locally connected if and only if $C_\epsilon(X)$ is locally connected for every $\epsilon > 0$. While it is true that the local connectivity of $C_\epsilon(X)$ for every $\epsilon > 0$ implies the local connectivity of X, we show in this note that the reverse implication is false.

Below we construct a locally connected continuum X in $\mathbf{R}^3$ for which $C_\epsilon(X)$ fails to be locally connected for some $\epsilon > 0$. The metric considered on X is the usual metric inherited from $\mathbf{R}^3$. All

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points (r, θ, z) are described using the standard cylindrical coordinate system, and all concepts and notation which are used without definition can be found in [3]. The example is similar to [4, Example 2].

Example 1. For each $n = 1, 2, \cdots$, let S_n denote the circle described by $\{(1, \theta, n^{-1}) : 0 \leq \theta < 2\pi\}$ and let $S_0 = \{(1, \theta, 0) : 0 \leq \theta < 2\pi\}$. For each $n = 1, 2, \cdots$ and each $i = 1, 2, \cdots, 2^n$, let A_i^n denote the straight line segment given by $\{(1, 2\pi i/2^n, z) : 0 \leq z \leq n^{-1}\}$. Define X to be the continuum given by

$$X = \left(\bigcup_{n=0}^{\infty} S_n \right) \cup \left(\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^n} A_i^n \right).$$

It is straightforward to show that X is a Peano continuum. We will now prove that $C_\epsilon(X)$ fails to be locally connected at the point S_0 when $\epsilon = 2$.

Let $\{U_1, \cdots, U_k\}$ be an open cover of S_0 with the property that for every $n = 0, 1, \cdots$ and every $i = 1, \cdots, k$ it is true that

(1) $S_n - U_i$ is connected and has arc length greater than $3\pi/2$.

Observe that $\mathcal{U} = \langle U_1, \cdots, U_k \rangle$ is an open subset of $C(X)$ that contains S_0 as well as all S_n for n sufficiently large. Select N such that $S_N \in \mathcal{U}$. We will prove that $C_\epsilon(X)$ fails to be locally connected at S_0 by showing that every arc in $\mathcal{U}$ with endpoints S_0 and S_N must contain a point of diameter greater than 2. Let $f : [0, 1] \to \mathcal{U}$ be an embedding for which $f(0) = S_0$ and $f(1) = S_N$. Let $\pi : X \to S_N$ denote the natural projection map. For any subset $S \subset X$ we say that $(1, \theta, z) \in S$ is an *antipodal point* of S provided that $(1, \theta + \pi, z')$ belongs to S for some z'. We will denote the set of antipodal points of S by $AP(S)$. We now show that

(2) $(1, \theta, z) \in AP(S)$ if and only if $(1, \theta, N^{-1}) \in AP(\pi(S))$.

To see (2), let $S \subset X$ and let $(1, \theta, z) \in AP(S)$. By definition it follows that $(1, \theta + \pi, z')$ belongs to S for some z'; thus, $\pi(1, \theta + \pi, z') = (1, \theta + \pi, N^{-1})$ belongs to $\pi(S)$. Since $(1, \theta, N^{-1}) = \pi(1, \theta, z) \in \pi(S)$, it follows that $(1, \theta, N^{-1}) \in AP(\pi(S))$. The argument for the converse is similar.

If $M \in \mathcal{U}$ and $M \subset S_N$, then there exists an arc A (possibly empty) such that M is the closure of $S_N - A$; thus, the only elements of $M - AP(M)$ are the points that are diametrically opposed to the interior points of A. Therefore, $AP(M)$ is either S_N (if $A = \emptyset$) or the union of two disjoint arcs. Since $f(t)$ is a continuum for each $0 \leq t \leq 1$, it follows from continuity that

(3) $AP(\pi(f(t)))$ is either S_N or the union of two disjoint arcs.

Continuity also shows that the intersection of $\pi^{-1}(AP(\pi(f(t))))$ and $f(t)$ is closed; moreover, it follows from (2) that this intersection is equal to $AP(f(t))$. Therefore, we have that

(4) $AP(f(t))$ is closed for every $0 \leq t \leq 1$.

Suppose that $(1, \theta, z) \in AP(f(t))$; then $(1, \theta + \pi, z') \in f(t)$ for some z'. If $z' \neq z$, then $(1, \theta, z)$ and $(1, \theta + \pi, z')$ are more than two units apart. Moreover, if $(1, \theta, z) \in AP(f(t)) - \bigcup_{n=0}^{\infty} S_n$, then it follows from the connectivity of $f(t)$ that there must exist some $z'' \neq z$ with $(1, \theta + \pi, z'') \in f(t)$. It follows that

(5) $$\text{if } AP(f(t)) - \bigcup_{n=0}^{\infty} S_n \neq \emptyset \text{ then } \operatorname{diam}(f(t)) > 2.$$

We now show that there exists some $t_0 \in [0, 1]$ for which the diameter of $f(t_0)$ is greater than 2. Begin by defining

$$t' = \min\{t : [0, 1] : AP(f(t)) \cap S_N \neq \emptyset\}.$$

Suppose that $t' = 1$. Choose $\gamma > 0$ small enough such that the γ-ball, $\mathcal{B}$, about S_N has the properties that $\mathcal{B} \subset \mathcal{U}$ and $S_n \cap (\cup \mathcal{B}) = \emptyset$ for all $n \neq N$. Choose $\delta > 0$ such that if $t \in (1 - \delta, 1]$ then $H_d(f(t), S_N) < \gamma$. Let $t_0 \in (1 - \delta, 1)$. By (3) we have that $AP(f(t_0)) \neq \emptyset$. However, since $t_0 < t'$ we have by the definition of t' and our choice of γ that $AP(f(t_0)) - \cup_{n=0}^{\infty} S_n \neq \emptyset$. Therefore, $\operatorname{diam}(f(t_0)) > 2$ by (5).

Now suppose that $t' < 1$. Let $q = (1, \theta, z) \in AP(f(t')) \cap S_N$ and let $q' \in f(t') \cap \pi^{-1}(1, \theta + \pi, z)$. We may assume that $q' = (1, \theta + \pi, z)$ since $d(q, q') > 2$ otherwise. Using (3), we have that $AP(\pi(f(t')))$ contains an arc I containing q. We suppose first that q is an isolated point of $AP(f(t'))$. Let $\{y_i\}_{i=1}^{\infty}$ be a sequence in I converging to q; then use (2) to select $x_i \in \pi^{-1}(y_i) \cap AP(f(t'))$ for each $i = 1, 2, \cdots$. We have by (4) that $AP(f(t'))$ is closed; hence, some subsequence of $\{x_i\}_{i=1}^{\infty}$ converges to a point x_0 of $AP(f(t'))$. Moreover, since $\{y_i\}_{i=1}^{\infty}$ converges to q, we have that $x_0 \in \pi^{-1}(q)$. Finally, since q

is an isolated point of $AP(f(t'))$, it follows that x_0 is a member of $f(t') \cup \pi^{-1}(q)$ that does not belong to S_N. Therefore, $d(x_0, q') > 2$, and thus, $\mathrm{diam}(f(t')) > 2$. On the other hand, if q is not an isolated point of $AP(f(t'))$, then we may assume that the arc I containing q belongs to $S_N \cap AP(f(t'))$. Choose $\gamma > 0$ small enough so that (i) no γ-ball about a point of I meets any S_n for $n \neq N$ and (ii) the midpoint $m = (1, \mu, z)$ of I is not contained in the γ-balls about the endpoints of I. Choose $\delta > 0$ such that if $t \in (t' - \delta, t']$, then $H_d((f(t), f(t')) < \gamma$. Let $t_0 \in (t' - \delta, t')$. Since $H_d(f(t_0), f(t')) < \gamma$, we have by (i), (ii), and the construction of X that $f(t_0)$ contains a point m' for which $\pi(m') = m$; furthermore, we have by (i) that $m' \in S_N$. Thus, $m' = (1, \mu, z) = m \in f(t_0)$. By a similar argument we can show that $(1, \mu + \pi, z) \in f(t_0)$. Therefore, $m \in AP(t_0)$, contrary to our assumption that $t_0 < t'$.

Example 2. K. Kuratowski [1, p. 268] describes a continuum, K, consisting of the segment $\{(x, 0) : 0 \leq x \leq 1\}$, of the vertical segments $\{(m/2^{n+1}, y) : 0 \leq m \leq 2^{n+1}, 0 \leq y \leq 1/2^n\}$ and of the level segments $\{(x, 1/2^n) : 0 \leq x \leq 1\}$, where $n = 1, 2, \cdots$. We note that K is similar in structure to the continuum in the previous example; however, $C_{\rho_1,\epsilon}(K)$ is locally connected when ρ_1 is the usual metric inherited from $\mathbf{R}^2$. (Informally, observe that if a subcontinuum A of K is contained in an open subset $\mathcal{U}$ of $C(X)$, then $\mathcal{U}$ also contains subsets of A with diameter smaller than that of A. By first shrinking A to a continuum with smaller diameter within $\mathcal{U}$, one can then continuously grow continua to include a subset of a target subcontinuum within $\mathcal{U}$ before continuously releasing A.)

Instead of considering the usual metric on K, let $h : K \to S^1 \times [0, 1]$ be an embedding which sends the leftmost vertical segment of K to $\{(1, 0, z) : 0 \leq z \leq 1\}$ and the rightmost vertical segment of K to $\{1, 3\pi/2, z) : 0 \leq z \leq 1\}$, and which preserves the vertical and horizontal orientations of all subsets of K. Let d denote the usual metric for $h(K)$ inherited from $\mathbf{R}^3$, and let ρ_2 denote the metric on K given by $\rho_2(x, y) = d(h(x), h(y))$. Then an argument essentially identical to the one given in Example 1 can be used to show that $C_{\rho_2,\epsilon}(X)$ fails to be locally connected for $\epsilon = 2$.

Noting that the small-point hyperspaces of the arc, circle, and simple triod are all locally connected, while the examples provided in this article admit non-locally connected small-point hyperspaces, the referee suggests the following question.

Question 1. *Are the small-point hyperspaces of an hereditarily locally connected continuum always locally connected?*

Recall that a continuum is said to be *cyclicly connected* provided that any two points of the continuum are contained in some simple closed curve. Theorem 3.11 of [2] states that $C_\epsilon(X)$ is cyclicly connected for every $\epsilon > 0$ whenever X is locally connected; however, the argument that is used to justify this assertion uses Proposition 3.1 of [2]. Therefore, the following question remains open.

Question 2. *If X is a locally connected continuum with metric ρ, must $C_{\rho,\epsilon}(X)$ be cyclicly connected for every $\epsilon > 0$?*

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