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by

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MORE PARACOMPACT SUBSPACES OF $\square(\omega+1)^\omega$

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Abstract. We develop machinery to find paracompact subspaces of $\nabla(\omega+1)^\omega$, the quotient topology (mod finite) of $\square(\omega+1)^\omega$.

1. The question

Definition 1.1. $\square_{i\in I}X_i$ is the topology on $\Pi_{i\in I}X_i$ whose basic open sets are all $\Pi_{i\in I}u_i$, where each u_i is open in X_i.

Which box products are paracompact? In fact, which are normal? This was one of the main questions in [12], and while there are many partial results, very few positive results are known in ZFC.

Partial answers, almost all modulo set theoretic hypotheses, appeared from the late 1970s to the mid 1980s, largely utilizing an associated topology and depending on set theoretic axioms.

Definition 1.2. If x and y are functions with domain I, $y =^* x$ if and only if $\{i \in I : y(i) \neq x(i)\}$ is finite.

Definition 1.3. $\nabla_{n\in\omega}X_n$ is the quotient topology of $\square_{i\in I}X_i$ under the equivalence classes $x^\nabla = \{y : y =^* x\}$.

Kenneth Kunen [5] showed that, if each X_n is compact, then $\square_{n\in\omega}X_n$ is paracompact if and only if $\nabla_{n\in\omega}X_n$ is paracompact. He also noted without attribution that a regular space in which every G_δ is open (such as $\nabla_{n\in\omega}X_n$) is paracompact if and only if it

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is ultraparacompact, i.e., every open cover has a pairwise disjoint covering refinement. So the ∇ product became an object of investigation, with the goal of finding out when it is ultraparacompact.

Definition 1.4. The unboundedness number $\mathfrak{b} = \inf\{|F| : F \subset \omega^\omega$ and $\forall g \in \omega^\omega\ \exists f \in F\ f \not\leq^* g\}$. The dominating number $\mathfrak{d} = \inf\{|F| : F \subset \omega^\omega$ and $\forall g \in \omega^\omega\ \exists f \in F\ g \leq^* f\}$.

A series of papers in the 1970s and 1980s showed that

(1) if $\mathfrak{d} = \omega_1$ and each X_n is compact with weight $\leq \omega_1$, then $\square_{n<\omega} X_n$ is paracompact [13];
(2) if $\mathfrak{b} = \mathfrak{d}$ and each X_n is compact metrizable, then $\square_{n<\omega} X_n$ is paracompact [3][1];
(3) if $\mathfrak{d} = \mathfrak{c}$ and each X_n is compact first countable, then $\square_{n<\omega} X_n$ is paracompact [9].

L. Brian Lawrence [8] showed that $\square(\omega+1)^{\omega_1}$ is not normal, and various results of Kunen [5], [6] and Eric K. van Douwen [1], [2] showed that either one slightly large non-compact factor or compact factors with large character preclude normality of the box products.[2] Lawrence [7] proved that $\square\mathbb{Q}^\omega$ is paracompact if either $\mathfrak{b} = \mathfrak{d}$ or $\mathfrak{d} = \mathfrak{c}$.[3] But results (1), (2), and (3) above delineate the landscape of most interest in this paper; in fact, we focus on the simplest form of the question: Is $\square(\omega+1)^\omega$ really paracompact? By (1), (2), and (3), any model in which $\square(\omega+1)^\omega$ is not paracompact (or not normal) must satisfy $\mathfrak{b} < \mathfrak{d} < \mathfrak{c}$.

That this is not sufficient follows from the fact that if the Hechler iteration [4] is used to add a dominating family of reals of order type $\kappa \times \lambda$ for any κ, λ regular, then $\square_{n<\omega} X_n$ is paracompact in this model if each X_n is compact first countable (see [10]).

A key observation used in many of the positive results is that the intersection of fewer than $\mathfrak{b}$ open subsets of $\nabla_{n<\omega} X_n$ is open.

While we want to know if $\square(\omega+1)^\omega$ is really paracompact, we don't answer this question. Instead, we ask, in ZFC, what subspaces

[1]This improves the pioneering result in [11], which assumed CH.

[2]Hence, one line of research asks, how nice can a space X be so $X \times \square(\omega+1)^\omega$ fails to be normal? See, e.g., [16].

[3]Another line of research involves generalizing this result, e.g., "Box products twenty-five years later" by Scott W. Williams (unpublished) and [14].

of $\nabla(\omega+1)^\omega$ are paracompact?[4] We develop machinery which enables us to find many paracompact subspaces of $\nabla(\omega+1)^\omega$. The hope is that these techniques will increase our understanding so we may eventually show that $\nabla(\omega+1)^\omega$ really is paracompact. And if it's consistently not, we will know that many of its subspaces are.

The organization is as follows: §2 gives the basic topological lemmas, §3 describes a space closely associated with $\nabla(\omega+1)^\omega$ which is much easier to work with, and §4 gives the basic facts about this space. §5 defines the notion of good equivalence relation and shows how resulting transversals are paracompact subspaces. §6 gives an example of a good equivalence relation, and §7 gives another example. §8 shows how to iteratively refine good equivalence relations so that piecing together the resulting transversals also gives paracompact subspaces. The paper ends with a conjecture that implies the paracompactness of $\square(\omega+1)^\omega$.

2. The basics

Recall the following familiar notions.

Definition 2.1. Let $Y \subseteq X$ a space. Y is discrete if and only if $\exists\{U_y : y \in Y\}$ an open family with each $U_y \cap Y = \{y\}$.

Definition 2.2. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ where X is a space. $\mathcal{A}$ is discrete if and only if $\forall x \in X\ \exists U$ a neighborhood of x so $U \cap \mathrm{cl}A \neq \emptyset$ for at most one $A \in \mathcal{A}$. $\mathcal{A}$ is closed discrete if, in addition, each $A \in \mathcal{A}$ is closed.

Now we formally define the kinds of subspaces we are looking for.

Definition 2.3. Y is an ultraparacompact subspace of X if and only if every open cover of X has a pairwise disjoint open refinement covering Y.

Note that this is slightly stronger than "is a subspace and is ultraparacompact." For example, a discrete subspace is ultraparacompact, but not necessarily an ultraparacompact subspace.

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[4]The reader can translate back to $\square(\omega+1)^\omega$ by noting that, because equivalence classes are countable and hence Lindelöf, if $\{x^\nabla : x \in X\}$ is paracompact for $X \subseteq (\omega+1)^\omega$, then X is paracompact.

Not surprisingly, we will get our ultraparacompact subspaces by piecing together spaces which are more than discrete.

Definition 2.4. Y is a separated subset of X if and only if there is a pairwise disjoint open cover $\{U_y : y \in Y\}$ where each $y \in U_y$.

Separated spaces are both discrete and ultraparacompact subspaces.

In general, we need a stronger notion than separated.

Definition 2.5. Y is a strongly separated subset of X if and only if there is a discrete pairwise disjoint clopen family $\{U_y : y \in Y\}$ where each $y \in U_y$.

Note that strongly separated subsets are closed.

In the right situation, we can piece strongly separated subspaces together to get something stronger than an ultraparacompact subspace.

Definition 2.6. Y is a strongly ultraparacompact subspace of X if and only if every open cover of X has a discrete clopen refinement covering Y.

And what is the right situation?

Definition 2.7. X is a P_κ-space if and only if the intersection of fewer than κ many open subsets of X is open.

Fact 2.8. *If X is a P_κ-space, $\lambda < \kappa$, $x \in X \setminus \bigcup \mathcal{A}$, and $\mathcal{A} = \{\mathcal{A}_\alpha : \alpha < \lambda\}$ is a collection of closed discrete families, then x has a neighborhood U so $U \cap \bigcup \mathcal{A} = \emptyset$.*

Proof: For all $\alpha < \lambda$, let U_α be a neighborhood of x so $U_\alpha \cap \bigcup \mathcal{A}_\alpha = \emptyset$. Let $U = \bigcap_{\alpha<\lambda} U_\alpha$. □

Here's how to put things together.

Lemma 2.9. *Suppose X is a 0-dimensional P_κ-space, and that $X \supseteq Y = \bigcup_{\alpha<\lambda} Y_\alpha$ where $\lambda < \kappa$, $\{Y_\alpha : \alpha < \lambda\}$ is pairwise disjoint, and each Y_α is a strongly separated subspace of X. Then Y is strongly separated.*

Proof: Each Y_α is closed, so let $\mathcal{N}_\alpha$ be a discrete clopen cover of Y_α so that $\bigcup \mathcal{N}_\alpha \cap \bigcup_{\beta\neq\alpha} Y_\beta = \emptyset$. Since X is P_κ, if $N \in \mathcal{N}_\alpha$, then $N \setminus \bigcup_{\beta\neq\alpha} \bigcup \mathcal{N}_\beta$ is open non-empty. Then $\mathcal{M} = \{N \setminus \bigcup_{\beta\neq\alpha} \bigcup \mathcal{N}_\beta : N \in \mathcal{N}_\alpha, \alpha < \lambda\}$ is a discrete clopen cover of Y. □

Lemma 2.10. (1) *Suppose X is a 0-dimensional P_κ-space, that $X \supseteq Y = \bigcup_{\alpha<\lambda} Y_\alpha$ where $\lambda \leq \kappa$, and each Y_α is strongly separated in $Y \setminus \bigcup_{\beta<\alpha} Y_\beta$. Then Y is an ultraparacompact subspace of X.*

(2) *Suppose X is a 0-dimensional P_κ-space, and that $X \supseteq Y = \bigcup_{\alpha\leq\lambda} Y_\alpha$ where $\lambda < \kappa$, and if $\alpha < \lambda$, then Y_α is strongly separated in $Y \setminus \bigcup_{\beta<\alpha} Y_\beta$, and Y_λ is separated in Y. Then Y is an ultraparacompact subspace of X.*

Proof: (1) Let $\mathcal{U}$ be an open cover of X. For each $\alpha < \lambda$, let $\mathcal{N}_\alpha = \{N_y : y \in Y_\alpha\}$ be a discrete (in $Y \setminus \bigcup_{\beta<\alpha} Y_\beta$) separating clopen cover of Y_α refining $\mathcal{U}$, with $\bigcup \mathcal{N}_\alpha \cap \bigcup_{\beta<\alpha} Y_\beta = \emptyset$.

We construct a pairwise disjoint cover $\mathcal{M} = \bigcup_{\alpha<\lambda} \mathcal{M}_\alpha$ of Y refining $\mathcal{U}$ by induction. At stage α let $S_\alpha = Y_\alpha \setminus \bigcup_{\beta<\alpha} \mathcal{M}_\beta$. By Fact 2.8, for each $y \in S_\alpha$, $M_y = N_y \setminus \bigcup_{\beta<\alpha} \mathcal{M}_\beta$ is non-empty. Let $\mathcal{M}_\alpha = \{M_y : y \in S_\alpha\}$.

(2) Note that you need only that Y_λ is separated to extend the above proof to stage λ. $\square$

Lemma 2.10 is implicit in many of the proofs of the 1970s and 1980s.

3. Modifying $\nabla(\omega+1)^\omega$

$\nabla(\omega+1)^\omega$ is a little awkward to talk about — since each $x \in x^\nabla$, you have to continually distinguish between places where x is finite and places where x is infinite. So instead, we consider the set $P = \{x : x \text{ is a partial function from } \omega \text{ to } \omega\}$. For $x, y \in P$, we define $x =^* y$ if and only if $\operatorname{dom} x =^* \operatorname{dom} y$ and $\{n \in \operatorname{dom} x \cap \operatorname{dom} y : x(n) \neq y(n)\}$ is finite. For $x \in P$, define $x^\nabla = \{y \in P : y =^* x\}$. If $E \subseteq P$, define $E^\nabla = \{z^\nabla : z \in E\}$.

Now consider the map $\varphi : (\omega+1)^\omega \to P$ defined by $\varphi(x) = x \cap (\omega \times \omega)$; i.e., $\varphi(x) = \{(n, x(n)) : x(n) < \omega\}$. We define u to be open in P if and only if $\bigcup u^\nabla = u$ and $(\varphi^{\leftarrow}[u])^\nabla$ is open in $\nabla(\omega+1)^\omega$.

P is not even T_0, but it is easy to talk about, and, as Fact 3.2 below shows, quite useful.

Definition 3.1. (1) $X \subset P$ is a fine transversal if and only if for all distinct $x, y \in X, x \neq^* y$.

(2) X is a fine transversal of $Q \subseteq P$ if and only if it is a fine transversal and $X \subseteq Q$.

(3) $X \subseteq Q$ is a maximal fine transversal of $Q \subseteq P$ if and only if X is a fine transversal of Q and $\forall y \in Q\ \exists x \in X\ x =^* y$. If $Q = P$, we say that X is a maximal fine transversal.

Note that fine transversals may be partial transversals; they need not meet every equivalence class.

The following is immediate.

Fact 3.2. *If X is a maximal fine transversal of $Q \subseteq P$, then $X \cong (\varphi^{\leftarrow}[Q])^{\nabla}$; hence, every maximal fine transversal of P is homeomorphic to $\nabla(\omega+1)^{\omega}$.*

So in looking for ultraparacompact subsets of $\nabla(\omega+1)^{\omega}$, it suffices to find ultraparacompact fine transversals of P.

Now we need some notation to describe basic open sets.

Definition 3.3. If x and y are functions whose range is an ordered set, $x \leq^* y$ iff $\{i \in \text{dom } x \cap \text{dom } y : x(i) > y(i)\}$ is finite; $x >^* y$ iff $\{i \in \text{dom } x \cap \text{dom } y : x(i) \leq y(i)\}$ is finite.[5]

Definition 3.4. Given $x \in P$ and $f \in \omega^{\omega}$, we define $N(x, f) = \{y \in P : x \subseteq^* y \text{ and } y \setminus x >^* f\}$.

The $N(x, f)$'s form a base for the topology on P.

Finally, we note that if π is a permutation of ω and $\pi(x)$ is defined by $\pi(x)(n) = x(\pi(n))$, then $Y \subseteq P$ is separated, strongly separated, a paracompact subspace, or a strongly ultraparacompact subspace if and only if $\{\pi(y) : y \in Y\}$ is. So every time we find a paracompact subspace, we find $\mathfrak{c}$ many.

For the rest of this paper, $x, y, z, w \in P$ and f, g, h, k are strictly increasing functions in ω^{ω}.

4. Basic facts

Definition 4.1. (1) x is incompatible with y if and only if $\{n \in \text{dom } x \cap \text{dom } y : x(n) \neq y(n)\}$ is infinite.

(2) x is incompatible with $N(y, f)$ if and only if x is incompatible with y or $x \setminus y \not>^* f$.

[5] An anomaly which will not affect us should be noted: if dom $x \cap$ dom y is finite, then $x \leq^* y$ and $x >^* y$.

In either context, "compatible" means "not incompatible."

We state three facts without proof. The first relates incompatibility to $\supset^*$ and $\geq^*$.

Fact 4.2. (1) *If x is incompatible with $N(y,g)$ and $z \supseteq^* x$, then z is incompatible with $N(y,g)$*

(2) *If x is compatible with y and incompatible with $N(y,g)$ and $z \subseteq^* y$, then x is incompatible with $N(z,g)$.*

(3) *If x is incompatible with $N(y,g)$ and $h \geq^* g$, then x is incompatible with $N(y,h)$.*

The second tells us that incompatibility suffices to show that basic open sets are disjoint.

Fact 4.3. (1) *x is incompatible with y if and only if every $N(x,f) \cap N(y,g) = \emptyset$.*

(2) *x is incompatible with $N(y,g)$ if and only if for every f, $N(x,f) \cap N(y,g) = \emptyset$.*

(3) *If $N(x,f) \cap N(y,g) = \emptyset$, then either x is incompatible with $N(y,g)$ or y is incompatible with $N(x,f)$.*

And the third tells us what happens when $x \notin N(y,g)$.

Fact 4.4. *If $x \notin N(y,g)$, then one of the following three conditions holds:*

(a) *x is incompatible with y, or*

(b) *x is compatible with $N(y,g)$ and $y \setminus x$ is infinite, or*

(c) *$x \supseteq^* y$ and x is incompatible with $N(y,g)$.*

Some simple observations on the general themes of strong separation and closed sets are presented below.

Theorem 4.5. (1) *Let $B = \{x \in P : x$ is bounded below some constant function$\}$. Then B is closed, and any fine transversal of B is strongly separated.*

(2) *Fix f. Let $Q_f = \{x \in P : x \leq^* f\}$. Q_f is closed, and any fine transversal of it is strongly separated.*

(3) *Fix z. Let $Q_z = \{x \in P : x \subseteq^* z\}$. Q_z is closed, and any fine transversal of it is strongly separated.*

Proof: First, note that if $A = B, Q_f$ or Q_y, and $x \subseteq^* y \in A$, then $x \in A$. Also, note that (1) is a consequence of (2), since if $x \in B$, then $x \leq^*$ the identity function.

(2) For closed, if $x \notin Q_f$, then $N(x,f) \cap Q_f = \emptyset$.

For separated, let $\mathcal{N} = \{N(y,f) : y \in Q_f\}$. We need to show that $\mathcal{N}$ is pairwise disjoint. If $x \neq^* y$ are distinct elements of Q_f, then, without loss of generality, $y \setminus x$ is infinite. So y is incompatible with $N(x,f)$.

For strongly separated, again, let $\mathcal{N} = \{N(y,f) : y \in Q_f\}$. We need to show that $\bigcup \mathcal{N}$ is closed. Suppose $x \notin Q_f$ and $y \in Q_f$. If $x \notin N(y,f)$, then either x is incompatible with $N(y,f)$ or, by Fact 4.4, $y \setminus x$ is infinite. Since $y \setminus x \not>^* f$, $N(x,f) \cap N(y,f) = \emptyset$. So either $x \in N(y,f)$ for some $y \in Q_f$ or $N(x,f) \cap N(y,f) = \emptyset$ for all $y \in Q_f$.

(3) For closed, if $x \notin Q_z$, then $x \not\subseteq^* y$ for all $y \in Q_z$, so $N(x,f) \cap Q_z = \emptyset$ for any f.

For separated, let $f >^* z$ and $\mathcal{N} = \{N(y,f) : y \in Q_z\}$. We need to show that $\mathcal{N}$ is pairwise disjoint. If $y, w \subseteq^* z$ and $y \neq^* w$, then, without loss of generality, $y \setminus w$ is infinite. So y is incompatible with $N(w,f)$.

For strongly separated, again, let $f >^* z$ and $\mathcal{N} = \{N(y,f) : y \in Q_z\}$. We need to show that $\bigcup \mathcal{N}$ is closed. Suppose $x \in P \setminus \bigcup \mathcal{N}$. Then $x \notin Q_z$ and $x \setminus y$ is infinite for all $y \in Q_z$. We will show that $N(y,f) \cap N(x,f) = \emptyset$, for all $y \in Q_z$. Suppose, by way of contradiction, that $N(y,f) \cap N(x,f) \neq \emptyset$ for some $y \in Q_z$. Then x is compatible with y and $x \setminus y >^* f$. Since $y \subseteq^* z <^* f$, $y \setminus x$ is finite. So $y \subseteq^* x \cap z$. Hence, $x \in N(y,f)$. □

Corollary 4.6. *Let $F \in [\omega^\omega]^{\leq \mathfrak{b}}$ and $X = \{x : \exists f \in F\ x \leq^* f\}$. Let Y be a fine transversal of X. Then Y is an ultraparacompact subspace of P.*

Proof: Each Q_f is closed and each $Y \cap Q_f$ is strongly separated, so by Lemma 2.10, we're done. □

Corollary 4.6 was implicit in the literature from the 1970s and 1980s.

The following machinery produces pairwise disjoint neighborhoods in more general situations.

Definition 4.7. Let $a \in [\omega]^\omega$. $n_a^+ = \inf(a \setminus (n+1))$, and if $a \cap n \neq \emptyset$, then $n_a^- = \sup(a \cap n)$.

Definition 4.8. (1) $\top(x) = \{n \in \text{dom } x : \forall m \in \text{dom } x, \text{ if } m \leq n, \text{ then } x(m) \leq x(n)\}$.

(2) $x^\top = x|_{\top(x)}$.

(3) $f \gg x$ iff $\forall m\ f(m) > x(m^+_{\top(x)})$. $f \gg^* x$ iff $\forall^\infty m\ f(m) > x(m^+_{\top(x)})$.

In particular, if $f \gg x$, then $f > x$; if $f \gg^* x$, then $f >^* x$.

Here's an example of how $\top$ works. Let $x(6n) = 3n+3, x(6n+1) = 3n+2, x(6n+2) = 3n+1, x(k)$ undefined otherwise. Then $\top(x) = \{6n : n < \omega\}$, and $x^\top(m) = \frac{m}{2}+3$ for all $m \in \top(x)$. If $f \gg^* x$, then for all $n \in [0,6), f(n) > 6$ (because $x(6) = 6$); for all $n \in [6,12), f(n) > 9$ (because $x(12) = 9$); for all $n \in [12,18), f(n) > 12$ (because $x(18) = 12$); and so on.

The $\top$ operation is problematic. If x is bounded by a constant function, then $\top(x)$ may be finite; for example, suppose $x(0) = 5$, but for all $n > 0$, $x(0) = 2$. Hence, if $x =^* y$ and x is bounded by a constant function, we might have $x^\top \neq^* y^\top$; in our example, let $y(n) = 2$ for all n.

But if x is not bounded by a constant function, then $\forall y =^* x\ x^\top =^* y^\top$. By Theorem 4.5(1) and the proof of Lemma 2.10, a fine transversal Y is a (strongly) paracompact subset of P if and only if $Y \setminus B$ is a (strongly) paracompact subset of P, where B is as in Theorem 4.5.

Also, if we define $Fin = \{x \in P : \text{dom } x \text{ is finite}\}$ and $\bar{\omega}$ to be the function in $(\omega+1)^\omega$ whose constant value is ω, then $x \in Fin$ iff $\varphi^\leftarrow(x) =^* \bar{\omega}$. Hence, a fine transversal Y is a (strongly) paracompact subset of P if and only if $Y \setminus (B \cup Fin)$ is a (strongly) paracompact subset of P.

This leads us to the following.

Definition 4.9. $P^* = P \setminus (B \cup Fin)$.

From now on we restrict ourselves to P^* and reserve the letters x, y, z, w for elements of P^*.

Let us get back to $\top$.

Lemma 4.10. *$T_x = \{y : y^\top =^* x\}$ is closed and any fine transversal of T_x is strongly separated.*

Proof: Let $f \gg x$. Let $\mathcal{N} = \{N(y,f) : y \in T_x\}$.

For closed, if $z \subseteq^* y$ and $\top(y) \subseteq^* \top(z)$, then$\top(y) =^* \top(z)$. Hence, if $z \notin T_x$, $y \in T_x$, and $y \supset^* z$, then $\top(y) \setminus \top(z)$ is infinite; hence, $y \not>^* f$. So if $z \notin T_x$, then $N(z, f) \cap T_x = \emptyset$.

For discrete, suppose $y, z \in T_x$. If $y \setminus z$ is infinite, then $\forall^\infty n \in$ dom $(y \setminus z)$, $n \notin$ dom $y^\top =^*$ dom x, and $y(n^-_{\top(y)}) = x(n^-_{\top(y)})$. For such n, let $m = n^-_{\top(y)}$. Then $y(n) < y(m) = x(m) \leq x(n^+_{\top(y)}) < f(n)$. So y is incompatible with $N(z, f)$ and $\mathcal{N}$ is a pairwise disjoint cover of T_x.

For strongly separated, suppose $z \notin \bigcup \mathcal{N}$. If $y \in T_x$ and $y \subseteq^* z$, then by Fact 4.4(c), z is incompatible with $N(y, f)$, so $N(z, f) \cap N(y, f) = \emptyset$. By the argument in the preceding paragraph, if $y \in T_x$ and $y \setminus (z \cup x)$ is infinite, then y is incompatible with $N(z, f)$. And if $y \in T_x$ and $y \setminus z \subseteq^* x$, then dom $(y \setminus z) \subseteq^* \top(y)$ and $\forall^\infty n \in \top(y)$ $y(n) < f(n)$. So y is incompatible with $N(z, f)$. □

Section 6 is based largely on the following lemma.

Lemma 4.11. *Let $INC = \{x$: for all but finitely many pairs $n < m \in$ from dom x $x(n) \leq x(m)\}$. INC is closed in P^* and every fine transversal of INC is strongly separated.*

Proof: For closed, if $x \notin INC$ and $y \supset^* x$, then $y \notin INC$, so every $N(x, f) \cap INC = \emptyset$.

For separated, for all $x \in INC$, let $f_x \gg x$. For $x, y \in INC$, suppose $E = x \setminus y$ is infinite. For $k \in E$, let $n_k = k^+_{\text{dom } y}$. If $x(k) > y(n_k)$, then, because $x \in INC$, $n_k \in$ dom $(y \setminus x)$ and $y(n_k) < x(k) \leq x((n_k)^+_{\text{dom } x}) < f(n_k)$. If $x(k) < y(n_k)$, then $x(k) < g(k)$. At least one case holds infinitely often, so $N(x, f_x) \cap N(y, f_y) = \emptyset$.

For strongly separated, for all $y \in INC$, let $f_y \gg y$. Fix $x \notin INC$ and let $g = 1 + f_{x^\top}$. We will show, for all $y \in INC$, that either $N(y, f_y) \cap N(x, g) = \emptyset$ or $x \in N(y, f_y)$. Since there is at most one $y \in INC$ with $x \in N(y, f_y)$, this will complete the proof.

So assume $y \in INC$ and y and x are compatible. Assume that $N(y, f_y) \cap N(x, g) \neq \emptyset$. Then we may assume, by finite modification, that $x|_{\text{dom } y \cap \text{dom } x} = y|_{\text{dom } y \cap \text{dom } x}$, $x \setminus y > f_y$, and $y \setminus x > g$. Suppose $m \in$ dom $(y \setminus x)$. If $m^+_{\top(x)} \in$ dom y, then $y(m) \leq y(m^+_{\top(x)}) = x(m^+_{\top(x)}) < g(m)$. Hence, $\forall m \in$ dom $(y \setminus x)$ $m^+_{\top(x)} \notin$ dom y. But for such m, $y(m) > g(m) > x(m^+_{\top(x)})$; hence, $x(m^+_{\top(x)}) < f_y(m) \leq$

$f_y(m^+_{\top(x)})$. But by assumption, $x \setminus y > f_y$. Hence, $y \subseteq^* x$. But then $x \in N(y, f_y)$. $\square$

Lemma 4.11 is one of the main theorems of [10].

5. Equivalence relations

Our main tool is a class of equivalence relations which are coarser than $=^*$.

Definition 5.1. Let $\approx$ be an equivalence relation on P^*. We say that $\approx$ is a good equivalence relation if and only if there is a partition $\{P_\alpha : \alpha < \lambda\}$ of P^*, $\lambda \leq \mathfrak{b}$, and there is a strict pre-order $\prec_\approx$ where

(a) if $x =^* y$, then $x \approx y$;
(b) each $\bigcup_{\beta<\alpha} P_\beta$ is a closed subset of P^*;
(c) if $x \in P_\alpha$ and $x \approx y$, then $y \in P_\alpha$;
(d) if $x \approx y$ and $z \prec_\approx x$, then $z \prec_\approx y$;
(e) if $x \in P_\alpha$ and $y \prec_\approx x$, then $y \in P_\beta$ for some $\beta < \alpha$;
(f) if $y, z \in P_\beta$ and $y, z \prec_\approx x$, then $y \approx z$;
(g) $\forall x \in P^* \exists f_x$ so if $N(x, f_x) \cap N(y, f_y) \neq \emptyset$, then either $x \prec_\approx y$ or $y \prec_\approx x$ or $x \approx y$;
(h) if $x \subseteq^* y$, then $y \not\prec_\approx x$.

If $x \in P_\alpha$, we write $\mathrm{rk}_\approx(x) = \alpha$. Note that, by conditions (g) and (h), if $y \in N(x, f_x)$, then either $x \prec_\approx y$ or $x \approx y$.

Lemma 5.2. *Let $\approx$ be a good equivalence relation and define $\mathcal{U}_\alpha = \{N(x, f_x) : x \in P_\alpha\}$ where f_x and P_α are as in Definition 5.1. Let Y_α be an $\approx$-transversal of P_α and $\mathcal{V}_\alpha = \{N(y, f_x) : y \in Y_\alpha\}$. Then*

(a) *$\bigcup_{\beta<\alpha} P_\beta \cap \bigcup \mathcal{U}_\alpha = \emptyset$;*
(b) *if $x, y \in U \cap P_\alpha$, $U \in \mathcal{U}_\alpha$, then $x \approx y$;*
(c) *if $x \in \bigcup_{\beta>\alpha} P_\beta \setminus \bigcup \mathcal{V}_\alpha$, then there is an open neighborhood W of x with $W \cap \bigcup \mathcal{V}_\alpha = \emptyset$.*

Hence, Y_α is strongly discrete in $Y_\alpha \cup (P^* \setminus \bigcup_{\beta\leq\alpha} P_\alpha)$.

Proof: (a) follows by condition (h) in Definition 5.1.

(b) By conditions (e), (g), and (h), if $x, y \in U \cap P_\alpha$ and $U \in \mathcal{U}_\alpha$, then $x \approx y$.

(c) By condition (f), if $x \in \bigcup_{\beta>\alpha} P_\beta \setminus \bigcup \mathcal{V}_\alpha$, then there is at most one $y \in Y_\alpha$ with $N(x, f_x) \cap N(y, f_y) \neq \emptyset$. $W = N(x, f_x) \setminus N(y, f_y)$ is the desired neighborhood. $\square$

An immediate corollary to Lemma 5.2 is the following.

Corollary 5.3. *Let $\approx$ be a good equivalence relation on P^*, where λ and P_α are as in Definition 5.1. Let Y_α be an $\approx$-transversal of P_α for each α.*

1. *Y_α is strongly separated in $Y_\alpha \cup (P^* \setminus \bigcup_{\beta\leq\alpha} P_\beta)$.*
2. *$\bigcup_{\alpha<\lambda} Y_\alpha$ is an ultraparacompact subspace of P^*.*

And a corollary to Corollary 5.3 is the following.

Corollary 5.4. *If $\approx$ is a good equivalence relation on P^* and $ht(\approx) < \mathfrak{b}$, then every $\approx$-transversal is a strongly ultraparacompact subspace of P^*.*

Proof: Given an $\approx$-transversal Y, define $Y_\alpha = Y \cap P_\alpha$, where $\{P_\alpha : \alpha < \mathrm{ht}\ (\approx)\}$ is as in Definition 5.1. We are done by Corollary 5.3. $\square$

In fact, we can enlarge the Y_α's of Lemma 5.2.

Definition 5.5. Let $\approx$ be an equivalence relation. For $y \in P^*$, define $y\downarrow_\approx = \{x : x \approx y \text{ and } x \subseteq^* y\}$. Let Y be an $\approx$-transversal of P^*, and for each $y \in Y$, suppose D_y is a fine transversal of $\{y\downarrow_\approx : y \in Y\}$ with $y \in D_y$. We call $\bigcup_{y\in Y} D_y$ a downward $\approx$-completion of Y.

Lemma 5.6. *Let $\approx$ be a good equivalence relation. Let Y_α be an $\approx$-transversal of P_α and let D_α be a downward $\approx$-completion of Y_α. Then D_α is strongly discrete in $D_\alpha \cup (P^* \setminus \bigcup_{\beta\leq\alpha} P_\alpha)$.*

Proof: Let $\mathcal{U}_\alpha = \{N(x, f_x) : x \in D_\alpha\}$, where f_x is as in Definition 5.1. If $y \in Y_\alpha$ and $y \approx x$, let $g_x > y, f_x$. As in the proof of Theorem 4.5(3), if $x, z \in y\downarrow$ and $x \neq^* z$, then $N(x, g_x) \cap N(z, g_z) = \emptyset$. As in the proof of Lemma 5.2, if $x, z \in D_\alpha$ and $x \not\approx z$, then $N(x, g_x) \cap N(z, g_z) = \emptyset$. So D_α is discrete. For strongly discrete, let $x \in P^* \setminus \bigcup_{\beta\leq\alpha} P_\alpha$ and let $N = N(x, f_x)$. Then $N \cap N(z, f_z) = \emptyset$ for all $z \in D_\alpha$ with $z \not\prec x$. There is at most one element Y_α with $y \prec x$. By Theorem 4.5(3), x has a clopen neighborhood M so $M \cap y\downarrow = \emptyset$. So $N \cap M$ is the desired neighborhood with $N \cap M \cap D_\alpha = \emptyset$. $\square$

An immediate corollary to Lemma 5.6 follows.

Corollary 5.7. *Let $\approx$ be a good equivalence relation on P^*, let Y_α be an $\approx$-transversal of P_α, where $\{P_\alpha : \alpha < \lambda\}$ is as in Definition 5.1, let D_α be a downward $\approx$-completion of Y, and let $D = \bigcup_{\alpha<\lambda} D_\alpha$.*

1. *D_α is strongly separated in $D \setminus \bigcup_{\beta<\alpha} P_\beta$.*
2. *$\bigcup_{\alpha<\lambda} D_\alpha$ is an ultraparacompact subspace of P^*.*

And below is a corollary to Corollary 5.7.

Corollary 5.8. *If $\approx$ is a good equivalence relation on P^* and $ht(\approx) < \mathfrak{b}$, then every downward $\approx$-completion of an $\approx$-transversal is a strongly ultraparacompact subspace of P^*.*

Definition 5.9. Let $\{P_\alpha : \alpha < \lambda\}$ be as in Definition 5.1 for the good equivalence class $\approx$. We say that λ is a height of $\approx$. We define $ht(\approx)$ to be the least such λ.

Hence, by Lemma 2.10, we have the following corollary.

Corollary 5.10. *Let $\approx$ be a good equivalence relation where $ht(\approx) < \mathfrak{b}$, and suppose $\{Z_\beta : \beta < \lambda\}$ is a pairwise disjoint family of downward $\approx$-completions of $\approx$-transversals, where $\lambda < \mathfrak{b}$. Then $Z = \bigcup_{\beta<\lambda} Z_\beta$ is a strongly ultraparacompact subspace of P^*.*

Proof: Let $\{P_\alpha : \alpha < ht(\approx)\}$ be as in Definition 5.1. Let $D_{\alpha,\beta} = Z_\beta \cap P_\alpha$ and $W_\alpha = \bigcup_{\beta<\lambda} D_{\alpha,\beta}$. It suffices to show that each W_α is strongly ultraparacompact in $Z \setminus \bigcup_{\beta<\alpha} P_\beta$. Each $D_{\alpha,\beta}$ is strongly separated in $D_{\alpha,\beta} \cup (P^* \setminus \bigcup_{\gamma<\alpha} P_\gamma)$. By condition (g) of Definition 5.1, if $\beta < \gamma$, then for each $y \in D_{\alpha,\gamma}$, there is at most one $z \in D_{\alpha,\beta}$ with $N(y, f_y) \cap N(z, f_z) \neq \emptyset$ (where f_y and f_z are as in condition (g) of Definition 5.1). So $\{N(y, f_y) \setminus \bigcup_{\beta<\gamma} \bigcup_{z\in D_{\alpha,\beta}} N(z, f_z) : y \in D_{\alpha,\gamma}\}$ is a closed discrete family separating W_α. $\square$

Now that we know how useful good equivalence relations are, we need to find some examples.

6. The $\perp$ decomposition

We make use of Lemma 4.11 to define our first equivalence relation.

Definition 6.1. Let $x \in P^*$. $\perp(x) = \{n \in \text{dom } x : \forall m \in \text{dom } x \text{ if } n \leq m, \text{ then } x(n) \leq x(m)\}; x^\perp = x|_{\perp(x)}$.

That is, $\perp$ is a sort of inverse of $\top$.

For example, again consider the function $x(6n) = 3n+3, x(6n+1) = 3n+2, x(6n+2) = 3n+1$. Then $\perp(x) = \{6n+2 : n < \omega\}$ and each $x^{\perp}(m) = \frac{m}{2}$.

Note that each $x^{\perp}$ is in INC and that $x \in INC$ iff $x =^* x^{\top} =^* x^{\perp}$.

Using $\perp$, we decompose elements of P^*.

Definition 6.2. For $x \in P^*$, we define x_n inductively: $x_0 = x^{\perp}$; if $x \setminus \bigcup_{i \leq n} x_i$ is infinite, then $x_{n+1} = (x \setminus \bigcup_{i \leq n} s_i)^{\perp}$; if $x \setminus \bigcup_{i \leq n} x_i$ is finite, then $x_{n+1} = \emptyset$.

Fact 6.3. $x =^* \bigcup_{n<\omega} x_n$.

Proof: For all $i < \omega$ inf range$(x_i) \geq i$. Hence, if $k \in \text{dom}(x_i)$ for some i, $i \leq \text{inf range}(x_i) \leq x(k)$. We have only to consider the case where there is $k \in \text{dom}(x)$ such that $\forall i < \omega$, $k \notin \text{dom}(x_i)$. Then $k \notin \bigcup_{i \leq x(k)} \text{dom}(x_i)$. If all $x_i \neq \emptyset$ for $i \leq x(k)$, then $x(k) >$ inf range$(x_{x(k)}) \geq x(k)$, a contradiction. So there is $i \leq x(k)$ with $x_i = \emptyset$, and by definition, $x =^* \bigcup_{i<\omega} x_i$. $\square$

The next fact says that taking subsets cannot increase the levels of the $\perp$ decomposition and taking supersets cannot decrease it.

Fact 6.4. (1) *If $w \subseteq x$ and $k \in$ dom $x_n \cap w$, then $k \in$ dom w_m for some $m \leq n$.*

(2) *If $x \subseteq w$ and $k \in$ dom x_n, then $k \in$ dom w_m for some $m \geq n$.*

The next lemma tells us what happens vis-a-vis initial segments and tails of functions.

Lemma 6.5. (1) *If $k \in$ dom x_n, then $\forall m \leq n$; if $j < k$ and $j \in$ dom x_m, then $x_m(j) < x_n(k) = x(k)$.*

(2) *Suppose $k \in$ dom x_n and let $w = x|_{\omega \setminus k}$. Then $k =$ inf dom w_n.*

Proof: (1) is immediate.

As for (2), by definition, each $w_i = x_i|_{\omega \setminus k}$. $\square$

Here is the equivalence relation.

Definition 6.6. Let $x, y \in P^*$. $x \approx_{\perp} y$ if and only if each $x_n =^* y_n$.

Note that if $x =^* y$, then $x \approx_\perp y$, so condition (a) of Definition 5.1 holds.

We define a rank function which will establish the stratification of Definition 5.1.

Definition 6.7. For $n < \omega$, $rk\ x = n$ if and only if each x_m is infinite for $m \leq n$ and $x =^* \bigcup_{m \leq n} x_m$; if x_n is infinite for all n, then $rk\ x = \omega$.

Here is the stratification.

Definition 6.8. For $n \leq \omega$, define $P_n = \{x \in P^* : \text{rk } x = n\}$.

We relate $\top$ and $\perp$.

Fact 6.9. (1) *If $rk\ x = n < \omega$, then $x^\top =^* x_n$ and $x =^* y$ for all $y \approx_\perp x$.*
(2) *If $f \gg x^\top$, then $f > x_n$ for all $n < rk\ x$.*

Proof: (1) is clear.

For (2), $\forall k \in \text{dom } x_n$, $f(k) > x^\top(k^+_{\top(x)}) \geq x(k) = x_n(k)$. $\square$

In [10], it is shown (using different notation) that any fine transversal of $\bigcup_{n<\omega} P_n \cup \{x \in P_\omega : \forall n < \omega \text{ inf dom } x_n < \text{inf dom } x_{n+1}\}$ is paracompact, along with some generalizations. In this paper, we concentrate on $\approx_\perp$.

Here is the associated pre-order.

Definition 6.10. Let $x, y \in P^*$. $x \prec_{\approx_\perp} y$ iff rk $x <$ rk y and $\forall m \leq$ rk x $x_m =^* y_m$.

$\prec_{\approx_\perp}$ is clearly transitive. Conditions (a), (c), (d), (e), (f), and (h) of Definition 5.1 clearly hold.

We prove condition (b).

Fact 6.11. (1) *If $y \supseteq^* x$, then $rk\ y \geq rk\ x$.*
(2) *Let $f \gg x^\top$. If $rk\ y < rk\ x$, then $y \notin N(x, f)$.*

Proof: (1) follows from Fact 6.4.

For (2), if $y \in N(x, f)$, then $y \supseteq^* x_n$; so rk $y \geq$ rk x by (1). $\square$

Condition (b) is immediate from Fact 6.11(2). It remains to prove condition (g).

Lemma 6.12. *If $n < \omega$ is least so $x_n \neq^* y_n$, then either $(x_n \setminus y_n) \cap (x \setminus y)$ is infinite or $(y_n \setminus x_n) \cap (y \setminus x)$ is infinite.*

Proof: By finite modification, we may assume $x|_{\mathrm{dom}\ x \cap \mathrm{dom}\ y} = y|_{\mathrm{dom}\ x \cap \mathrm{dom}\ y}$ and $\forall j < n\ x_j = y_j$. Hence, $(x_n \cup y_n) \cap \bigcup_{j<n} x_j$ is empty and $\bigcup_{j<n} x_j = \bigcup_{j<n} y_j \subseteq x \cap y$.

Suppose there are infinitely many pairs m and k, where $m \in \mathrm{dom}\ x_n$, $m < k \in \mathrm{dom}\ y_n$, and $x_n(m) > y_n(k)$. Then $k \notin \mathrm{dom}\ x_j$ for all $j \geq n$, and we already know that $k \notin \mathrm{dom}\ x_j$ for any $j < n$, so $(y_n \setminus x_n) \cap (y \setminus x)$ is infinite.

Similarly, if there are infinitely many pairs m and k, where $m \in \mathrm{dom}\ y_n$, $m < k \in \mathrm{dom}\ x_n$, and $y_n(m) > x_n(k)$, then $(x_n \setminus y_n) \cap (x \setminus y)$ is infinite.

So we may assume that $x_n \cup y_n \in INC$. Suppose $x_n \setminus y_n$ is infinite. Fix $m \in \mathrm{dom}\ x_n \setminus \mathrm{dom}\ y_n$. Let $j = m^-_{\mathrm{dom}\ y_n}$ and $k = m^+_{\mathrm{dom}\ y_n}$. For all $i > n$, if $s \in (j, k) \cap \mathrm{dom}\ y_i$, then $y_i(s) > y_n(k)$. So $m \notin \mathrm{dom}\ y$. Hence, $(x_n \setminus y_n) \cap (x \setminus y)$ is infinite. $\square$

Lemma 6.13. *Suppose for all $m < n$, $x_m =^* y_m$ and $(x_n \setminus y_n) \cap (x \setminus y)$ is infinite. Suppose $f \gg^* x_n$ and $g \gg^* y_n$. Then $N(x, f) \cap N(y, g) = \emptyset$.*

Proof: Let $z = (x_n \setminus y_n) \cap (x \setminus y)$. For $i \in \mathrm{dom}\ z$, let $m_i = i^-_{\mathrm{dom}\ y_n}$. Define $E = \{i \in \mathrm{dom}\ z : x_n(i) \leq y_n(m_i)\}, G = \mathrm{dom}\ z \setminus E$.

For all $^\infty i \in E$, $x(i) = x_n(i) \leq y_n(m_i) < g(i)$. Hence, if E is infinite, x is incompatible with $N(y, g)$.

So suppose E is finite. For all $i \in G$, $y(m_i) = y_n(m_i) < x_n(i)$; since $i = (m_i)^+_{\mathrm{dom}\ x}$, $\forall^\infty i \in G\ x_n(i) < f(m_i)$.

If there are infinitely many $i \in G$ with $m_i \notin \mathrm{dom}\ x_n$, then y is incompatible with $N(x, f)$.

Otherwise, let $k_i = i^+_{\mathrm{dom}\ y_n}$ for all $i \in G$. If there are infinitely many $i \in G$ with $x_n(i) \geq y_n(k_i)$, then infinitely many $k_i \notin \mathrm{dom}\ x$ and y is incompatible with $N(x, f)$.

The only remaining possibility is that $\forall^\infty i \in G$, $x_n(i) < y_n(k_i)$, in which case, x is incompatible with $N(y, g)$. $\square$

Corollary 6.14. *If $rk\ x \leq rk\ y$, $f \gg^* x_n$ for all n and $g \gg^* y_n$ for all n, and $N(x, f) \cap N(y, g) \neq \emptyset$, then $x \prec_{\approx_\perp} y$ or $x \approx_\perp y$.*

Proof: Otherwise, the least n so $x_n \neq^* y_n$ satisfies the hypothesis of Lemma 6.13 (possibly exchanging the roles of x and y), so $N(x, f) \cap N(y, g) = \emptyset$. $\square$

Corollary 6.14 immediately proves condition (g). Hence,

Theorem 6.15. $\approx_\perp$ *is good.*

Applying corollaries 5.4, 5.8, and 5.10, we have the following theorem.

Theorem 6.16. (1) *An* $\approx_\perp$*-transversal is strongly ultraparacompact.*

(2) *If* Y *is an* $\approx_\perp$*-transversal, then* $Y\downarrow$ *is strongly ultraparacompact.*

(3) *If* $\lambda < \mathfrak{b}$ *and* Z *is the union of* λ *many downward* $\approx_\perp$*-completions of* $\approx_\perp$*-transversals, then* Z *is strongly ultraparacompact.*

7. An unbounded family and its equivalence relation

In this section, we define a different decomposition and use it to define a different good equivalence relation.

Fix $\vec{h} = \{h_\beta : \beta < \mathfrak{b}\}$ an unbounded subset of ω^ω where each h_β is strictly increasing, and if $\beta < \gamma$, then $h_\beta <^* h_\gamma$.

The next definition uses this unbounded family to decompose each element of P^*.

Definition 7.1. (1) For $x \in \omega^\omega$, $L(x, f) = \{n : x(n) \leq f(n)\}$.

(2) $\beta(x, 0)$ is the least β with $L(x, h_\beta)$ infinite; $x_{\beta(x,0)} = x|_{L(x, h_{\beta(x,0)})}$.

(3) We define $\beta(x,\gamma)$ by induction: it is the least β with $L(x \setminus \bigcup_{\rho<\gamma} x_{\beta(x,\rho)}, h_\beta)$ infinite; $x_{\beta(x,\gamma)} = (x|_{L(x\setminus\bigcup_{\rho<\gamma} x_{\beta(x,\rho)}, h_{\beta(x,\gamma)})})$.

(4) $E_x = \{\beta < \mathfrak{b} : \exists\gamma\ \beta = \beta(x,\gamma)\}$.

For $\beta = \beta(x,\gamma) \in E_x$, we write x_β instead of $x_{\beta(x,\gamma)}$.

Fact 7.2. (1) *If* $\beta \neq \gamma \in E_x$*, then* $x_\beta \cap x_\gamma = \emptyset$*.*

(2) E_x *is countable.*

(3) $x =^* \bigcup_{\beta\in E_x} x_\beta$*.*

Proof: (1) is by definition.

For (2), $x \supseteq \bigcup_{\beta\in E_x} x_\beta$ and x is countable.

For (3), let $y = x \setminus \bigcup_{\beta\in E_x} x_\beta$. If y is infinite, then there is $\alpha \notin E_x$ $\alpha \geq \sup E_x$ with $L(y, h_\alpha)$ infinite, contradicting the definition of E_x. □

Given Fact 7.2, we write $rk\ x =$ order type of E_x, $\delta(x) = \inf \mathfrak{b} \setminus E_x$. Note that E_x is a proper end-extension of E_y if and only if $E_x \cap \delta(y) = E_y$.

Given $\vec{h}$ as above, we use this decomposition to define an equivalence relation $\approx_{\vec{h}}$ and a strict pre-order $\prec_{\vec{h}}$, and we show that $\approx_{\vec{h}}$ is good.

Definition 7.3. (1) $x \approx_{\vec{h}} y$ if and only if $E_x = E_y$ and $\forall \beta \in E_x\ x_\beta =^* y_\beta$.

(2) $y \prec_{\vec{h}} x$ if and only if E_x is a proper end-extension of E_y, and, for all $\beta \in E_y$, $x_\beta =^* y_\beta$.

(3) $P_\alpha = \{x \in P^* : rk\ x = \alpha\}$.

Note that E_x is finite if and only if ($x \approx_{\vec{h}} y$ iff $x =^* y$).

By definition, we have the following fact.

Fact 7.4. *Conditions* (a), (c), (d), (e), (f), (h) *of Definition 5.1 hold.*

We relate this decomposition to the topology.

Lemma 7.5. *For all x, let $f_x > h_{\delta(x)}$.*

(a) *If $y \in N(x, f_x)$, then $x \approx_{\vec{h}} y$ or $x \prec_{\vec{h}} y$.*

(b) *If $N(x, f_x) \cap N(y, f_y) \neq \emptyset$, then either $x \prec_{\vec{h}} y$ or $y \prec_{\vec{h}} x$ or $x \approx_{\vec{h}} y$.*

Proof: (a) Assume $y \in N(x, f_x)$. $y \supset^* x$ and, for all $\beta < \delta(x)$, if $y \neq^* x$, then $y \setminus x >^* h_\beta$, so $E_y \cap \delta(x) = E_x$ and $\forall \beta \in E_x\ x_\beta =^* y_\beta$.

(b) Let $z \in N(x, f_x) \cap N(y, f_y)$. By (a), either (i) $x, y \prec_{\vec{h}} z$, or (ii) $y \prec_{\vec{h}} z$ and $x \approx_{\vec{h}} z$, or (iii) $x \prec_{\vec{h}} z$ and $y \approx_{\vec{h}} z$. We need only to consider (i): then by the definition of $\prec_{\vec{h}}$, either $x \prec_{\vec{h}} y$ or $y \prec_{\vec{h}} x$ or $x \approx_{\vec{h}} y$. □

Theorem 7.6. *$\approx_{\vec{h}}$ is good.*

Proof: Let f_x be as in Lemma 7.5. Condition (g) follows from Lemma 7.5(b). For condition (b): by Lemma 7.5(a), if $rk\ x > \beta$, then $N(x, f_x) \cap P_\beta = \emptyset$. □

As in the proof of Theorem 6.16,

Theorem 7.7. (1) *An $\approx_{\vec{h}}$-transversal is strongly ultraparacompact.*

(2) *If Y is an $\approx_{\vec{h}}$-transversal, then $Y \downarrow$ is strongly ultraparacompact.*

(3) *If $\lambda < \mathfrak{b}$ and Z is the union of λ many downward $\approx_{\vec{h}}$-completions of $\approx_{\vec{h}}$-transversals, then Z is strongly ultraparacompact.*

8. Refining equivalence relations

In this section, we show how to iteratively refine equivalence relations so that we get bigger paracompact subspaces.

Suppose we have a good equivalence relation $\approx$. Fix some y and consider $y^{\nabla\approx} = \{z : z \approx y\}$. How can we partition each of these equivalence classes further so that we can get bigger paracompact subspaces? One way is to apply $\approx$ to $\{z \setminus y : z \in y^{\nabla\approx}\}$.

Definition 8.1. For a good equivalence relation $\approx$, and $y \in P^*$, we define $z \approx^y w$ if and only if $z \approx w \approx y$, $z \cap y =^* w \cap y$, and $z \setminus y \approx w \setminus y$. If Y is a maximal $\approx$-transversal, we define $\approx^Y$ as $z \approx^Y w$ if and only if $\exists y \in Y\ z \approx^y w$. If $\prec$ is the partial order associated with $\approx$ as in Definition 5.1, we write $z \prec^Y w$ if and only if $z \prec y$, or $z \approx w$ and $z \cap y =^* w \cap y$ and $z \setminus y \prec w \setminus y$.

Each $\approx^y$ is an equivalence relation on $y^{\nabla\approx}$. Since Y is a maximal $\approx$-transversal, the definition of $\approx^Y$ is unambiguous: For each z there is exactly one y so that $z \approx^y w$ is possible for some w. Hence, $\approx^Y$ is an equivalence relation which refines $\approx$.

Now we generalize the definition of good so that the notion of rank has two parameters.

Definition 8.2. Let $\approx$ be a good equivalence relation via $\{P_\alpha : \alpha < \lambda\}$. Let Y be a maximal $\approx$-transversal. For $\alpha, \beta < \lambda$, $P_{\alpha,\beta} = \{x \in P_\alpha : \exists y \in Y\ x \setminus y \in P_\beta\}$.

Definition 8.3. Given a good equivalence relation $\approx$ via $\{P_\alpha : \alpha < \lambda\}$ and Y an $\approx$-transversal, we say that the ordered pair $(\approx, \approx^Y)$ is 2-good if and only if

(a) if $x =^* y$, then $x \approx^Y y$;
(b) each $\bigcup_{\rho<\alpha} P_\rho \cup \bigcup_{\gamma<\beta} P_{\alpha,\gamma}$ is a closed subset of P^*;
(c) if $x \in P_{\alpha,\beta}$ and $x \approx^Y w$, then $w \in P_{\alpha,\beta}$;
(d) if $x \approx^Y w$ and $z \prec_{\approx^Y} x$, then $z \prec_{\approx^Y} w$;
(e) if $x \in P_{\alpha,\gamma}$ and $y \prec_{\approx^Y} x$, then $y \in P_{\alpha,\beta} \cup \bigcup_{\rho<\alpha} P_\rho$ for some $\beta < \gamma$;
(f) if $w, z \in P_{\alpha,\beta}$ and $w, z \prec_{\approx^Y} x$, then $w \approx^Y z$.

(g) $\forall x \in P^* \exists g_x$ so if $N(x, g_x) \cap N(y, g_y) \neq \emptyset$, then either $x \prec_{\approx^Y} y$ or $y \prec_{\approx^Y} x$ or $x \approx^Y y$;
(h) if $x \subseteq^* y$; then $y \not\prec_{\approx^Y} x$.

Theorem 8.4. *Suppose* $\approx$ *is good and* Y *is a maximal* $\approx$*-transversal. Then* $(\approx, \approx^Y)$ *is 2-good.*

Before giving the proof, we need to look at projections of subsets of P^*.

Definition 8.5. Let $A \in [\omega]^\omega$. $x_A = x|_{\mathrm{dom} x \cap A}$. $f_A = f|_A$. If $Q \subseteq P^*$, then $Q_A = \{x_A : x \in Q\}$.

Now we prove Theorem 8.4.

Proof: Again, (a), (c), (d), (e), (f), (h) in Definition 8.3 are immediate.

For (b): if $x \notin \bigcup_{\rho \leq \alpha} P_\rho$, then since $\approx$ is good, there is f with $N(x, f) \cap \bigcup_{\rho \leq \alpha} P_\rho = \emptyset$, and we're done.

Otherwise, $x \in P_\alpha \setminus \bigcup_{\gamma < \beta} P_{\alpha,\gamma}$. There is f so $N(x, f) \cap \bigcup_{\beta < \alpha} P_\beta = \emptyset$ and if $z \in N(x, f) \cap P_\alpha$, then $z \approx x$. There is a unique $y \in Y$ with $x \approx y$. Let $A = \mathrm{dom}\ (x \setminus y)$. There are two cases.

Case 1. A is infinite. By hypothesis, $x \setminus y \notin \bigcup_{\gamma < \beta} P_\gamma$. There is $g > f$ so $N(x, g)_A \cap (\bigcup_{\gamma < \beta} P_\gamma)_A = \emptyset$. Then $N(x, g)$ is the desired neighborhood disjoint from $\bigcup_{\rho < \alpha} P_\rho \cup \bigcup_{\gamma < \beta} P_{\alpha,\gamma}$.

Case 2. A is finite. Then $x \subsetneq y$. Let $g > y, f$. Again, $N(x, g)$ is the desired neighborhood.

For (g): for each x, let $y(x)$ be the unique element of Y with $x \approx y(x)$, and define $g_x \gg y(x), f_x, f_{x \setminus y(x)}$, where $f_x, f_{x \setminus y(x)}$ satisfy condition (g) in Definition 5.1. Suppose $N(x, g_x) \cap N(w, g_w) \neq \emptyset$. Either $x \prec w$ or $w \prec x$ or $x \approx w$. In the first two cases, we're done. In the latter case, $y(x) \approx x \approx w \approx y(w)$, so set $y = y(x) = y(w)$. Since each $g_x \gg y(x)$, by the technique of the proof of Theorem 4.5(3), if $z \in N(x, g_x) \cap N(w, g_w)$, then $x \cap y =^* z \cap y =^* w \cap y$. Hence, either $x \setminus y \prec w \setminus y$ or $w \setminus y \prec x \setminus y$ or $x \setminus y \approx w \setminus y$, which completes the proof. $\square$

To get strongly paracompact subspaces out of a 2-good sequence $(\approx, \approx^Y)$, we need another definition.

Definition 8.6. For $E \subseteq P^*$, $E^{\nabla\approx} = \{y^{\nabla\approx} : y \in E\}$

Theorem 8.7. *If $\approx$ is good, $ht(\approx) < \mathfrak{b}$, Y is a maximal $\approx$-transversal, $(\approx, \approx^Y)$ is 2-good, and $Z \supset Y$ is a maximal $\approx^Y$-transversal of P^*, then Z is strongly ultraparacompact.*

Proof: Y is strongly ultraparacompact; it remains to show that $Z \setminus Y^{\nabla_{\approx^Y}}$ is strongly ultraparacompact in $P^* \setminus Y^{\nabla_{\approx^Y}}$. Define $Z_\alpha = (Z \setminus Y^{\nabla_{\approx^Y}}) \cap P_\alpha, Z_{\alpha,\beta} = (Z \setminus Y^{\nabla_{\approx^Y}}) \cap P_{\alpha,\beta}$. Imitating the proofs in section 5, for each α, $\beta < \lambda$, $Z_{\alpha,\beta}$ is ultraparacompact in $P^* \setminus (Y^{\nabla_{\approx^Y}} \cup \bigcup_{\gamma<\alpha} P_\gamma \cup \bigcup_{\gamma<\beta} P_{\alpha,\gamma})$. Hence, for each $\alpha < \lambda$, Z_α is strongly ultraparacompact in $P \setminus \bigcup_{\gamma<\alpha} Z_\gamma$. So $Z \setminus Y^{\nabla_{\approx^Y}} = \bigcup_{\alpha<\lambda} Z_\alpha$ is strongly ultraparacompact in $P^* \setminus Y^{\nabla_{\approx^Y}}$. $\square$

As in corollaries 5.4 and 5.8, we have the following corollary.

Corollary 8.8. *If $\approx$ is good, $ht(\approx) < \mathfrak{b}$, Y is a maximal $\approx$-transversal, $(\approx, \approx^Y)$ is 2-good, $Z \supset Y$ is a maximal $\approx^Y$-transversal of P^*, and D is a downward $\approx^Y$-completion of Z, then D is strongly ultraparacompact.*

Now we iterate this process: By induction, we define the notion of α-good for $\alpha \le \mathfrak{b}$. Note that 1-good = good.

First, we provide some notation.

Definition 8.9. (1) If $\tau, \sigma \in \lambda^\beta$, then $\tau < \sigma$ iff $\exists \gamma < \beta\ \tau|_\gamma = \sigma|_\gamma$ and $\tau(\gamma) < \sigma(\gamma)$ (lexicographic order). We write $\Delta(\sigma, \tau) = \gamma$.

(2) If $\sigma \in \lambda^\beta$ and $\gamma < \lambda$, then $\sigma^\frown\gamma : \beta + 1 \to \lambda$, $\sigma^\frown\gamma|_\beta = \sigma$, and $\sigma^\frown\gamma(\beta) = \gamma$.

(3) If $\sigma \in \lambda^\beta$, then $||\sigma|| = \beta$.

(4) If $\sigma \in \lambda^\beta$, then $S_\sigma = \{\tau : \tau < \sigma \text{ and } \Delta(\sigma, \tau) = ||\tau|| - 1\}$.

Fact 8.10. *If $\sigma \in \lambda^\beta$ and $\beta \le \lambda$, then $|S_\sigma| \le \lambda$.*

Proof: Let $S_\alpha = \{\tau < \sigma : \tau|_\alpha = \sigma|_\alpha \text{ and } \tau(\alpha) < \sigma(\alpha)\}$. Each $|S_\alpha| < \lambda$, and $S_\sigma = \bigcup_{\alpha<\beta} S_\alpha$. $\square$

Definition 8.11. Given a sequence of equivalence relations $\{\approx_\beta : \beta < \alpha\}$, we say that $\{\approx_\beta : \beta < \alpha\}$ is coherent if and only if there is a pairwise disjoint family of sets $\{Y_\beta : \beta < \alpha\}$ where each $\bigcup_{\gamma\le\beta} Y_\gamma$ is a maximal $\approx_\beta$-transversal and

(a) if $\gamma < \beta$, then $\approx_\beta$ refines $\approx_\gamma$;

(b) each $\approx_{\beta+1} = \approx_\beta^{\bigcup_{\gamma\le\beta} Y_\gamma}$;

(c) if β is a limit, then $x \approx_\beta w$ if and only if $\forall \gamma < \beta\ x \approx_\gamma w$;

(d) $x \prec_{\beta+1} w$ if and only if either $\exists\gamma \leq \beta\ x \prec_\gamma w$ or $\exists y \in \bigcup_{\gamma\leq\beta} Y_\gamma\ x \approx_\beta y \approx_\beta w, x \cap y =^* w \cap y$, and $x \setminus y \prec_\beta w \setminus y$;
(e) if β is a limit, then $x \prec_\beta w$ if and only if $\exists\gamma < \beta\ x \prec_\gamma w$.

Definition 8.12. A coherent sequence of equivalence relations $\{\approx_\beta : \beta < \alpha\}$ is α-good if and only if there is $\lambda < \mathfrak{b}$ and a family $\{P_\sigma : \sigma \in \lambda^{<\alpha}\}$ where

(a) if $x =^* y$, then $x \approx_\beta y$ for all β;
(b) each $\bigcup_{||\tau||<\alpha} P_\tau$ is a closed subset of P^*;
(c) each $\bigcup_{||\tau||\leq||\sigma||} P_\tau \cup \bigcup_{\gamma<\beta} P_{\sigma\frown\gamma}$ is a closed subset of P^*;
(d) if $x \in P_{\sigma\frown\beta}$ and $y \prec_{\approx_{||\sigma||+1}} x$, then $y \in P_{\sigma\frown\gamma}$ for some $\gamma < \beta$ or $y \in P_\tau$ for some $\tau \subsetneq \sigma$;
(e) if $w, z \in P_\sigma$ and $w, z \prec_{\approx_{||\sigma||}} x$, then $w \approx_{||\sigma||} z$;
(f) $\forall\beta\ \forall x \in P^*\ \exists g_{x,\beta}$ so if $N(x, g_{x,\beta}) \cap N(y, g_{y,\beta}) \neq \emptyset$, then either $x \prec_{\approx_\beta} y$ or $y \prec_{\approx_\beta} x$ or $x \approx_\beta y$;
(g) if $x \subseteq^* y$, then $y \not\prec_{\approx_\beta} x$ for all β.

The following is immediate.

Lemma 8.13. (1) *If $\{\approx_\beta : \beta \leq \alpha\}$ is coherent via $\{Y_\beta : \beta \leq \alpha\}$ and is α-good, then $\{\approx_\beta : \beta \leq \alpha\}$ is $\alpha+1$-good, where $\approx_\alpha$ is defined as in Definition 8.11 (with respect to $\bigcup_{\beta\leq\alpha} Y_\beta$).*

(2) *If α is a limit and for all $\beta < \alpha$ there is Y_γ so $\{\approx_\gamma : \gamma < \beta\}$ is coherent via $\{Y_\gamma : \gamma < \beta\}$ and is β-good, then $\{\approx_\beta : \beta < \alpha\}$ is α-good.*

(3) *If α is a limit and for all $\beta < \alpha$ there is Y_β so $\{\approx_\beta : \beta < \beta < \alpha\}$ is coherent via $\{Y_\beta : \beta < \alpha\}$ and is α-good, $Y_\alpha \cap \bigcup_{\beta<\alpha} Y_\beta = \emptyset, Y_\alpha \cup \bigcup_{\beta<\alpha} Y_\beta$ is a maximal $\approx_\alpha$-transversal, and $\approx_{\alpha+1}$ is defined as in Definition 8.11 with respect to $Y_\alpha \cup \bigcup_{\beta<\alpha} Y_\beta$, then $\{\approx_\beta : \beta \leq \alpha\}$ is $\alpha + 1$-good.*

Theorem 8.14. *If $\alpha \leq \mathfrak{b}$ and $\{\approx_\beta : \beta < \alpha\}$ is α-good via $\{Y_\beta : \beta < \alpha\}$, then $\bigcup_{\beta<\alpha} Y_\beta$ is ultraparacompact; if $\alpha < \mathfrak{b}$, then $\bigcup_{\beta<\alpha} Y_\beta$ is strongly ultraparacompact; if D_β is a downward $\approx_\beta$-completion of Y_β for each $\beta < \alpha$, then $\bigcup_{\beta<\alpha} D_\beta$ is ultraparacompact and strongly ultraparacompact if $\alpha < \mathfrak{b}$.*

Proof: At successor stages, the proof that each $Y_{\beta+1}$ is strongly paracompact in $P^* \setminus \bigcup_{\gamma\leq\beta} Y_\gamma^{\nabla_{\beta+1}}$ is similar that of Theorem 8.7.

Consider β a limit, and by induction, assume that $\bigcup_{\gamma<\beta} Y_\gamma$ is strongly paracompact. For each $y \in Y_\beta$, let y_γ be the unique element of Y_γ with $y_\gamma \approx_\gamma y$, and let $f_{y,\gamma}$ so that $\{N(y_\gamma, f_{y,\gamma}) : y \in Y_\beta\}$ is a closed separated cover of Y_γ. Let $f_y >^* f_{y,\gamma}$ for all γ. Then $\mathcal{N} = \{N(y, f_y) : y \in Y_\beta\}$ is a pairwise disjoint cover of Y_β by condition (f). $\bigcup \mathcal{N}$ is closed, since, for each $x \in P^*$, there is at most one $y(x) \in Y_\beta$ with $x \approx_\beta y(x)$. If x is compatible with $N(y, f_y) \in \mathcal{N}$, then $y = y(x)$. So either $x \in N(y(x), f_{y(x)})$ or there is a neighborhood M of X with $M \cap N(y(x), f_{y(x)}) = \emptyset$.

For downward $\approx_\beta$-completions, use the techniques of Corollary 5.8. $\square$

So the following conjecture would prove that $\square(\omega+1)^\omega$ is paracompact.

Conjecture. *There is a $\mu \leq \mathfrak{b}$ and a μ-good sequence of equivalence relations $\{\approx_\beta : \beta < \mu\}$ so that if $\{Y_\beta : \beta < \mu\}$ witnesses $\{\approx_\beta : \beta < \mu\}$ is μ-good and D_β is a downward $\approx_\beta$-extension of Y_β for each $\beta < \mu$, then $\bigcup_{\beta<\mu} D_\beta$ is a maximal fine transversal.*

By diagonalization, if $\mathfrak{b} = \mathfrak{c}$, the conjecture is true (for $\mu = \mathfrak{b}$).

References

[1] Eric K. van Douwen, *The box product of countably many metrizable spaces need not be normal*, Fund. Math. **88** (1975), no. 2, 127–132.

[2] ———, *Another nonnormal box product*, General Topology and Appl. **7** (1977), no. 1, 71–76.

[3] ———, *Covering and separation properties of box products*, in Surveys in General Topology. Ed. G. M. Reed. New York-London-Toronto, Ont.: Academic Press, 1980. 55–129

[4] Stephen H. Hechler, *On the existence of certain cofinal subsets of ${}^\omega\omega$*, in Axiomatic Set Theory (Proc. Sympos. Pure Math., Vol. XIII, Part II, Univ. California, Los Angeles, Calif., 1967). Providence, R.I.: Amer. Math. Soc., 1974. 155–173

[5] Kenneth Kunen, *Paracompactness of box products of compact spaces*, Trans. Amer. Math. Soc. **240** (1978), 307–316.

[6] ———, *Box products of ordered spaces*, Topology Appl. **20** (1985), no. 3, 245–250.

[7] L. Brian Lawrence, *The box product of countably many copies of the rationals is consistently paracompact*, Trans. Amer. Math. Soc. **309** (1988), no. 2, 787–796.

[8] ———, *Failure of normality in the box product of uncountably many real lines*, Trans. Amer. Math. Soc. **348** (1996), no. 1, 187–203.

[9] Judy Roitman, *More paracompact box products*, Proc. Amer. Math. Soc. **74** (1979), no. 1, 171–176.

[10] ———, *Paracompactness of box products and their subspaces*, Topology Proc. **31** (2007), no. 1, 265–281.

[11] Mary Ellen Rudin, *The box product of countably many compact metric spaces*, General Topology and Appl. **2** (1972), 293–298.

[12] ———, *Lectures on Set Theoretic Topology.* Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 23. Providence, R.I.: American Mathematical Society, 1975.

[13] Scott W. Williams, *Box products*, in Handbook of Set-Theoretic Topology. Ed. Kenneth Kunen and Jerry E. Vaughan. Amsterdam: North-Holland Publishing Co., 1984. 169–200

[14] Louis Wingers, *Box products of σ-compact spaces*, Topology Appl. **56** (1994), no. 2, 185–197.

[15] ———, *Box products and Hurewicz spaces*, Topology Appl. **64** (1995), no. 1, 9–21.

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