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# ON HEREDITARILY DECOMPOSABLE HOMOGENEOUS CONTINUA

SERGIO MACÍAS AND SAM B. NADLER, JR.

ABSTRACT. Professors J. Krasinkiewicz and P. Minc asked independently, *Is the simple closed curve the only nondegenerate hereditarily decomposable homogeneous continuum?* We present some partial answers to this question.

## 1. INTRODUCTION

J. Krasinkiewicz [7, Problem 156] (and, independently, P. Minc [19, Problem 81]) asked, *Is the simple closed curve the only nondegenerate hereditarily decomposable homogeneous continuum?* Charles L. Hagopian answered the question in the affirmative when the continuum is planar [13, Theorem 2]; he did this in the process of generalizing a theorem of R. H. Bing [3]. T. Maćkowiak and E. D. Tymchatyn gave an affirmative answer to the question when the continuum is atriodic [21, (14.8)]. Here, we obtain a number of diverse sufficient conditions for an affirmative answer to

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*Key words and phrases.* continuous decomposition, continuum, essential map, hereditarily decomposable continuum, hereditarily weakly confluent map, homogeneous continuum, irreducible map, monotone map, property of Kelley, property of Kelley hereditarily, set function $\mathcal{T}$, set function $\mathcal{Y}$, upper semicontinuous decomposition, weakly confluent map.

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Krasinkiewicz's and Minc's question (Theorem 4.1). We also obtain several results that would be of interest if there is a negative answer to the question (section 5).

Regarding Krasinkiewicz's and Minc's question, we observe that if $X$ is a decomposable homogeneous continuum such that each proper subcontinuum of $X$ is locally connected, then $X$ itself is locally connected; the reason is that $X$ is the union of two locally connected continua [18, Theorem 1, p. 230]. We also observe that $X$ is one-dimensional [23, 13.57]. Hence, since the simple closed curve and the Menger curve are the only two locally connected one-dimensional homogeneous continua [2, Theorem XIII], $X$ is a simple closed curve. We extend this result in part (5) of Theorem 4.1.

In our proofs, we leave to the reader the case when the space is degenerate.

## 2. Definitions and notation

If $(Z, d)$ is a metric space, then given $A \subset Z$ and $\varepsilon > 0$, the open ball about $A$ of radius $\varepsilon$ is denoted by $\mathcal{V}_\varepsilon(A)$, the interior of $A$ is denoted by $Int_Z(A)$, the boundary of $A$ is denoted by $Bd_Z(A)$, and the closure of $A$ is denoted by $Cl_Z(A)$. We use dim to denote dimension.

Let $Z$ be a metric space and let $\mathcal{G}$ be a decomposition of $Z$. The group of homeomorphisms, $H(Z)$, of $Z$ *respects* $\mathcal{G}$ if for each $G \in \mathcal{G}$ and each $h \in H(Z)$, $h(G) \in \mathcal{G}$.

A *continuum* is a nonempty compact, connected metric space. A continuum $X$ is *colocally connected at a point* $p \in X$ provided that there is a local base about $p$ of open sets whose complements are connected. A continuum $X$ is *almost connected im kleinen at a point* $p \in X$ if each open subset of $X$ containing $p$ also contains a subcontinuum with nonempty interior. The continuum $X$ is *connected im kleinen at a point* $p$ provided that there is a local base about $p$ of connected neighborhoods. A continuum $X$ is *locally connected* at a point $p$ if there is a local base about $p$ of open connected sets.

A *graph* is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only at one or both of their endpoints. A *tree* is a graph that does not contain simple closed curves. A continuum $X$ is *tree-like* provided that for each $\varepsilon > 0$, there exist a tree $T$ and a surjective continuous function $f\colon X \to T$ such that $\operatorname{diam}\bigl(f^{-1}(t)\bigr) < \varepsilon$ for each $t \in T$.

A point $p$ of a continuum $X$ is a *local separating point* of $X$ provided that there exists a compact neighborhood $R$ of $p$ in $X$ such that $R \setminus \{p\} = M_1 \cup M_2$, $M_1 \cap C \neq \emptyset$ and $M_2 \cap C \neq \emptyset$, where $M_1$ and $M_2$ are mutually separated and $C$ is the component of $R$ containing $p$. A continuum $X$ is a *rational curve* if it has a basis of open sets whose boundaries are at most countable.

A *map* means a continuous function. A surjective map $f\colon X \to Y$ between continua is said to be *irreducible* if for each proper subcontinuum $K$ of $X$, $f(K) \neq Y$. A map $f$ is *essential* provided that it is not homotopic to a constant map.

A continuum $X$ is *homogeneous* if for each pair of points $x_1$ and $x_2$ of $X$, there is a homeomorphism $h\colon X \to X$ such that $h(x_1) = x_2$. Given a continuum $X$ and $\varepsilon > 0$, a homeomorphism $h\colon X \to X$ is an *$\varepsilon$-homeomorphism* provided that $d(x, h(x)) < \varepsilon$ for every $x \in X$. A continuum $X$ has the *property of Effros* provided that for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $x_1$ and $x_2$ are two points of $X$ and $d(x_1, x_2) < \delta$, then there exists an $\varepsilon$-homeomorphism $h\colon X \to X$ such that $h(x_1) = x_2$. The number $\delta$ is called an *Effros number* for the given $\varepsilon$. It is known that homogeneous continua have the property of Effros [20, 4.2.31].

If $f\colon X \to Y$ is a map between continua, a *stable value* of $f$ is a value $q$ of $f$ for which there exists $\varepsilon > 0$ such that for any map $g\colon X \to Y$ such that $\rho(f, g) < \varepsilon$, $q \in g(X)$, where $\rho$ is the supremum metric. We say that a continuum $Y$ is in $Class(S)$ provided that every map of any continuum onto $Y$ has a stable value [24].

A continuum $X$ has the *property of Kelley at a point $p$* provided that for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $q \in X$, $d(p, q) < \delta$, and $P$ is a subcontinuum of $X$ containing $p$, there exists a subcontinuum $Q$ of $X$ containing $q$ such that $\mathcal{H}(P, Q) < \varepsilon$, where $\mathcal{H}$ is the Hausdorff metric [22, (0.1)]. The continuum $X$ has the *property of Kelley* if it has the property of Kelley at each of its points. The continuum $X$ has the *property of Kelley hereditarily* provided that each of its subcontinua has the property of Kelley.

A continuum $X$ is *aposyndetic at a point $p \in X$* provided that for each point $q \in X \setminus \{p\}$, there exists a subcontinuum of $X$

containing $p$ in its interior and not containing $q$. The continuum $X$ is *aposyndetic* if it is aposyndetic at each of its points.

In 1948, F. Burton Jones defined a set function, which is now commonly denoted by $\mathcal{T}$, while studying aposyndetic continua [16]. Given a continuum $X$, the function $\mathcal{T}\colon \mathcal{P}(X) \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of $X$, is defined as follows. If $A \in \mathcal{P}(X)$, then

$$\mathcal{T}(A) = X \setminus \{x \in X \mid \text{there is a subcontinuum } W \text{ of } X \setminus A \text{ such that } x \in Int_X(W)\}.$$

Let us observe that a continuum $X$ is aposyndetic if and only if $\mathcal{T}(\{x\}) = \{x\}$ for every $x \in X$.

The set function $\mathcal{Y}$ was defined by Eugene L. VandenBoss [28] as follows:

Given a continuum $X$, $\mathcal{Y}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by:

For all $A \in \mathcal{P}(X)$,

$$\mathcal{Y}(A) = X \setminus \{x \in X \mid \text{there is a subcontinuum } W \text{ of } X \setminus A \text{ such that } x \in Int_X(W) \text{ and } Int_X(W) \text{ is connected}\}.$$

Other terminology and notation are standard and can be found in [18] and [23].

## 3. Preliminary results

We prove some preliminary results needed for the proof of our main theorem.

The following result is a generalization of [1, Theorem 2.1].

**Proposition 3.1.** *If $X$ is a continuum with the property of Kelley at the point $p$ and $p$ is a local separating point, then $X$ is connected im kleinen at $p$.*

*Proof:* Let $\varepsilon_0 > 0$ be given. Let $R$ be a compact neighborhood of $p$ in $X$ such that $R \setminus \{p\} = M_1 \cup M_2$, and $M_j \cap C \neq \emptyset$, $j \in \{1, 2\}$, where $M_1$ and $M_2$ are mutually separated and $C$ is the component of $R$ containing $p$. Let $\varepsilon \in (0, \frac{\varepsilon_0}{2})$ be such that $Cl\,(\mathcal{V}_{2\varepsilon}(p)) \subset R$. Let $\delta \in (0, \varepsilon)$ be given by the property of Kelley of $X$ at $p$ for $\varepsilon$. Let $L_p$ be the component of $Cl\,(\mathcal{V}_{\varepsilon}(p))$ containing $p$. Note that $L_p \subset C$ and $L_p \cap M_j \neq \emptyset$, $j \in \{1, 2\}$. Let $x \in \mathcal{V}_\delta(p) \setminus \{p\}$. Then there exists a subcontinuum $L_x$ of $X$ such that $x \in L_x$ and $\mathcal{H}(L_x, L_p) < \varepsilon$. Observe that $L_x \subset \mathcal{V}_{2\varepsilon}(p) \subset R$, and $L_x \cap M_j \neq \emptyset$, $j \in \{1, 2\}$. Thus, $p \in L_x$ and $L_x \subset C$. Let $K = \bigcup\{L_x \mid x \in \mathcal{V}_\delta(p)\}$. Hence, $K$ is a connected neighborhood of $p$ in $X$ and $K \subset \mathcal{V}_{\varepsilon_0}(p)$. Therefore, $X$ is connected im kleinen at $p$. □

**Theorem 3.2.** *Let $X$ be a decomposable homogeneous continuum. If $f\colon X \to \mathcal{S}^1$ is an irreducible map, then $f$ is essential.*

*Proof:* Suppose $f\colon X \to \mathcal{S}^1$ is inessential. Then there exists a map $\varphi\colon X \to \mathbb{R}$ such that $f(x) = e^{i\varphi(x)}$ [31, Theorem (6.2), p. 225]. Since $f$ is irreducible, without loss of generality, we assume that $\varphi(X) = [0, 2\pi]$. Let $x_0 \in \varphi^{-1}(0)$ and let $x_{2\pi} \in \varphi^{-1}(2\pi)$. Let $K$ be an irreducible subcontinuum of $X$ between $x_0$ and $x_{2\pi}$ [31, (11.2), p. 17]. Note that $\varphi(K) = [0, 2\pi]$. Hence, $f(K) = e^{i\varphi(K)} = \mathcal{S}^1$. Thus, since $f$ is an irreducible map, $K = X$. Therefore, since $K$ is irreducible between $x_0$ and $x_{2\pi}$, $X$ is irreducible between $x_0$ and $x_{2\pi}$. Thus, since $X$ is homogeneous, every point of $X$ is a point of irreducibility. Hence, $X$ is indecomposable [23, Corollary 11.19], a contradiction. Therefore, $f$ is essential. □

**Theorem 3.3.** *If $X$ is an aposyndetic continuum and $p$ is a noncut point of $X$ such that every subcontinuum of $X$ containing $p$ has the property of Kelley, then $X$ is connected im kleinen at $p$.*

*Proof:* Suppose $X$ is not connected im kleinen at the point $p$. Then there exist an open set $U$ of $X$ containing $p$ and a sequence $\{K_n\}_{n=1}^{\infty}$ of components of $Cl_X(U)$ converging to a continuum $K$ such that $p \in K \subset Cl_X(U)$ and $K_n \cap K = \emptyset$ for every positive integer $n$ [31, (12.1), p. 18]. Since $X$ is aposyndetic and $p$ is a noncut point of $X$, by [20, 3.1.30] and [31, (4.14), p. 50], $X$ is colocally connected at $p$. Thus, there exists an open set $V$ of $X$ such that $p \in V \subset Cl_X(V) \subset U$ and $X \setminus V$ is connected. Observe that $K \cap (X \setminus V) \neq \emptyset$. Let $q \in Bd_X(V) \cap K$. Let $r < \frac{1}{2}\min\{d(p,q), d\,(Cl_X(V), X \setminus U)\}$. Let $\{q_\ell\}_{\ell=1}^{\infty}$ be a sequence of points in $(\mathcal{V}_r(q) \cap Bd_X(V)) \setminus K$ converging to $q$. For each positive integer $\ell$, let $R_\ell$ be the component of $Cl_X\,(\mathcal{V}_r(q))$ containing $q_\ell$. Note that $(X \setminus V) \cup R_\ell$ is connected [23, 5.4] and $Cl_X\,(\cup_{\ell=1}^{\infty} R_\ell) \setminus \cup_{\ell=1}^{\infty} R_\ell \subset K$. Let $L = K \cup (X \setminus V) \cup (\cup_{\ell=1}^{\infty} R_\ell)$. Then $L$ is a subcontinuum of $X$ containing $p$ without the property of Kelley. □

**Corollary 3.4.** *If $X$ is an aposyndetic continuum with the property of Kelley hereditarily, then $X$ is locally connected.*

*Proof:* Suppose $X$ is not connected im kleinen at the point $p$. Then there exist an open set $U$ of $X$ containing $p$ and a sequence $\{K_n\}_{n=1}^{\infty}$ of components of $Cl_X(U)$ converging to a continuum $K$ such that $p \in K \subset Cl_X(U)$ and $K_n \cap K = \emptyset$ for every positive integer $n$ [31, (12.1), p. 18]. Note that $K$ is a convergence continuum [23, 5.11]. Hence, $K$ contains uncountably many noncut points of $X$ [23, 6.29(b)]. Without loss of generality, we assume that $p$ is a noncut point of $X$. Now the corollary follows from Theorem 3.3. □

**Theorem 3.5.** *If $X$ is a homogeneous continuum, then the following are equivalent.*

(1) *$X$ is locally connected;*
(2) *$X$ is locally connected at some point;*
(3) *$X$ is connected im kleinen at some point;*
(4) *$X$ is almost connected im kleinen at every point of $X$;*
(5) *$X$ is almost connected im kleinen at some point of $X$.*

*Proof:* We only prove that (5) implies (1), the other implications are clear.

Suppose $X$ is almost connected im kleinen at a point $x$. Let $\varepsilon > 0$ and let $\delta > 0$ be an Effros number for $\frac{\varepsilon}{2}$. Without loss of generality, we assume $\delta < \frac{\varepsilon}{2}$. Since $X$ is almost connected im kleinen at $x$, there exists a subcontinuum $W$ of $X$ such that $Int_X(W) \neq \emptyset$ and $W \subset \mathcal{V}_\delta(x)$. Let $w \in Int_X(W)$. Then $d(x, w) < \delta$. Hence, there exists an $\frac{\varepsilon}{2}$–homeomorphism $h\colon X \to X$ such that $h(w) = x$. Thus, $x \in Int_X(h(W))$ and $h(W) \subset \mathcal{V}_\varepsilon(x)$. Therefore, $X$ is connected im kleinen at $x$. Since $X$ is homogeneous, $X$ is connected im kleinen at every point; hence, $X$ is locally connected [20, 1.7.12]. □

The following result is by VandenBoss [28, Theorem 28]; we include its proof for the convenience of the reader.

**Theorem 3.6.** *Let $X$ be a continuum, and let $A$ and $B$ be two nonempty closed subsets of $X$. If $A \subset Int_X(B)$ and $\mathcal{T}(B) = B$, then $\mathcal{Y}(A) \subset B$.*

*Proof:* First we show that the components of $X \setminus B$ are open. Let $L$ be a component of $X \setminus B$, and let $x \in L$. Since $x \notin \mathcal{T}(B)$, there exists a subcontinuum $W$ of $X$ such that $x \in Int_X(W) \subset W \subset B$. Hence, $W \subset L$ and $x \in Int_X(L)$. Therefore, $L$ is open.

To show $\mathcal{Y}(A) \subset B$, let $p \in X \setminus B$, and let $L$ be the component of $X \setminus B$ containing $p$. Then $W = Cl_X(L)$ is a subcontinuum of $X$ such that $p \in Int_X(W) \subset W \subset X \setminus Int_X(B) \subset X \setminus A$, and $Int_X(W)$ is connected. Thus, $p \in X \setminus \mathcal{Y}(A)$. Therefore, $\mathcal{Y}(A) \subset B$. □

The following is a modification of a result of VandenBoss [28, Corollary 30].

**Theorem 3.7.** *Let $X$ be a continuum such that $\mathcal{T}(\mathcal{T}(A)) = \mathcal{T}(A)$ for every nonempty closed subset $A$ of $X$. Then $\mathcal{Y}(A) = \mathcal{T}(A)$ for each nonempty closed subset $A$ of $X$.*

*Proof:* Let $A$ be a nonempty closed subset of $X$. We know, from the definitions, that $\mathcal{T}(A) \subset \mathcal{Y}(A)$. Let $x \in X \setminus \mathcal{T}(A)$. Then there exists an open subset $U$ of $X$ such that $A \subset U$ and $x \in X \setminus \mathcal{T}\left(Cl_X(U)\right)$ [20, 3.1.20]. Since $\mathcal{T}\left(\mathcal{T}\left(Cl_X(U)\right)\right) = \mathcal{T}\left(Cl_X(U)\right)$, and since $A \subset U \subset Int_X\left(\mathcal{T}\left(Cl_X(U)\right)\right)$, by Theorem 3.6, $\mathcal{Y}(A) \subset \mathcal{T}\left(Cl_X(U)\right)$. Hence, $x \in X \setminus \mathcal{Y}(A)$. Therefore, $\mathcal{Y}(A) = \mathcal{T}(A)$. □

**Corollary 3.8.** *If $X$ is a homogeneous continuum, then $\mathcal{Y}(A) = \mathcal{T}(A)$ for each nonempty closed subset $A$ of $X$.*

*Proof:* Since $X$ is a homogeneous continuum, $\mathcal{T}(\mathcal{T}((A)) = \mathcal{T}(A)$ for every nonempty closed subset $A$ of $X$ [20, 4.2.32]. Then the corollary now follows from Theorem 3.7. □

**Theorem 3.9.** *Let $X$ be an aposyndetic continuum such that $\mathcal{T}(\mathcal{T}(A)) = \mathcal{T}(A)$ for every nonempty closed subset $A$ of $X$. If $\mathcal{T}(W) = W$ for each subcontinuum $W$ of $X$ such that $Int_X(W)$ is nonempty and connected, then $X$ is locally connected.*

*Proof:* Note that, by [20, 3.1.32], it is enough to show that $\mathcal{T}(A) = A$ for each subcontinuum $A$ of $X$. Let $A$ be a subcontinuum of $X$. It is clear from the definition of $\mathcal{T}$ that $A \subset \mathcal{T}(A)$. Let $x \in X \setminus A$. By Theorem 3.7, $\mathcal{T}(A) = \mathcal{Y}(A)$; thus, since $X$ is aposyndetic, for each $a \in A$ there exists a subcontinuum $W_a$ of $X$ such that $a \in Int_X(W_a) \subset W_a \subset X \setminus \{x\}$ and $Int_X(W_a)$ is connected. Since $A$ is compact, there exist $a_1, \ldots, a_n \in A$ such that $A \subset \bigcup_{j=1}^{n} Int_X(W_{a_j}) \subset \bigcup_{j=1}^{n} W_{a_j} \subset X \setminus \{x\}$. Observe that $\bigcup_{j=1}^{n} Int_X(W_{a_j})$ is connected. Let $W = Cl_X\left(\bigcup_{j=1}^{n} Int_X(W_{a_j})\right)$. Then $W$ is a subcontinuum of $X$ such that $A \subset W \subset X \setminus \{x\}$ and $Int_X(W)$ is connected. By hypothesis, $\mathcal{T}(W) = W$. Hence,

$\mathcal{T}(A) \subset \mathcal{T}(W) = W \subset X \setminus \{x\}$. Thus, $x \in X \setminus \mathcal{T}(A)$. Therefore, $\mathcal{T}(A) = A$. □

Let us recall the following definition of hyperspaces. If $X$ is a continuum, then let

$$\mathcal{C}(X) = \{A \subset X \mid A \text{ is a subcontinuum of } X\}.$$

$\mathcal{C}(X)$ is called the *hyperspace of subcontinua of* $X$ and is topologized using the Hausdorff metric [22, (0.1)]. We refer the reader to [22] and [15] for information about hyperspaces.

**Theorem 3.10.** *Let* $X$ *be an aposyndetic continuum. If* $\dim(\mathcal{C}(X)) < \infty$*, then* $X$ *is a finite graph, hence, locally connected.*

*Proof:* Since $\dim(\mathcal{C}(X)) < \infty$, it follows easily that there exists a positive integer $n$ such that $\mathcal{C}(X)$ does not contain an $n$-cell. Hence, $X$ does not contain $n$-ods [15, Theorem 70.1]. Note that this implies that the complement of any proper subcontinuum of $X$ has at most $n-1$ components.

To show that $X$ is locally connected, by [20, 3.1.32], it suffices to prove that $\mathcal{T}(A) = A$ for ever subcontinuum $A$ of $X$. Let $A$ be a subcontinuum $A$ of $X$; we know that $A \subset \mathcal{T}(A)$. Let $x \in X \setminus A$. As in the proof of Theorem 3.9, we can construct a subcontinuum $W$ of $X$ such that $A \subset Int_X(W) \subset W \subset X \setminus \{x\}$. By the previous paragraph, $X \setminus W$ has only finitely many components. Let $K$ be the component of $X \setminus W$ containing $x$. Then, since $X \setminus W$ is open and has only finitely many components, $K$ is open in $X$. Thus, $Cl_X(K)$ is a subcontinuum of $X$ such that $x \in Int_X(Cl_X(K)) \subset X \setminus Int_X(W) \subset X \setminus A$. Hence, $x \in X \setminus \mathcal{T}(A)$. Therefore, $\mathcal{T}(A) = A$, and $X$ is locally connected. By [22, (1.109)], $X$ is a finite graph. □

**Lemma 3.11.** *Let* $X$ *be a continuum which is not separated by any subcontinuum. If* $f\colon X \to \mathcal{S}^1$ *is a surjective monotone map with nowhere dense fibers, then* $f$ *is irreducible.*

*Proof:* Let $L$ be a subcontinuum of $X$ such that $f(L) = \mathcal{S}^1$. Let $x \in X$ and let $B$ be an arc in $\mathcal{S}^1$ such that $f(x) \in Int_{\mathcal{S}^1}(B)$. Then $x \in f^{-1}(f(x)) \subset f^{-1}(Int_{\mathcal{S}^1}(B)) \subset L$ [12, Lemma 1]. Hence, $L = X$. Therefore, $f$ is irreducible. □

## 4. Main result

We state and prove our main theorem.

**Theorem 4.1.** *Let $X$ be a nondegenerate hereditarily decomposable homogeneous continuum. Then $X$ is a simple closed curve if $X$ satisfies any one of the following conditions.*

(1) *$X$ has a local separating point.*
(2) *$X$ is a rational curve.*
(3) *There is an irreducible map of $X$ onto $\mathcal{S}^1$.*
(4) *$\mathcal{C}(X)$ is finite-dimensional.*
(5) *Each proper subcontinuum of $X$ has the property of Kelley.*
(6) *$X$ is almost connected im kleinen at some point.*
(7) *For each subcontinuum $W$ of $X$, $X \setminus W$ has at most countably many components.*
(8) *$\mathcal{T}(W) = W$ for all subcontinua $W$ of $X$ such that $Int_X(W)$ is nonempty and connected.*

*Proof:* Let us note first that, since $X$ is a hereditarily decomposable continuum, $\dim(X) = 1$ [23, 13.57].

(1) Since $X$ is homogeneous, $X$ has the property of Kelley [30, (2.5)]. Hence, by Proposition 3.1, $X$ is locally connected. Note that all save a countable number of the local separating points are points of order two [31, (9.2), p. 61]. Hence, since $X$ is homogeneous, all the points of $X$ are of order two. Therefore, $X$ is a simple closed curve [23, 9.6].

(2) Since $X$ is a rational curve, it has a local separating point [31, (9.43), p. 63]. Hence, by (1), $X$ is a simple closed curve.

(3) Note that, by Theorem 3.2, $f$ is essential. Thus, $f$ is a weakly confluent map [9, Lemma 6]. Hence, since the image of each proper subcontinuum of $X$ under $f$ is an arc, that restriction is weakly confluent [25, p. 236]. Thus, $f$ is a hereditarily weakly confluent map which is irreducible. Then $f$ is monotone and $f^{-1}(z)$ is a nowhere dense subcontinuum of $X$ for every $z \in \mathcal{S}^1$. Furthermore, $f$ is constant on any given nowhere dense subcontinuum of $X$ [8, Theorem 3.1]. Note that, in this case, $X$ is not unicoherent. Let

$$\mathcal{G}_f = \left\{ f^{-1}(z) \mid z \in \mathcal{S}^1 \right\}.$$

Then $\mathcal{G}_f$ is an upper semicontinuous decomposition of $X$. Let $z \in \mathcal{S}^1$ and let $h \colon X \to X$ be a homeomorphism. Then $h\left(f^{-1}(z)\right)$ is a nowhere dense subcontinuum of $X$. Thus, there exists $z' \in \mathcal{S}^1$ such that $h\left(f^{-1}(z)\right) \subset f^{-1}(z')$. Hence, since $h$ is a homeomorphism, $f^{-1}(z) \subset h^{-1}\left(f^{-1}(z')\right)$. Since, $h^{-1}\left(f^{-1}(z')\right)$ is a nowhere dense subcontinuum of $X$, we have, in fact, that $f^{-1}(z) = h^{-1}\left(f^{-1}(z')\right)$, and $h\left(f^{-1}(z)\right) = f^{-1}(z')$. Therefore, the group of homeomorphisms of $X$ respects $\mathcal{G}_f$. Thus, $\mathcal{G}_f$ is a continuous decomposition, and the elements of $\mathcal{G}_f$ are homogeneous, mutually homeomorphic, and indecomposable continua [26, Theorem 1]. Hence, $f^{-1}(z)$ is degenerate for each $z \in \mathcal{S}^1$. Therefore, $f$ is a homeomorphism and $X$ is a simple closed curve.

(4) Since $X$ is homogeneous and hereditarily decomposable, $X$ is aposyndetic [20, 5.1.21]. By Theorem 3.10, $X$ is locally connected. Therefore, $X$ is a simple closed curve [2, Theorem XIII].

(5) Since $X$ is homogeneous, $X$ has the property of Kelley [30, (2.5) Theorem]. Since $X$ is homogeneous and hereditarily decomposable, $X$ is aposyndetic [20, 5.1.21]. Thus, $X$ is locally connected, by Corollary 3.4. Therefore, $X$ is a simple closed curve [2, Theorem XIII].

(6) By Theorem 3.5, $X$ is locally connected. Therefore, $X$ is a simple closed curve [2, Theorem XIII].

(7) A hereditarily decomposable homogeneous continuum is aposyndetic [20, 5.1.21]. Since $X$ is aposyndetic, $X$ is semilocally connected [20, 3.1.30]. Recall that each continuum has at least two nonseparating points [23, 6.6]; thus, since $X$ is homogeneous, $X$ does not have a separating point. Hence, $X$ is colocally connected [31, (4.14), p. 50]. Now, we show that $X$ is almost connected im kleinen. Let $x$ be a point of $X$ and let $U$ be an open subset of $X$ such that $x \in U$. Let $U'$ be an open subset of $X$ such that $x \in U' \subset Cl_X(U') \subset U$. Since $X$ is colocally connected, there exists an open subset $V$ of $X$ such that $x \in V \subset U'$ and $X \setminus V$ is connected. Hence, $X \setminus V$ is a subcontinuum of $X$. By the hypothesis, $V$ has at most countably many components. Hence, by the Baire Category Theorem, at least one of these components, say $C$, has nonempty interior. Thus, $Cl_X(C)$ is a subcontinuum with nonempty interior lying in $U$. This proves $X$ is almost connected im kleinen at $x$. Therefore, $X$ is a simple closed curve by (6).

(8) Since $X$ is a homogeneous continuum, $\mathcal{T}(\mathcal{T}(A)) = \mathcal{T}(A)$ for every nonempty closed subset $A$ of $X$ [20, 4.2.32]. Also, since $X$ is homogeneous and hereditarily decomposable, $X$ is aposyndetic [20, 5.1.21]. Then, by Theorem 3.9, $X$ is locally connected. Therefore, $X$ is a simple closed curve [2, Theorem XIII]. □

Another way to prove that (2) of Theorem 4.1 implies that $X$ is a simple closed curve is as follows: It follows from a theorem of Bing [3, Theorem 2] that a nonlocally connected homogeneous continuum has an open subset with uncountably many components; thus, a homogeneous rational continuum must be locally connected. Hence, since the continuum $X$ is hereditarily decomposable, $X$ must be a simple closed curve [2, Theorem XIII].

Regarding part (5) of Theorem 4.1, we remark that there exists a continuum $X$ such that every nondegenerate proper subcontinuum of $X$ has the property of Kelley, but $X$ itself does not have that property. The example is the modification to the Knaster continuum made by Jack T. Goodykoontz, Jr., by internalizing a simple triod [10, Diagram 2, p. 360]. Another example can be found in [5, Example 7].

## 5. Other results

We present several results of independent interest that give information about the structure of hereditarily decomposable homogeneous continua. The results would be of interest if such continua do not have to be simple closed curves.

**Lemma 5.1.** *If $X$ is a decomposable one-dimensional homogeneous continuum, then there exists an essential map $f\colon X \to \mathcal{S}^1$.*

*Proof:* Let $X$ be a decomposable one-dimensional homogeneous continuum. Since homogeneous tree-like continua are indecomposable [17, Theorem 1], $X$ is not tree-like. Thus, since acyclic homogeneous one-dimensional continua are tree-like [27, Corollary 6.5], there exists an essential map $f\colon X \to \mathcal{S}^1$. □

Let us recall that J. H. Case and R. E. Chamberlin, in [4, Theorem 1], characterized tree-like continua as such continua $X$ for which for each graph $G$, every map $f\colon X \to G$ is inessential. They also constructed a one-dimensional acyclic continuum which is not tree-like [4, pp. 78–83].

Since hereditarily decomposable continua are one-dimensional [23, 13.57], we have the following theorems.

**Theorem 5.2.** *If $X$ is a hereditarily decomposable homogeneous continuum, then there exists an essential map $f\colon X \to \mathcal{S}^1$.*

**Theorem 5.3.** *If $M$ is a hereditarily decomposable continuum not separated by any subcontinuum, then $M$ belongs to $Class(S)$.*

*Proof:* Let $f: X \to M$ be a surjective map. By [29, Theorem 2], there exists an upper semicontinuous decomposition $\mathcal{G}$ of $M$ such that $M/\mathcal{G}$ is a simple closed curve and $Int_M(G) = \emptyset$ for all $G \in \mathcal{G}$. Let $q: M \twoheadrightarrow \mathcal{S}^1$ be the quotient map.

Let $Z$ be an arc in $\mathcal{S}^1$ with $z_1$ and $z_2$ as its two end points. Let $R = Cl_M\left(q^{-1}(Z \setminus \{z_1, z_2\})\right)$. Then $R$ is an irreducible continuum between $q^{-1}(z_1)$ and $q^{-1}(z_2)$, with respect of intersecting them [12, Lemma 1], and $q|_R$ is a monotone map from $R$ to an arc. Hence, the set $\{z \in Z \mid (q|_R)^{-1}(z)$ is a layer of continuity [18, p. 201]$\}$ is a dense $G_\delta$ subset of $Z$ [18, p. 202]. Thus, if $(q|_R)^{-1}(z)$ and $(q|_R)^{-1}(z')$ are two layers of continuity of $Z$, then both $(q|_R)^{-1}(z)$ and $(q|_R)^{-1}(z')$ are terminal subcontinua of $M$ (by [12, Lemma 1] and [11, Lemma 7]), $M \setminus \{(q|_R)^{-1}(z), (q|_R)^{-1}(z')\}$ is not connected, and $M$ belongs to $Class(S)$ [14, Theorem 1]. □

**Corollary 5.4.** *If $X$ is a hereditarily decomposable homogeneous continuum, then each point of $X$ belongs to a subcontinuum $M$ of $X$ in $Class(S)$ which is nonunicoherent; furthermore, there exists an irreducible hereditarily weakly confluent map $q: M \to \mathcal{S}^1$ such that the fibers of $q$ are nowhere dense subcontinua of $M$.*

*Proof:* First note that, by Theorem 5.2, there exists an essential map $r: X \to \mathcal{S}^1$. Hence, by [31, (5.4), p. 222], there exists a subcontinuum $M$ of $X$ such that $r|_M: M \to \mathcal{S}^1$ is essential, $r|_A: A \to \mathcal{S}^1$ is inessential for any proper nonempty closed subset $A$ of $M$, and $M \setminus L$ is connected for all subcontinua $L$ of $M$. Then, by Theorem 5.3, $M$ belongs to $Class(S)$. Since $M$ is hereditarily decomposable and not separated by any subcontinuum, $M$ admits an upper semicontinuous decomposition such that the quotient space is a simple closed curve and the elements of the decomposition are nowhere dense subcontinua of $M$ [29, Theorem 2]. Let $q: M \to \mathcal{S}^1$ be the quotient map. By Lemma 3.11, $q$ is irreducible. Hence, by [8, Corollary 2.4], $q$ is hereditarily weakly confluent. It follows easily that $M$ is not unicoherent. The corollary now follows from the homogeneity of $X$. □

We pose the following question, after which we explain its connection with several results in the paper.

**Question 5.5.** If $X$ is a hereditarily decomposable homogeneous continuum containing a simple closed curve, then is $X$ a simple closed curve?

Suppose we could prove that the continuum $M$ obtained in Corollary 5.4 were homogeneous. Then, since the map $q$ in Corollary 5.4 is irreducible, we see from part (3) of Theorem 4.1 that $M$ would be a simple closed curve. Hence, if Question 5.5 has an affirmative answer, we would then know that the simple closed curve is the only homogeneous hereditarily decomposable continuum. Finally, we note that the only homogeneous plane continuum that contains a simple closed curve is a simple closed curve [6, Theorem 2].

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