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PRZYMUSIŃSKI'S CHARACTERIZATION OF COUNTABLY KATĚTOV SPACES

HARUTO OHTA

Abstract. We determine a subset A of a space X such that $A \times M$ is C-embedded in $X \times M$ for every σ-locally compact, metric space M with $w(M) \leq \kappa$. Using this, we give a proof of a theorem, announced by Teodor C. Przymusiński, which asserts that a space X is countably Katětov if and only if $X \times M$ is rectangularly normal for every σ-locally compact, metric space M.

1. Introduction

A space X is called *countably Katětov* if it is normal and for every closed set A in X, every countable, locally finite open cover $\mathcal{G}$ of A can be extended to a locally finite open cover $\mathcal{H}$ of X; i.e., there is a bijection $e : \mathcal{G} \to \mathcal{H}$ such that $e[G] \cap A = G$ for each $G \in \mathcal{G}$ (see [9]). A space is called *σ-locally compact* if it is the union of countably many locally compact, closed subspaces. A subset A of a space X is said to be *C-embedded* in X if every real-valued continuous function on A extends continuously over X. Recall from [8] that a product space $X \times Y$ is *rectangularly normal* if every closed rectangle, i.e., a closed set of the form $A \times B$, in $X \times Y$ is C-embedded in $X \times Y$. In

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[7, Theorem 4], Teodor C. Przymusiński announced the following theorem.

Theorem 1 (Przymusiński)**.** *The following are equivalent.*

(i) $X \times M$ *is rectangularly normal for every* σ*-locally compact, metric space* M;
(ii) X *is countably Katětov.*

The implication (i) $\to$ (ii) follows from [8, Theorem 2.3], and (ii) $\to$ (i) was proved in [7] assuming that $\dim M = 0$. Przymusiński stated in [7] that he has a very complicated proof that eliminates the assumption of $\dim M = 0$, but his proof has not appeared anywhere until now. He also asked in [7] how to eliminate the assumption of $\dim M = 0$ from the proof in a reasonably simple way. The purpose of this note is to prove the following theorem, which implies Theorem 1 without the assumption that $\dim M = 0$, by applying the technique developed in [6].

Theorem 2. *For a subset* A *of a space* X *and an infinite cardinal* κ*, the following are equivalent.*

(1) $A \times M$ *is* C*-embedded in* $X \times M$ *for every* σ*-locally compact, metric space* M *with* $w(M) \leq \kappa$;
(2) $A \times J(\kappa)$ *is* C*-embedded in* $X \times J(\kappa)$;
(3) A *is* C*-embedded in* X *and every locally finite map* $\mathcal{G} : \omega \to \mathcal{T}_0(A)^\kappa$ *admits a locally finite expansion* $\mathcal{H} : \omega \to \mathcal{T}_0(X)^\kappa$.

Here, we give the definitions of terms and symbols used in the above theorem. A *zero-set* in a space X is a set of the form $f^{-1}(0)$ for some real-valued continuous function f on X and a *cozero-set* is the complement of a zero-set. For a space X, let $\mathcal{T}(X)$ denote the set of all open sets in X and let $\mathcal{T}_0(X)$ denote the set of all cozero-sets in X. Let κ be an infinite cardinal. Following [6], for a map $\mathcal{H} : \Gamma \to \mathcal{T}_0(X)^\kappa$, we define a map $\langle \mathcal{H}, \kappa \rangle : \Gamma \to \mathcal{T}(X)$ by $\langle \mathcal{H}, \kappa \rangle(\gamma) = \bigcup_{\alpha<\kappa} \mathcal{H}(\gamma)(\alpha)$ for $\gamma \in \Gamma$. We say that a map $\mathcal{H} : \Gamma \to \mathcal{T}_0(X)^\kappa$ is *locally finite* if the family $\{\langle \mathcal{H}, \kappa \rangle(\gamma) : \gamma \in \Gamma\}$ is locally finite in X. For a subset A of a space X and two maps $\mathcal{G} : \Gamma \to \mathcal{T}_0(A)^\kappa$ and $\mathcal{H} : \Gamma \to \mathcal{T}_0(X)^\kappa$, we say that $\mathcal{H}$ is an *expansion* of $\mathcal{G}$ if $\mathcal{G}(\gamma)(\alpha) \subseteq \mathcal{H}(\gamma)(\alpha)$ for every $\gamma \in \Gamma$ and $\alpha < \kappa$. Finally, let $J(\kappa)$ be the hedgehog of spininess κ, i.e., $J(\kappa) = \{\theta\} \cup (\kappa \times \omega)$ topologized by declaring that each point of $\kappa \times \omega$ is isolated and a basic neighborhood of the point θ is a set of the form $B(n) = \{\theta\} \cup (\kappa \times (\omega \setminus n))$ for $n < \omega$ (see [4, Example 4.1.5]).

In section 2, we prove Theorem 2, from which we deduce Theorem 1. Section 3 is devoted to applications and remarks. Throughout this note, $|A|$ denotes the cardinality of a set A and ω denotes the first infinite cardinal. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. Other terms and notation will be used as in [4] and [5].

2. Proofs of theorems 1 and 2

Let κ be an infinite cardinal and let $\kappa^{<\omega} = \bigcup_{n<\omega} k^n$. For every $\sigma \in \kappa^n$ and $\alpha < \kappa$, $\sigma\hat{}\alpha \in \kappa^{n+1}$ is defined by $(\sigma\hat{}\alpha)|_n = \sigma$ and $(\sigma\hat{}\alpha)(n) = \alpha$. For a metric space M, a map $\mathcal{S} : \kappa^{<\omega} \to \mathcal{T}(M)$ is called a *strong sieve* on M if

(i) $\mathcal{S}(\emptyset) = M$;
(ii) $\mathcal{S}(\sigma) = \bigcup_{\alpha<\kappa} \mathcal{S}(\sigma\hat{}\alpha)$ for every $\sigma \in \kappa^{<\omega}$;
(iii) $\emptyset \notin \mathcal{S}[\kappa^{<\omega}]$;
(iv) $\mathcal{S}[\kappa^n]$ is locally finite in M for every $n < \omega$;
(v) for each $z \in M$, $\{\mathcal{S}_n(z) : n < \omega\}$ is a local base at z, where $\mathcal{S}_n(z) = \bigcup\{\mathcal{S}(\sigma) : \sigma \in \kappa^n \text{ and } z \in \mathrm{cl}_M\mathcal{S}(\sigma)\}$; and
(vi) diameter $\mathcal{S}(\sigma) \leq 1/n$ for each $\sigma \in \kappa^n$ and $n < \omega$.

A space M is called *weight-homogeneous* if $w(U) = w(M)$ for every nonempty open set U in M. By [6, Lemma 5.5], a metric space M has a strong sieve $\mathcal{S} : \kappa^{<\omega} \to \mathcal{T}(M)$ if it is weight-homogeneous, nowhere locally compact, and $w(M) = \kappa$. The following lemma is a consequence of König's lemma and local finiteness of $\mathcal{S}[\kappa^n]$ for all $n < \omega$ (see also [4, Theorem 3.2.13]).

Lemma 1. *Let $\mathcal{S} : \kappa^{<\omega} \to \mathcal{T}(M)$ be a strong sieve on M. Then, for each $z \in M$, there exists $t \in \kappa^\omega$ such that $\{\mathcal{S}(t|_n) : n < \omega\}$ is a local base at z in M.*

For a space Y, we say that a map $\mathcal{H} : \kappa^{<\omega} \to \mathcal{T}_0(Y)^\kappa$ is *decreasing* if for every $\sigma \in \kappa^{<\omega}$ and $\alpha < \kappa$, the family $\mathcal{H}(\sigma\hat{}\alpha)[\kappa]$ refines $\mathcal{H}(\sigma)[\kappa]$; i.e., each member of $\mathcal{H}(\sigma\hat{}\alpha)[\kappa]$ is included in some member of $\mathcal{H}(\sigma)[\kappa]$. For a strong sieve $\mathcal{S} : \kappa^{<\omega} \to \mathcal{T}(M)$ on M, we say that a map $\mathcal{H} : \kappa^{<\omega} \to \mathcal{T}_0(Y)^\kappa$ is *$\mathcal{S}$-free* if

$$\bigcap_{n<\omega} (\mathrm{cl}_Y\langle\mathcal{H}, \kappa\rangle(t|_n) \times \mathrm{cl}_M\mathcal{S}(t|_n)) = \emptyset \quad \text{for every } t \in \kappa^\omega.$$

The following lemma, which is a special case of [6, Theorem 5.1], is a key to our proof of Theorem 2.

Lemma 2. *Let A be a C-embedded subset of a space X and let M be a weight-homogeneous, nowhere locally compact, metric space with $w(M) = \kappa$. Then $A \times M$ is C-embedded in $X \times M$ provided that for every strong sieve $\mathcal{S} : \kappa^{<\omega} \to \mathcal{T}(M)$ on M, every decreasing $\mathcal{S}$-free map $\mathcal{G} : \kappa^{<\omega} \to \mathcal{T}_0(A)^\kappa$ satisfying that*

$$(2.1) \quad (\forall \sigma, \tau \in \kappa^{<\omega})(\textit{if } \mathcal{S}(\sigma) \subseteq \mathcal{S}(\tau), \textit{ then } \langle \mathcal{G}, \kappa\rangle(\sigma) \subseteq \langle \mathcal{G}, \kappa\rangle(\tau))$$

admits an $\mathcal{S}$-free expansion $\mathcal{H} : \kappa^{<\omega} \to \mathcal{T}_0(X)^\kappa$.

Remark 1. In [6, Theorem 5.1], it is not required that $\mathcal{G}$ satisfies (2.1); however, we can add this condition without any change of the proof. Indeed, the maps $\mathcal{G}_k : \kappa^{<\omega} \to \mathcal{T}_0(A)^\kappa$, $k < \omega$, defined in the proof of [6, Theorem 5.1], satisfy (2.1).

Proof of Theorem 2: (1) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (3): We use an argument similar to those in the proofs of Proposition 2.2 in [8] and Theorem 2.1 in [11]. Let $\mathcal{G} : \omega \to \mathcal{T}_0(A)^\kappa$ be a locally finite map. For each $n < \omega$ and $\alpha < \kappa$, there exist zero-sets $Z_{n,\alpha,i}$, $i < \omega$, in A such that $\mathcal{G}(n)(\alpha) = \bigcup_{i<\omega} Z_{n,\alpha,i}$. For each $i < \omega$, there exists a continuous function $f_{n,\alpha,i} : A \to [0,1]$ such that $f_{n,\alpha,i}[Z_{n,\alpha,i}] = \{1\}$ and $f_{n,\alpha,i}[A \setminus \mathcal{G}(n)(\alpha)] = \{0\}$. Take a bijection $\varphi : \kappa \times \omega \to \kappa$ and define a function $f : A \times J(\kappa) \to [0,1]$ by

$$f(\langle x, z\rangle) = \begin{cases} 0 & \text{if } z = \theta, \\ f_{n,\alpha,i}(x) & \text{if } z = \langle \varphi(\alpha, i), n\rangle. \end{cases}$$

Then f is continuous by local finiteness of $\{\langle \mathcal{G}, \kappa\rangle(n) : n < \omega\}$. By (2), f extends to a continuous function $g : X \times J(\kappa) \to [0,1]$. Define $\mathcal{H} : \omega \to \mathcal{T}_0(X)^\kappa$ by

$$\mathcal{H}(n)(\alpha) = \bigcup_{i<\omega} \{x \in X : |g(\langle x, \theta\rangle) - g(\langle x, \langle \varphi(\alpha, i), n\rangle\rangle)| > 2/3\}$$

for $n < \omega$ and $\alpha < \kappa$. Since $\mathcal{H}(n)(\alpha) \cap A = \mathcal{G}(n)(\alpha)$ for each $n < \omega$ and $\alpha < \kappa$, $\mathcal{H}$ is an expansion of $\mathcal{G}$. To show that $\mathcal{H}$ is locally finite, let $x \in X$, and choose a neighborhood U of x in X and $m < \omega$ such that diameter $g[U \times B(m)] < 1/3$. Then, by the definition of $\mathcal{H}$, $U \cap \langle \mathcal{H}, \kappa\rangle(n) = \emptyset$ for all $n \geq m$. Hence, $\mathcal{H}$ is locally finite.

(3) $\Rightarrow$ (1): Let M be a σ-locally compact, metric space with $w(M) \leq \kappa$. We have to show that $A \times M$ is C-embedded in $X \times M$.

Claim 1. We may assume that M is a weight-homogeneous, nowhere locally compact, σ-locally compact, metric space with $w(M) = \kappa$.

Proof of Claim 1: We consider κ the discrete space, and let T be the subspace of the product κ^ω consisting of all $t \in \kappa^\omega$ satisfying that $t(i) \leq t(j)$ whenever $i < j$, and $t|_{\omega \setminus n}$ is constant for some $n < \omega$. Then T is a weight-homogeneous, nowhere locally compact, σ-locally compact, metric space with weight κ, and hence, so is the product space $M_0 = M \times T$. If we prove that $A \times M_0$ is C-embedded in $X \times M_0$, then $A \times M$ is also C-embedded in $X \times M$, since $A \times M$ is homeomorphic to $A \times M \times \{t\}$, where t is an arbitrary fixed point in T. Hence, we may consider the space M_0 instead of M.

By Claim 1, we can apply Lemma 2. Let $\mathcal{S} : \kappa^{<\omega} \to \mathcal{T}(M)$ be a strong sieve on M and consider a decreasing $\mathcal{S}$-free map $\mathcal{G} : \kappa^{<\omega} \to \mathcal{T}_0(A)^\kappa$ satisfying (2.1). It is enough to show that $\mathcal{G}$ has an $\mathcal{S}$-free expansion. Since M is a σ-locally compact, metric space, there exist discrete collections $\{M_{i,\lambda} : \lambda < \xi(i)\}$, $i < \omega$, of non-empty compact sets such that $M = \bigcup_{i<\omega} \bigcup_{\lambda<\xi(i)} M_{i,\lambda}$. For each $\sigma \in \kappa^{<\omega}$, there exist unique $k(\sigma) < \omega$ and $\psi(\sigma) < \xi(k(\sigma))$ such that $\mathrm{cl}_M \mathcal{S}(\sigma) \cap M_{k(\sigma),\psi(\sigma)} \neq \emptyset$, and $\mathrm{cl}_M \mathcal{S}(\sigma) \cap M_{i,\lambda} = \emptyset$ if either $i < k(\sigma)$, or $i = k(\sigma)$ and $\lambda < \psi(\sigma)$. For each $n < \omega$, $i < \omega$, and $\lambda < \xi(i)$, put

$$\Sigma_n(i,\lambda) = \{\sigma \in \kappa^n : k(\sigma) = i, \psi(\sigma) = \lambda\} \text{ and}$$
$$G_n(i,\lambda) = \bigcup\{\langle \mathcal{G}, \kappa\rangle(\sigma) : \sigma \in \Sigma_n(i,\lambda)\}.$$

Claim 2. For each $i < \omega$ and $\lambda < \xi(i)$, $\{G_n(i,\lambda) : n < \omega\}$ is locally finite in A.

Proof of Claim 2: Fix $i < \omega$ and $\lambda < \xi(i)$. For each $n < \omega$, put

$$\Sigma_n^*(i,\lambda) = \{\sigma \in \kappa^n : \mathrm{cl}_M \mathcal{S}(\sigma) \cap M_{i,\lambda} \neq \emptyset\} \text{ and}$$
$$U_n(i,\lambda) = \bigcup\{\langle \mathcal{G}, \kappa\rangle(\sigma) : \sigma \in \Sigma_n^*(i,\lambda)\}.$$

Then $G_n(i,\lambda) \subseteq U_n(i,\lambda)$, because $\Sigma_n(i,\lambda) \subseteq \Sigma_n^*(i,\lambda)$. Hence, it suffices to show that $\{U_n(i,\lambda) : n < \omega\}$ is locally finite in A. If $m < n$, then for each $\sigma \in \kappa^n$, $\mathcal{S}(\sigma) \subseteq \mathcal{S}(\sigma|_m)$ and $\langle \mathcal{G}, \kappa\rangle(\sigma) \subseteq$

$\langle \mathcal{G}, \kappa \rangle(\sigma|_m)$ since $\mathcal{G}$ is decreasing, which implies that $U_n(i,\lambda) \subseteq U_m(i,\lambda)$. Hence, $\{U_n(i,\lambda) : n < \omega\}$ is decreasing. Now, suppose that there exists a point $x \in \bigcap_{n<\omega} \mathrm{cl}_A U_n(i,\lambda)$. Then, for each $n < \omega$, since $\Sigma_n^*(i,\lambda)$ is finite, there exists $\sigma_n \in \Sigma_n^*(i,\lambda)$ such that $x \in \mathrm{cl}_A \langle \mathcal{G}, \kappa \rangle(\sigma_n)$. Since $M_{i,\lambda}$ is compact, $\{\mathcal{S}(\sigma_n) : n < \omega\}$ has an accumulation point $z \in M$. By Lemma 1, there exists $t \in \kappa^\omega$ such that $\{\mathcal{S}(t|_n) : n < \omega\}$ is a local base at z in M. Since diameter $\mathcal{S}(\sigma_n) \to 0$, for each $n < \omega$, there exists $m > n$ such that $\mathcal{S}(\sigma_m) \subseteq \mathcal{S}(t|_n)$, and thus, $x \in \mathrm{cl}_A \langle \mathcal{G}, \kappa \rangle(\sigma_m) \subseteq \mathrm{cl}_A \langle \mathcal{G}, \kappa \rangle(t|_n)$ by (2.1). Hence,

$$\langle x, z \rangle \in \bigcap_{n<\omega} (\mathrm{cl}_A \langle \mathcal{G}, \kappa \rangle(t|_n) \times \mathrm{cl}_M \mathcal{S}(t|_n)),$$

which contradicts the fact that $\mathcal{G}$ is $\mathcal{S}$-free. Consequently, $\{U_n(i,\lambda) : n < \omega\}$ is locally finite in A, since $\bigcap_{n<\omega} \mathrm{cl}_A U_n(i,\lambda) = \emptyset$.

Fix $i < \omega$ and $\lambda < \xi(i)$ for a while. Since each $G_n(i,\lambda)$ is empty or the union of κ many cozero-sets in A, it follows from (3) that for each $n < \omega$, there exists a set of the form

$$H_n(i,\lambda) = \bigcup \{H_{\sigma,\alpha} : \sigma \in \Sigma_n(i,\lambda), \alpha < \kappa\}$$

such that $\mathcal{G}(\sigma)(\alpha) \subseteq H_{\sigma,\alpha} \in \mathcal{T}_0(X)$ for each $\sigma \in \Sigma_n(i,\lambda)$ and $\alpha < \kappa$, and the collection $\{H_n(i,\lambda) : n < \omega\}$ is locally finite in X. Taking such sets for all $i < \omega$ and $\lambda < \xi(i)$, we define a map $\mathcal{H} : \kappa^{<\omega} \to \mathcal{T}_0(X)^\kappa$ by

$$\mathcal{H}(\sigma)(\alpha) = H_{\sigma,\alpha} \quad \text{for each } \sigma \in \kappa^{<\omega} \text{ and } \alpha < \kappa.$$

This is well-defined, since each $\sigma \in \kappa^{<\omega}$ belongs to the unique $\Sigma_n(i,\lambda)$. Since $\mathcal{H}$ is an expansion of $\mathcal{G}$, it remains to show that $\mathcal{H}$ is $\mathcal{S}$-free. Let $t \in \kappa^\omega$ be fixed and suppose that there exists a point $z \in \bigcap_{n<\omega} \mathrm{cl}_M \mathcal{S}(t|_n)$. We have to show that

$$\bigcap_{n<\omega} \mathrm{cl}_X \langle \mathcal{H}, \kappa \rangle(t|_n) = \emptyset. \tag{2.2}$$

Now, there exist $j < \omega$ and $\mu < \xi(j)$ such that $z \in M_{j,\mu}$ and $z \notin M_{i,\lambda}$ if either $i < j$, or $i = j$ and $\lambda < \mu$. If we put $F = \bigcup \{M_{i,\lambda} : (i < j) \text{ or } (i = j \text{ and } \lambda < \mu)\}$, then F is closed in M and $z \notin F$. Since diameter $\mathcal{S}(t|_n) \to 0$, there exists $m < \omega$ such that $\mathrm{cl}_M \mathcal{S}(t|_n) \cap F = \emptyset$ for all $n > m$. Since $z \in \mathrm{cl}_M \mathcal{S}(t|_n) \cap M_{j,\mu}$

for each $n < \omega$, this means that $t|_n \in \Sigma_n(j,\mu)$ for all $n > m$, and hence,

$$\langle \mathcal{H}, \kappa \rangle (t|_n) = \bigcup_{\alpha<\kappa} H_{t|_n,\alpha} \subseteq H_n(j,\mu) \text{ for all } n > m.$$

Therefore, we have (2.2) by local finiteness of $\{H_n(j,\mu) : n < \omega\}$, which completes the proof. $\square$

Next, we deduce Theorem 1 from Theorem 2. We need some lemmas and definitions; the first one is due to Ernest Michael (see [10, Theorem 4]).

Lemma 3 (Michael)**.** *Let B be a closed set in a metric space M. Then $X \times B$ is C-embedded in $X \times M$ for every space X.*

For an infinite cardinal κ, recall that a set is *κ-open* if it is the union of less than κ many cozero-sets, and a *κ-open cover* is a cover consisting of κ-open sets. The smallest cardinal greater than κ is denoted by κ^+. We consider the following three conditions on a subspace A of a space X.

- $(1)_\kappa$ Every locally finite map $\mathcal{G} : \omega \to \mathcal{T}_0(A)^\kappa$ admits a locally finite expansion $\mathcal{H} : \omega \to \mathcal{T}_0(X)^\kappa$;
- $(2)_\kappa$ every countable, locally finite κ^+-open cover of A can be extended to a locally finite κ^+-open cover of X;
- $(3)_\kappa$ every countable, locally finite κ^+-open cover of A can be extended to a locally finite open cover of X.

Lemma 4. *Let κ be an infinite cardinal. For a closed set A in a normal space X, the conditions $(1)_\kappa$, $(2)_\kappa$, and $(3)_\kappa$ are equivalent to each other.*

Proof: $(1)_\kappa \Rightarrow (2)_\kappa$: Let $\mathcal{G} = \{G_n : n < \omega\}$ be a countable, locally finite κ^+-open cover of A. Since each G_n is the union of at most κ many cozero-sets in A, we can consider $\mathcal{G}$ as a locally finite map $\mathcal{G} : \omega \to \mathcal{T}_0(A)^\kappa$, and thus, $\mathcal{G}$ has a locally finite expansion $\mathcal{H} : \omega \to \mathcal{T}_0(X)^\kappa$ by $(1)_\kappa$. Put $H_n = \langle \mathcal{H}, \kappa \rangle(n)$ for each $n < \omega$. Since X is normal and A is closed, we may assume that $H_n \cap A = G_n$ for each $n < \omega$ and we can take a cozero-set U in X such that $X \setminus \bigcup_{n<\omega} H_n \subseteq U$ and $U \cap A = \emptyset$. Define $U_0 = H_0 \cup U$ and $U_n = H_n$ for each $n \geq 1$. Then $\{U_n : n < \omega\}$ is a locally finite κ^+-open cover of X which extends $\mathcal{G}$.

$(2)_\kappa \Rightarrow (3)_\kappa$: Obvious.

$(3)_\kappa \Rightarrow (1)_\kappa$: Let $\mathcal{G} : \omega \to \mathcal{T}_0(A)^\kappa$ be a locally finite map. Then $\{\langle \mathcal{G}, \kappa\rangle(n) : n < \omega\} \cup \{A\}$ is a κ^+-open cover of A. Hence, by $(3)_\kappa$, there exists a locally finite open cover $\{V_n : n < \omega\} \cup \{V\}$ of X such that $V_n \cap A = \langle \mathcal{G}, \kappa\rangle(n)$ for each $n < \omega$ and $V \cap A = A$. For each $n < \omega$ and $\alpha < \kappa$, $\mathcal{G}(n)(\alpha)$ is an F_σ-set in X, since it is a cozero-set in A and A is closed in X. By the normality of X, we can find a cozero-set $V_{n,\alpha}$ in X such that $V_{n,\alpha} \cap A = \mathcal{G}(n)(\alpha)$ and $V_{n,\alpha} \subseteq V_n$. Then we obtain a locally finite expansion $\mathcal{H} : \omega \to \mathcal{T}_0(X)^\kappa$ of $\mathcal{G}$ by letting $\mathcal{H}(n)(\alpha) = V_{n,\alpha}$ for each $n < \omega$ and $\alpha < \kappa$. $\square$

Remark 2. Conditions $(1)_\kappa$ and $(2)_\kappa$ are not equivalent for a C-embedded subset A of a space X (see Remark 4 below). An essential difference between them is that if $\mu < \kappa$, then $(1)_\kappa$ implies $(1)_\mu$, but, in general, $(2)_\kappa$ does not imply $(2)_\mu$.

For an infinite cardinal κ, Kaori Yamazaki [12] defined a space X to be (ω, κ)-*Katětov* if it is normal and every closed set A in X satisfies condition $(2)_\kappa$. A space X is countably Katětov if and only if X is (ω, κ)-Katětov for $\kappa = w(X)$, since every open set in a normal space X is $w(X)^+$-open. Hence, Theorem 1 is included in the following theorem; the "if" part was essentially proved by Przymusiński [8] and also follows from [12, Theorem 3.1].

Theorem 3. *Let κ be an infinite cardinal. A space X is (ω, κ)-Katětov if and only if $X \times M$ is rectangularly normal for every σ-locally compact, metric space M with $w(M) \leq \kappa$.*

Proof: By Lemma 3, if M is a metric space, then $X \times M$ is rectangularly normal if and only if $A \times M$ is C-embedded in $X \times M$ for every closed set A in X. Hence, this follows from Theorem 2 and Lemma 4. $\square$

3. Applications and remarks

The following theorem, which is the case $\kappa = \omega$ of Theorem 2 above, improves Yamazaki [12, Corollary 2.5], where the equivalences (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) were proved. Let $\mathbb{Q}$ denote the space of rational numbers.

Theorem 4. *For a subset A of a space X, the following are equivalent.*

(1) *$A \times M$ is C-embedded in $X \times M$ for every σ-locally compact, separable, metric space M;*
(2) *$A \times \mathbb{Q}$ is C-embedded in $X \times \mathbb{Q}$;*
(3) *$A \times J(\omega)$ is C-embedded in $X \times J(\omega)$;*
(4) *$A \times M$ is C-embedded in $X \times M$ for some non-locally compact, metric space M;*
(5) *every countable, locally finite cozero-set cover of A can be extended to a locally finite cozero-set cover of X.*

Proof: It is obvious that (1) $\Rightarrow$ (2) $\Rightarrow$ (4).

Since every non-locally compact, metric space contains a closed copy of $J(\omega)$, it follows from Lemma 3 that (4) implies (3).

Finally, it is not difficult to prove that if $\kappa = \omega$, then condition (3) in Theorem 2 is equivalent to condition (5) above. Hence, we have (3) $\Rightarrow$ (5) $\Rightarrow$ (1) by Theorem 2. $\square$

Recall from [3] that a subset A of a space X is P^γ(*locally finite*)-*embedded* in X if every locally finite partition of unity α on A with $|\alpha| \leq \gamma$ extends to a locally finite partition of unity on X, where γ is an infinite cardinal. It is known ([9] and [11]) that (5) in Theorem 4 is equivalent to the statement that A is P^ω(locally finite)-embedded in X.

Corollary 1. *Let A be a P^ω(locally finite)-embedded subset of a space X and let M be a σ-locally compact, separable, metric space. Then $A \times M$ is P^ω(locally finite)-embedded in $X \times M$.*

Proof: For every σ-locally compact, separable, metric space T, $A \times (M \times T)$ is C-embedded in $X \times (M \times T)$ by Theorem 4. Hence, $A \times M$ is P^ω(locally finite)-embedded in $X \times M$, again, by Theorem 4. $\square$

The author does not know if Corollary 1 remains true if ω is replaced by any uncountable cardinal γ.

Remark 3. Let us consider the countable sequential fan $S(\omega) = ((\omega + 1) \times \omega)/(\{\omega\} \times \omega)$. There exists a perfectly normal space X with a closed subset A such that $A \times S(\omega)$ is not C-embedded in $X \times S(\omega)$. Note that every perfectly normal space is countably Katětov, since it is countably paracompact (see also [9, Theorem 9]). Hence, in Theorem 2 and Theorem 4, the hedgehog cannot be replaced by the sequential fan. This can be seen from the following fact proved by Tôru Chiba and Keiko Chiba [2]: Let Y be a space with a point y_0 such that $\chi(y_0, Y) > d(Y) \geq \omega$, and let $\{B_p : p \in P\}$ be a neighborhood base at y_0 with $|P| = \chi(y_0, Y)$. On the other hand, let X be the perfectly normal space constructed from the uncountable set P in [1, Example H], and let $X_P = \{x_p : p \in P\}$ be the closed discrete set of nonisolated points in X. In part (i) of the proof of Theorem 3 in [2], they proved that disjoint closed sets

$$C = X_P \times \{y_0\} \quad \text{and} \quad D = \bigcup_{p \in P} (\{x_p\} \times (Y \setminus B_p))$$

in $X \times Y$ cannot be separated by disjoint open sets in $X \times Y$. Now, consider the space $S(\omega)$ as the space Y above, and let y_0 be the unique non-isolated point in $S(\omega)$. Then $\chi(y_0, S(\omega)) > d(S(\omega)) = \omega$. Since the sets C and D are disjoint zero-sets in $X_P \times S(\omega)$, it follows from [5, Theorem 1.18] that $X_P \times S(\omega)$ is not C-embedded in $X \times S(\omega)$.

Remark 4. We show that for every uncountable cardinal κ with uncountable cofinality, there is an example of a closed C-embedded subset A of a space X which satisfies condition $(2)_\kappa$ but not $(1)_\kappa$. This answers Yamazaki's question in [12, Remark 2.4(a)] negatively.

Fix an uncountable cardinal κ with uncountable cofinality. We consider a cardinal a space with the usual order topology. First, we show that if there exists a space Q, with $|Q| = \kappa$, having a point q and a countable, locally finite collection $\{G_n : n < \omega\}$ of cozero-sets in $Q \setminus \{q\}$ satisfying conditions (a), (b), and (c) below, then we have a required example.

(a) $Q \setminus \{q\}$ is C-embedded in Q;
(b) for each $n < \omega$, there is a homeomorphism $\psi_n : \kappa \times \omega \to G_n$ such that every neighborhood of q in Q intersects $\psi_n[(\kappa \setminus \alpha) \times (\omega \setminus i)]$ for each $\alpha < \kappa$ and each $i < \omega$; and
(c) there exists a collection $\{O_\alpha : \alpha < \kappa\}$ of open-closed sets in Q such that $O_\beta \subseteq O_\alpha$ whenever $\alpha < \beta$, and $\bigcap_{\alpha < \kappa} O_\alpha = \{q\}$.

Assume that such a space Q exists. Then, we define

$$X = (Q \times (\kappa + 1)) \setminus \{\langle q, \kappa \rangle\} \quad \text{and} \quad A = (Q \times \{\kappa\}) \cap X.$$

Since A is C-embedded in $Q \times \{\kappa\}$ by (a) and $Q \times \{\kappa\}$ is a retract of $Q \times (\kappa + 1)$, A is C-embedded in X. First, we show that A does not satisfy $(1)_\kappa$. Since $(1)_\kappa$ implies $(1)_\omega$, it is enough to show that the locally finite collection $\{G_n \times \{\kappa\} : n < \omega\}$ of cozero-sets in A cannot be extended to any locally finite collection of cozero-sets in X. Suppose on the contrary that there exists a locally finite collection $\{H_n : n < \omega\}$ of cozero-sets in X such that $H_n \cap A = G_n \times \{\kappa\}$ for each $n < \omega$. Note that every real-valued continuous function on κ^2 is constant on $(\kappa \setminus \alpha)^2$ for some $\alpha < \kappa$, since the cofinality of κ is uncountable. Thus, for each $n < \omega$ and each $i < \omega$, we can find $\alpha_{n,i} < \kappa$ such that

$$\psi_n[(\kappa \setminus \alpha_{n,i}) \times \{i\}] \times (\kappa \setminus \alpha_{n,i}) \subseteq H_n. \tag{3.1}$$

Put $\alpha = \sup\{\alpha_{n,i} : n < \omega, i < \omega\}$. Then $\alpha < \kappa$, and by (b), $\langle q, \alpha\rangle \in \mathrm{cl}_X H_n$ for each $n < \omega$, which contradicts local finiteness of $\{H_n : n < \omega\}$. Next, to prove that A satisfies $(2)_\kappa$, note that every open set in A or X is κ^+-open in A or X, respectively, since $|X| = \kappa$. Hence, it suffices to show that every countable, locally finite open cover $\{U_n : n < \omega\}$ of A can be extended to a locally finite open cover of X. For each $n < \omega$, put $V_n = \mathrm{pr}_Q^{-1}[\mathrm{pr}_Q[U_n]]$, where pr_Q is the projection from $Q \times (\kappa + 1)$ to Q. Then each V_n is open in X and $\{V_n : n < \omega\}$ is locally finite at each point of $X \setminus (\{q\} \times \kappa)$. Put

$$K = \bigcup_{\alpha<\kappa} (O_\alpha \times \alpha) \quad \text{and}$$
$$L = \bigcup_{\alpha<\kappa} ((Q \setminus O_\alpha) \times ((\kappa + 1) \setminus (\alpha + 1))).$$

Then K and L are disjoint open sets in X such that $\{q\} \times \kappa \subseteq K$ and $A \subseteq L$. Thus, putting $W_0 = V_0 \cup (X \setminus A)$ and $W_n = V_n \cap L$ for each $n > 0$, we obtain a locally finite open cover $\{W_n : n < \omega\}$ of X such that $W_n \cap A = U_n$ for each $n < \omega$.

Finally, we show that such a space Q exists. Let P be the quotient space obtained from $(\kappa + 1)^2 \setminus (\{\kappa\} \times \omega)$ by identifying points $\langle \alpha, \omega\rangle$ and $\langle \alpha, \kappa\rangle$ for each $\alpha \le \kappa$. As a set, P can be written as

$$P = (\kappa + 1)^2 \setminus (((\kappa + 1) \times \{\omega\}) \cup (\{\kappa\} \times \omega)).$$

When we write P as above, a basic neighborhood of a point $\langle \alpha, \kappa\rangle$ for $\alpha \le \kappa$ is a set of the form

$$B_{\beta,\gamma,n} = [((\alpha + 1) \setminus (\beta + 1)) \times (((\kappa + 1) \setminus (\gamma + 1)) \cup (\omega \setminus n))] \cap P$$

for $\beta < \alpha$, $\gamma < \kappa$, and $n < \omega$, and $P \setminus ((\kappa+1) \times \{\kappa\})$ has the subspace topology induced from the product topology on $(\kappa+1)^2$. Put $p = \langle \kappa, \kappa \rangle$, $G = \kappa \times \omega$, and

$$N_\alpha = [((\kappa+1) \setminus (\alpha+1)) \times (((\kappa+1) \setminus (\alpha+1)) \cup \omega)] \cap P$$

for each $\alpha < \kappa$. Then, by a reason similar to that of (3.1), $P \setminus \{p\}$ is C-embedded in P. Moreover, G is a cozero-set in P and $\{N_\alpha : \alpha < \kappa\}$ is a decreasing family of open-closed sets in P such that $\bigcap_{\alpha<\kappa} N_\alpha = \{p\}$. Finally, let Q be the quotient space obtained from $P \times \omega$ by collapsing the set $\{\langle\langle \kappa, \alpha \rangle, n\rangle : n < \omega\}$ to a point q_α for each α with $\omega < \alpha \leq \kappa$, and $\varphi : P \times \omega \to Q$ the quotient map. Put $q = q_\kappa$, $G_n = \varphi[G \times \{n\}]$ for each $n < \omega$, and $O_\alpha = \varphi[N_\alpha \times \omega]$ for each $\alpha < \kappa$. Then $\{G_n : n < \omega\}$ is a countable, locally finite collection of cozero-sets in $Q \setminus \{q\}$, and O_α is open-closed in Q for each $\alpha < \kappa$. The proof that Q, q, $\{G_n : n < \omega\}$, and $\{O_\alpha : \alpha < \kappa\}$ satisfy conditions (a), (b), and (c) is left to the reader.

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