TOPOLOGY PROCEEDINGS

Volume 34, 2009

Pages 167–173

http://topology.auburn.edu/tp/

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Electronically published on June 3, 2009

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS
Volume 34 (2009)
Pages 167-173

E-Published on June 3, 2009

ON THE CONTINUITY OF THE SET FUNCTION $\mathcal{K}$

SERGIO MACÍAS

ABSTRACT. David P. Bellamy asked, *If $\mathcal{T}$ is continuous for a continuum S, is $\mathcal{K}$ also continuous for S?* We present a class of continua for which the answer to this question is positive.

1. INTRODUCTION

David P. Bellamy asked the following question [12, Question 149]: *If $\mathcal{T}$ is continuous for S, is $\mathcal{K}$ also continuous for S?* We present a positive answer to this question when the continuum S is point $\mathcal{T}$-symmetric and $\mathcal{T}(\{s\})$ is a terminal continuum for each $s \in S$ (Corollary 3.6). This follows from the formula obtained in Theorem 3.4. We also show that for a point $\mathcal{T}$-symmetric continuum S for which $\mathcal{T}$ is continuous, $\mathcal{T}(A) = \mathcal{K}(A)$ for each subcontinuum A of S (Theorem 3.8). Let us note that the same question was also asked by Sandra Gorka in her dissertation [5, p. 118] under the supervision of Bellamy. Gorka makes an extensive study of several set functions; in particular, she studies the set functions $\mathcal{K}$ and $\mathcal{T}$.

F. Burton Jones defined the set functions $\mathcal{T}$ and $\mathcal{K}$ in [9] to study aposyndetic continua. Since then many properties related to these functions have been studied. Also, these functions have been applied to the study of continua. For example, the set function $\mathcal{K}$

2000 *Mathematics Subject Classification.* Primary 54B20; 54C60.

Key words and phrases. continuum, hyperspace, point $\mathcal{T}$-symmetric continuum, set functions $\mathcal{T}$ and $\mathcal{K}$, terminal continuum.

The author was partially supported by DGAPA, UNAM. The author thanks The University of Toledo for the support given during this research.

has been used to study plane continua ([6], [7], and [8]). It has been used to study monotone decompositions of continua [20], and continua such that for each pair of their points there exists an irreducible continuum between those two points which is decomposable [21]. Regarding the set function $\mathcal{T}$, for example, it has been used to study contractibility of continua [3], continua which can be mapped onto their cones [4], and symmetric products of continua [13].

2. Definitions

If Z is a topological space, then given $A \subset Z$, the interior of A is denoted by $Int_Z(A)$. The *power set of* Z is denoted by $\mathcal{P}(Z)$.

A *continuum* is a nonempty compact connected Hausdorff space. A continuum is *decomposable* if it is the union of two of its proper subcontinua. A continuum is *indecomposable* if it is not decomposable. A subcontinuum Y of a continuum X is *terminal* if for any subcontinuum W of X with $W \cap Y \neq \emptyset$, we have that either $W \subset$ or $Y \subset W$.

Given a continuum X, 2^X and $\mathcal{C}(X)$ denote the *hyperspaces of closed subsets* and *subcontinua of* X, respectively, topologized with the Vietoris topology [19]. If $f\colon X \to Y$ is a map between continua, $2^f\colon 2^X \to 2^Y$ given by $2^f(A) = f(A)$ is called the *induced map of* f. Note that 2^f is continuous [19, (1.168)].

Given a continuum X, we define the set functions $\mathcal{T}$ and $\mathcal{K}$ as follows: If $A \in \mathcal{P}(X)$, then

$$\mathcal{T}(A) = X \setminus \{x \in X \mid \text{there exists } W \in \mathcal{C}(X) \text{ such that } x \in Int(W) \subset W \subset X \setminus A\}$$

and

$$\mathcal{K}(A) = \bigcap \{W \mid W \in \mathcal{C}(X) \text{ and } A \subset Int_X(W)\}.$$

If $\mathcal{L} \in \{\mathcal{T}, \mathcal{K}\}$, we write $\mathcal{L}_X$ if there is a possibility of confusion. Let us observe that for any subset A of X, $\mathcal{L}(A)$ is a closed subset of X and $A \subset \mathcal{L}(A)$. The continuum X is said to be *point* $\mathcal{L}$*-symmetric* provided that for any two points x_1 and x_2 of X if $x_1 \in \mathcal{L}(\{x_2\})$, then $x_2 \in \mathcal{L}(\{x_1\})$. We say that a set function $\mathcal{L}$ is *continuous for a continuum* X provided that $\mathcal{L}\colon 2^X \to 2^X$ is continuous.

3. Continuity of $\mathcal{K}$

Let us recall that a surjective map $f\colon X \to Y$ between continua is *monotone* provided that $f^{-1}(y)$ is connected for each $y \in Y$.

We begin with a formula relating monotone open maps with terminal fibers and the set function $\mathcal{K}$.

Theorem 3.1. *Let X and Y be continua, and let $f\colon X \to Y$ be a monotone, open map such that $f^{-1}(y)$ is a terminal subcontinuum of X for all $y \in Y$. Then $\mathcal{K}_X(A) = f^{-1}\mathcal{K}_Y f(A)$ for all $A \in 2^X$.*

Proof: Let $A \in 2^X$. Let $x \in X \setminus f^{-1}\mathcal{K}_Y f(A)$. Then $f(x) \in Y \setminus \mathcal{K}_Y f(A)$. Thus, there exists $W \in \mathcal{C}(Y)$ such that $f(x) \in Y \setminus W$ and $f(A) \subset Int_Y(W)$. Hence, since f is monotone, $f^{-1}(W)$ is a subcontinuum of X such that $x \in f^{-1}f(x) \subset X \setminus f^{-1}(W)$ and $A \subset f^{-1}f(A) \subset Int_X\left(f^{-1}(W)\right)$. Therefore, $x \in X \setminus \mathcal{K}_X(A)$.

Next, let $x \in X \setminus \mathcal{K}_X(A)$. Then there exists $W \in \mathcal{C}(X)$ such that $x \in X \setminus W$ and $A \subset Int_X(W)$. Note that since $f^{-1}f(x)$ is a nowhere dense terminal continuum, $f^{-1}f(x) \cap W = \emptyset$. Hence, since f is continuous and open, $f(W)$ is a subcontinuum of Y such that $f(x) \in Y \setminus f(W)$ and $f(A) \subset Int_Y(f(W))$. This implies that $f(x) \in Y \setminus \mathcal{K}_Y f(A)$. Thus, $x \in f^{-1}f(x) \subset X \setminus f^{-1}\mathcal{K}_Y f(A)$.

Therefore, $\mathcal{K}_X(A) = f^{-1}\mathcal{K}_Y f(A)$. □

Remark 3.2. Observe that in Theorem 3.1 the hypothesis of the fibers of f being terminal continua cannot be removed. Let X be the Knaster continuum [11, p. 204]. Then X is an indecomposable metric continuum [11, Remark to Theorem 8, p. 213]. Since every proper subcontinuum of X has empty interior [11, Theorem 2, p. 207], it follows that $\mathcal{K}(A) = X$ for all $A \in 2^X$. Let $Z = X \times [0,1]$, let $\pi_X\colon Z \to X$ be the projection map, and let $(x,t) \in Z$. Note that π_X is a monotone and open map, but its fibers are not terminal. Then it is easy to see that $\mathcal{K}_Z(\{(x,t)\}) = \{(x,t)\}$. On the other hand, $\pi_X^{-1}\mathcal{K}_X\pi_X(\{(x,t)\}) = \pi_X^{-1}\mathcal{K}_X(\{x\}) = \pi_X^{-1}(X) = Z$. Therefore, $\mathcal{K}_Z(\{(x,t)\}) \neq \pi_X^{-1}\mathcal{K}_X\pi_X(\{(x,t)\})$. Let us also observe that since Z is not locally connected, $\mathcal{T}_Z$ is not continuous [14, 3.3.12]

The following result is the part we need from [17, Theorem 3.8].

Theorem 3.3. *Let X be a nondegenerate decomposable point $\mathcal{T}$-symmetric continuum for which $\mathcal{T}$ is continuous. Then $\mathcal{G} =$*

$\{\mathcal{T}(\{x\}) \mid x \in X\}$ is a continuous monotone decomposition of X such that $X/\mathcal{G}$ is locally connected and the elements of $\mathcal{G}$ are nowhere dense.

Theorem 3.4. *Let X be a decomposable point $\mathcal{T}_X$-symmetric continuum for which $\mathcal{T}_X$ is continuous and $\mathcal{T}_X(\{x\})$ is terminal for each $x \in X$. If $\mathcal{G} = \{\mathcal{T}_X(\{x\}) \mid x \in X\}$, then $\mathcal{K}_X(A) = q^{-1}\mathcal{K}_{X/\mathcal{G}}q(A)$ for every $A \in 2^X$, where $q\colon X \to X/\mathcal{G}$ is the quotient map.*

Proof: Since X is decomposable, X is nondegenerate. Since X is a nondegenerate decomposable point $\mathcal{T}_X$-symmetric continuum for which $\mathcal{T}_X$ is continuous, $\mathcal{G} = \{\mathcal{T}_X(\{x\}) \mid x \in X\}$ is a continuous decomposition, by Theorem 3.3. Thus, the quotient map $q\colon X \to X/\mathcal{G}$ is monotone and open. Since $q^{-1}q(x) = \mathcal{T}_X(\{x\})$ for each $x \in X$, by hypothesis, $q^{-1}(\chi)$ is a terminal subcontinuum of X for each $\chi \in X/\mathcal{G}$. Now Theorem 3.4 follows from Theorem 3.1. □

Question 3.5. Is Theorem 3.4 true without the assumption that the fibers of the map are terminal continua?

Corollary 3.6. *Let X be a point $\mathcal{T}_X$-symmetric continuum for which $\mathcal{T}_X$ is continuous and $\mathcal{T}_X(\{x\})$ is terminal for each $x \in X$. Then $\mathcal{K}_X$ is continuous.*

Proof: The result is clear if X is degenerate. So, we assume X is nondegenerate. If X is indecomposable, since every proper subcontinuum of X has empty interior [11, Theorem 2, p. 207], clearly, $\mathcal{K}_X$ is a constant map, hence, continuous. We assume for the rest of the proof that X is decomposable. Since X is a point $\mathcal{T}_X$-symmetric continuum for which $\mathcal{T}_X$ is continuous, by Theorem 3.3, $\mathcal{G} = \{\mathcal{T}_X(\{x\}) \mid x \in X\}$ is a decomposition and $X/\mathcal{G}$ is a locally connected continuum. Hence, $\mathcal{K}_{X/\mathcal{G}}$ is continuous [5, Theorem 36]. Let $q\colon X \to X/\mathcal{G}$ be the quotient map, and let $\Im\colon 2^{X/\mathcal{G}} \to 2^X$ be given by $\Im(\Gamma) = q^{-1}(\Gamma)$. Since q is continuous and open, we have that 2^q and $\Im$ are continuous ([19, (1.168)] and [10, Theorem 2], respectively). Note that, by Theorem 3.4, $\mathcal{K}_X = \Im \circ \mathcal{K}_{X/\mathcal{G}} \circ 2^q$. Therefore, $\mathcal{K}_X$ is continuous. □

Remark 3.7. Let us observe that all the known examples of continua for which the set function $\mathcal{T}$ is continuous are point $\mathcal{T}$-symmetric and the images of singletons under $\mathcal{T}$ are terminal continua [15], [16], and [18]. Also note that the continuity of $\mathcal{K}$ does not imply the continuity of $\mathcal{T}$. It is easy to see that $\mathcal{T}$ is not continuous for the suspension over the Cantor set $\Sigma(C)$, but $\mathcal{K}$ is the identity map on $2^{\Sigma(C)}$ [5, Theorem 26].

Now we show that the restrictions of $\mathcal{T}$ and $\mathcal{K}$ to the hyperspace of subcontinua of a point $\mathcal{T}$–symmetric continuum for which $\mathcal{T}$ is continuous coincide. Observe that it is not necessary to assume that the images of the singletons under $\mathcal{T}$ are terminal continua.

Theorem 3.8. *Let X be a point $\mathcal{T}_X$–symmetric continuum for which $\mathcal{T}_X$ is continuous. Then $\mathcal{K}_X(A) = \mathcal{T}_X(A)$ for each $A \in \mathcal{C}(X)$.*

Proof: Let us note that if X is indecomposable, then $\mathcal{K}$ is a constant map. Hence, the theorem is true when X is indecomposable. Suppose that X is decomposable. Since X is decomposable, X is nondegenerate. Since X is a nondegenerate decomposable point $\mathcal{T}_X$–symmetric continuum for which $\mathcal{T}_X$ is continuous, $\mathcal{G} = \{\mathcal{T}_X(\{x\}) \mid x \in X\}$ is a decomposition, by Theorem 3.3. Let $q\colon X \to X/\mathcal{G}$ be the quotient map. Let $A \in \mathcal{C}(X)$. Note that $\mathcal{T}_X(A) \in \mathcal{C}(X)$ [2, Theorem 4]. By [1, Lemma 9], $\mathcal{T}_X(A) = \cup\{\mathcal{T}_X(\{a\}) \mid a \in A\}$. Hence, by [17, Lemma 3.2], $\mathcal{T}_X(A) \subset \mathcal{K}_X(A)$. Let $x \in X \setminus \mathcal{T}_X(A)$. Since $\mathcal{T}_X(A) = \cup\{\mathcal{T}_X(\{a\}) \mid a \in A\}$, we have that $\mathcal{T}_X(\{x\}) \cap \mathcal{T}_X(A) = \emptyset$. Thus, $q(x) \in X/\mathcal{G} \setminus q\mathcal{T}_X(A)$. Since $X/\mathcal{G}$ is locally connected by Theorem 3.3, and $q\mathcal{T}_X(A)$ is a subcontinuum of $X/\mathcal{G}$, there exists $\mathcal{W} \in \mathcal{C}(X/\mathcal{G})$ such that $q(x) \in X/\mathcal{G} \setminus \mathcal{W}$ and $q\mathcal{T}_X(A) \subset Int_{X/\mathcal{G}}(\mathcal{W})$. Hence, since q is monotone, $q^{-1}(\mathcal{W})$ is a subcontinuum of X such that $\mathcal{T}_X(A) \subset Int_X\left(q^{-1}(\mathcal{W})\right)$ and $x \in q^{-1}(q(x)) \subset X \setminus q^{-1}(\mathcal{W})$. Therefore, $x \in X \setminus \mathcal{K}_X(A)$ and $\mathcal{T}_X(A) = \mathcal{K}_X(A)$. □

Remark 3.9. Let us note that the suspension over the Cantor set $\Sigma(C)$ is a point $\mathcal{K}$-symmetric continuum for which $\mathcal{K}$ is continuous (in fact, $\mathcal{K}$ is the identity map on $2^{\Sigma(C)}$ [5, Theorem 26]), but $\mathcal{K}$ and $\mathcal{T}$ do not agree on every subcontinuum of X (if A is an arc containing the two vertices of $\Sigma(C)$, then $\mathcal{T}(A) = \Sigma(C)$).

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