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DENDRITES WITH UNIQUE SYMMETRIC PRODUCTS

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Abstract. Let X be a metric continuum and n a positive integer. Let $F_n(X)$ be the hyperspace of all nonempty subsets of X with at most n points, metrized by the Hausdorff metric. Let X be a dendrite whose set of end points is closed, let Y be a continuum, and let $n \in \mathbb{N}$. In this paper, we prove that if $F_n(X)$ is homeomorphic to $F_n(Y)$, then X is homeomorphic to Y.

1. Introduction and general notions

A *continuum* is a nondegenerate, compact, connected metric space. For a given continuum X and $n \in \mathbb{N}$, we consider the following hyperspaces of X:

$$F_n(X) = \{A \subset X : A \text{ is nonempty and has at most } n \text{ points}\}$$

and

$$C_n(X) = \{A \subset X : A \text{ is closed nonempty and has at most } n \text{ components}\}.$$

Both $F_n(X)$ and $C_n(X)$ are metrized by the Hausdorff metric (see [14, Definition 0.1]) and are also known as the *n-th symmetric product of* X and the *n-fold hyperspace of* X, respectively. When $n = 1$, it is customary to write $C(X)$ instead of $C_1(X)$ and to refer to $C(X)$ as the *hyperspace of subcontinua of* X.

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If X is a continuum, $U_1, \dots, U_m \subset X$, and $n \in \mathbb{N}$, we define

$$\langle U_1, \dots, U_m \rangle_n = \left\{ A \in F_n(X) \colon A \subset \bigcup_{i=1}^{m} U_i \text{ and } A \cap U_i \neq \emptyset, \text{ for each } i \right\}.$$

It is known that the sets of the form $\langle U_1, \dots, U_m \rangle_n$, where $U_1, \dots, U_m$ are open in X, form a basis of the topology of $F_n(X)$, i.e., a basis for the topology induced by the Hausdorff metric on $F_n(X)$ (see [14, Theorem 0.13]).

Let $\mathcal{H}(X)$ be any one of the hyperspaces defined above and let $\mathcal{G}$ be a class of continua. We say that $X \in \mathcal{G}$ has *unique hyperspace* $\mathcal{H}(X)$ *in* $\mathcal{G}$ if, whenever $Y \in \mathcal{G}$ is such that $\mathcal{H}(X)$ is homeomorphic to $\mathcal{H}(Y)$, it follows that X is homeomorphic to Y. If $\mathcal{G}$ is the class of all continua, we simply say that X has unique hyperspace $\mathcal{H}(X)$.

A *dendrite* is a locally connected continuum (hence, arcwise connected) which contains no simple closed curves. We denote by $\mathfrak{D}$ the class of dendrites whose set of end points is closed. Let $X \in \mathfrak{D}$ and $n \in \mathbb{N}$; it is known that X has unique hyperspace $C_n(X)$, as we summarize in the following results.

Theorem 1.1 (see [8, Theorem 10])**.** *Let $X \in \mathfrak{D}$ such that X is not an arc. Then X has unique hyperspace $C(X)$.*

Theorem 1.2 (see [2, Theorem 5.1])**.** *Let X be a dendrite whose set of end points is not closed. Then there exists an uncountable class $\{Y_\alpha : \alpha \in \Gamma\}$ of dendrites such that, for each $\alpha, \beta \in \Gamma$, we have*

(a) *X is not homeomorphic to Y_α;*
(b) *Y_α is not homeomorphic to Y_β if $\alpha \neq \beta$;*
(c) *$C(X)$ is homeomorphic to $C(Y_\alpha)$.*

In particular, X does not have unique hyperspace $C(X)$ in the class of dendrites.

If X is an arc and Y is a continuum such that $C(X)$ is homeomorphic to $C(Y)$, then Y is either an arc or a simple closed curve (see [1, Lemma 11]). Thus, an arc X has unique hyperspace $C(X)$ in the class of dendrites. Combining this and theorems 1.1 and 1.2, we obtain the following result.

Theorem 1.3. *A dendrite X has unique hyperspace $C(X)$ if and only if $X \in \mathfrak{D}$ and X is not an arc.*

For $n = 2$, [9, Theorem 10] shows the following theorem.

Theorem 1.4. *If $X \in \mathfrak{D}$, then X has unique hyperspace $C_2(X)$ in the class of dendrites.*

In [12, Theorem 3], Alejandro Illanes proves the following theorem.

Theorem 1.5. *If $X \in \mathfrak{D}$, then X has unique hyperspace $C_2(X)$.*

For $n \geq 3$, the following theorem is known.

Theorem 1.6 (see [10, Theorem 5.6])**.** *If $X \in \mathfrak{D}$ and $n \geq 3$, then X has unique hyperspace $C_n(X)$.*

Question 1.7. Let X be a dendrite and $n \geq 2$. If X has unique hyperspace $C_n(X)$, is X in $\mathfrak{D}$?

In the First Workshop in Hyperspaces and Continuum Theory, celebrated in the city of Puebla, Mexico, July 2–13, 2007, Illanes asked if every element $X \in \mathfrak{D}$ has unique hyperspace $F_n(X)$. During the Workshop, Gerardo Acosta, Rodrigo Hernández-Gutiérrez, and Verónica Martínez-de-la-Vega showed that if $X \in \mathfrak{D}$, $n \in \mathbb{N}$, and Y is a continuum such that $F_n(X)$ is homeomorphic to $F_n(Y)$, then $Y \in \mathfrak{D}$ (see [3, Theorem 5.2]). In the same Workshop, the authors of this paper proved that any dendrite $X \in \mathfrak{D}$ has unique hyperspace $F_n(X)$ in $\mathfrak{D}$ for each $n \in \mathbb{N}$. We give a proof of this fact in this paper. Combining these results, it follows that every $X \in \mathfrak{D}$ has unique hyperspace $F_n(X)$, answering Illanes's question.

To show the main theorem of this paper, i.e., that any dendrite $X \in \mathfrak{D}$ has unique hyperspace $F_n(X)$ for each $n \in \mathbb{N}$, we proceed as follows. Let $X \in \mathfrak{D}$, let Y be a continuum, and let $n \in \mathbb{N}$ be such that $F_n(X)$ is homeomorphic to $F_n(Y)$. For $n = 1$, it is known that continua A and B are homeomorphic if and only if $F_1(A)$ and $F_1(B)$ are homeomorphic. Thus, X has unique hyperspace $F_1(X)$. For $n = 2$, we use the work done in [11] to show that X is homeomorphic to Y. For $n \geq 3$, we use the work done in [5] for finite graphs, together with some results from [3], to guarantee that X is homeomorphic to Y.

All spaces considered in this paper are assumed to be metric. If A is a subset of a space X, then $cl_X(A)$, $Bd(A)$, and $int_X(A)$ denote the closure, the boundary, and the interior of A in X, respectively. The cardinality of A is denoted by $|A|$. The letters $\mathbb{R}$

and $\mathbb{N}$ represent the sets of real numbers and of positive integers, respectively. We let $\mathbb{R}^2$ denote $\mathbb{R} \times \mathbb{R}$.

We recall that a *finite graph* is a continuum that can be written as the union of finitely many arcs, each two of which either are disjoint or intersect only in one or both of their end points. A *tree* is a finite graph that contains no simple closed curves. In fact, all concepts not defined here will be taken as in [14].

2. DENDRITES AND THE CLASS $\mathfrak{D}$

In this section, the letter X represents a dendrite. It is known that every subcontinuum of X is a dendrite (see [15, Corollary 10.6]). A subset C of X is said to be *continuumwise dense* in X provided that $C \cap A \neq \emptyset$ for each nondegenerate subcontinuum A of X. If $p \in X$, then by the *order of* p *in* X, denoted by ord_pX, we mean the Menger-Urysohn order (see [15, Definition 9.3]) or, equivalently, the order in the classical sense, i.e., the number of arcs emanating from p and disjoint out of p (see [6, p. 229]). If ord_pX is finite, then it is equal to the number of components of $X - \{p\}$ (see [16, (1.1), (iv), p. 88]). If ord_pX is infinite, then it is countable and the diameters of components of $X - \{p\}$ tend to zero (see [16, (2.6), p. 92]). In this case, we write $ord_pX = \omega$. Thus, $ord_pX \leq \omega$ for every $p \in X$.

We say that $p \in X$ is an *end point of* X if $ord_pX = 1$. The set of all end points of X is denoted by $E(X)$. It is known that $E(X)$ is zero-dimensional (see [13, Theorem 2, p. 292]). Let

$$E_a(X) = \{p \in E(X) : \text{ there is a sequence in } E(X) - \{p\} \text{ that converges to } p\}.$$

If $p \in E(X) - E_a(X)$, we say that p is an *isolated end point of* X. If $ord_pX = 2$, we say that p is an *ordinary point of* X. The set of all ordinary points of X is denoted by $O(X)$. It is known that $O(X)$ is continuumwise dense in X (see [15, 10.42]). In particular, $O(X)$ is dense in X. If $ord_pX \geq 3$, we say that p is a *ramification point of* X. The set of all ramification points of X is denoted by $R(X)$. It is known that $R(X)$ is countable (see [15, Theorem 10.23]). Clearly, $X = E(X) \cup O(X) \cup R(X)$.

If $p, q \in X$, then there is only one arc in X joining p and q. We denote such arc by $[p, q]$ and assume that $[p, p] = \{p\}$. We also consider the sets $(p,q) = [p,q] - \{p,q\}$, $[p,q) = [p,q] - \{q\}$, and $(p,q] = [p,q] - \{p\}$.

Let $[p,q]$ be an arc in X such that $(p,q) \subset O(X)$. We say that $[p,q]$ is an *internal arc* if $p,q \in R(X)$ and an *external arc* if one end point of $[p,q]$ is an end point of X and the other end point of $[p,q]$ is a ramification point of X. If $[p,q]$ is an internal arc in X, then $int_X([p,q]) = (p,q)$, and if $[p,q]$ is an external arc in X and $p \in E(X)$, then $int_X([p,q]) = [p,q)$. Let

$$\mathcal{A}(X) = \{[p,q]\colon [p,q] \text{ is either an internal arc or an external arc in } X\}.$$

If $p,q \in X$ and $[p,q] \in \mathcal{A}(X)$, we say that p is adjacent to q or that p and q are adjacent.

Given $n \in \mathbb{N}$, we define

$$\begin{array}{lcl} EA_n(X) & = & \{A \in F_n(X)\colon A \cap E_a(X) \neq \emptyset\}, \\ R_n(X) & = & \{A \in F_n(X)\colon A \cap R(X) \neq \emptyset\}, \text{ and} \\ \Lambda_n(X) & = & F_n(X) - (R_n(X) \cup EA_n(X)). \end{array}$$

Notice that $A \in \Lambda_n(X)$ if and only if $|A| \leq n$ and the points of A are either ordinary points of X or isolated end points of X.

We recall that $\mathfrak{D}$ is the class of all dendrites whose set of end points is closed.

We denote by $\mathcal{O}$ the class of all dendrites whose set of ordinary points is open.

We consider the dendrites F_ω, W, and W_0. The first one is the dendrite with only one ramification point whose order is ω. Let $R(F_\omega) = \{r\}$ and $E(F_\omega) = \{r_n : n \in \mathbb{N}\}$. Then

$$F_\omega = \bigcup_{n=1}^{\infty} [r, r_n].$$

Thus, $F_\omega \in \mathcal{O}$.

The dendrites W and W_0 will be constructed in $\mathbb{R}^2$. If $p,q \in \mathbb{R}^2$, then $\overline{pq}$ represents the straight line segment in $\mathbb{R}^2$ joining p and q. Let $b = (0,0)$, $c = (1,0)$, and, for each $n \in \mathbb{N}$, let $b_n = \left(-\frac{1}{n}, 0\right)$ and $e_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$. Then

$$W = \overline{b_1 c} \cup \left(\bigcup_{n=1}^{\infty} \overline{b_n e_n} \right) \text{ and } W_0 = W - (b, c]. \tag{2.1}$$

Theorem 2.1 (see [4, Theorem 3.3])**.** *A dendrite is in $\mathfrak{D}$ if and only if it contains no copy of F_ω or of W.*

As a consequence of this result, we have the following theorem.

Theorem 2.2. *If $X \in \mathfrak{D}$, then the order of every point of X is finite.*

From the proof of Proposition 3.4 in [4], we have the following.

Theorem 2.3. *Let $X \in \mathfrak{D}$, and let $\{s_n\}_{n=1}^{\infty}$ be a sequence in $R(X)$ that converges to s such that $s_n \neq s_m$ if $n \neq m$ and assume that $s \neq s_1$. Then*

(a) *there is a sequence $\{r_n\}_{n=1}^{\infty}$ in $R(X) \cap [s, s_1]$ that converges to s;*

(b) *there is a sequence $\{e_n\}_{n=1}^{\infty}$ in $E(X) - \{s\}$ that converges to s, so $s \in E_a(X)$.*

Corollary 2.4. *The class $\mathfrak{D}$ is a proper subset of $\mathcal{O}$.*

Proof: Let $X \in \mathfrak{D}$ and assume, on the contrary, that $O(X)$ is not open. Then there exist $r \in O(X)$ and a sequence $\{r_n\}_{n=1}^{\infty}$ in $X - O(X)$ that converges to r. Notice that $r_n \in E(X) \cup R(X)$ for every $n \in \mathbb{N}$. If $r_n \in E(X)$ for infinitely many n, then, since $E(X)$ is closed, $r \in E(X)$, a contradiction. This shows that $r_n \in R(X)$ for infinitely many n. By Theorem 2.3, $r \in E_a(X)$, contradicting our assumption that $r \in O(X)$. So, $X \in \mathcal{O}$. This shows that $\mathfrak{D} \subset \mathcal{O}$. Since $F_\omega \in \mathcal{O}$, by Theorem 2.1, we have $F_\omega \notin \mathfrak{D}$. Therefore, $\mathfrak{D} \subsetneq \mathcal{O}$. □

Lemma 2.5. *Let $X \in \mathfrak{D}$, and let $\{e_n\}_{n=1}^{\infty}$ be a sequence in $E(X)$ that converges to e such that $e_n \neq e_m$ if $n \neq m$ and assume that $e \neq e_1$. Then*

(a) *there is a sequence $\{r_n\}_{n=1}^{\infty}$ in $R(X) \cap [e, e_1]$ that converges to e;*

(b) *the continuum $[e, r_1] \cup (\bigcup_{n=1}^{\infty}[r_n, e_n])$ is a copy of W_0.*

Proof: To show (a), notice that $e \in E(X)$ and that e is an end point of the arc $[e, e_1] \cap [e, e_2]$; let r_1 be the other end point of this arc. Thus, $r_1 \in R(X)$. We will show that $[e, e_n] \cap [e, r_1] = [e, r_1]$ for only a finite number of end points e_n. Assume, on the contrary, if $[e, e_{n_k}] \cap [e, r_1] = [e, r_1]$ for each $k \in \mathbb{N}$, then $[e, r_1] \subset lim[e, e_{n_k}]$. By [7, Corollary 4], X is a smooth dendrite at e, so $lim[e, e_{n_k}] = \{e\}$.

Thus, $[e, r_1] = \{e\}$, which is a contradiction. Let $e_{n_1}, \dots, e_{n_k}$ be the only points such that $[e, e_{n_i}] \cap [e, r_1] = [e, r_1]$ for each $i \in \{1, \dots, k\}$, and assume that $n_1 < \dots < n_k$. Then $[e, e_n] \cap [e, r_1] \subsetneq [e, r_1]$ for each $n > n_k$. Let $[e, r_2] = [e, e_{n_k+1}] \cap [e, r_1]$, where $r_2 \in R(X) - \{r_1\}$. We proceed by induction to obtain a sequence $\{r_n\}_{n=1}^{\infty}$ in $R(X) \cap [e, e_1]$ with $r_n \neq r_m$ if $n \neq m$, such that $[e, r_{n+1}] \subset [e, r_n] \subset [e, e_1]$. Thus, $\{r_n\}_{n=1}^{\infty}$ converges to e.

To show (b), in (2.1), replace b by e and b_n by r_n for each $n \in \mathbb{N}$. □

Some results proved in [3] will be used. For convenience, we collect them here.

Theorem 2.6 (see [3, Theorem 3.1]). *Let $X \in \mathfrak{D}$ and $n \in \mathbb{N}$. If $A \in F_n(X) - EA_n(X)$, then there is a tree T in X such that $A \subset int_X(T)$ and $T \cap E_a(X) = \emptyset$.*

Theorem 2.7 (see [3, Theorem 3.2]). *Let $X \in \mathfrak{D}$ and $n \in \mathbb{N}$. Then $\Lambda_n(X)$ is an open subset of $F_n(X)$ such that $\mathcal{C}$ is a component of $\Lambda_n(X)$ if and only if $\mathcal{C}$ is of the form*

$$\langle int_X(I_1), \dots, int_X(I_m) \rangle_n,$$

where $m \leq n$ and $I_1, \dots, I_m$ is a finite collection of pairwise disjoint subsets of X such that $I_j \in \mathcal{A}(X)$ for every $j \in \{1, ..., m\}$.

If $n \in \mathbb{N}$, then an *n-cell* is a space homeomorphic to the cartesian product $[0, 1]^n$. Given a continuum X and $n \in \mathbb{N}$, we consider the class

$$\mathcal{E}_n(X) = \{A \in F_n(X) \colon A \text{ has a neighborhood in } F_n(X) \text{ which is an } n\text{-cell}\}.$$

Theorem 2.8 (see [3, Theorem 4.5]). *Let $X \in \mathfrak{D}$ and $n \in \mathbb{N}$. Then*

(a) $\mathcal{E}_n(X) \subset \Lambda_n(X)$;
(b) *if* $n \in \{2, 3\}$, *then* $\mathcal{E}_n(X) = \Lambda_n(X)$;
(c) *if* $n \geq 4$, *then* $\mathcal{E}_n(X) = \Lambda_n(X) - F_{n-1}(X)$.

Theorem 2.9. *Let $X \in \mathfrak{D}$, $n \in \mathbb{N}$, and $A \in EA_n(X)$. Then for every basis $\mathcal{B}$ of open neighborhoods of A in $F_n(X)$ and each $\mathcal{V} \in \mathcal{B}$, the set $\mathcal{V} \cap \mathcal{E}_n(X)$ has infinitely many components.*

Proof: Let $A \in EA_n(X)$ and let $\mathcal{B}$ be a basis of open neighborhoods of A in $F_n(X)$. We consider that $|A| = m$, and we write

$A = \{x_1, \ldots, x_m\}$, where $x_1 \in E_a(X)$. Let $U_1, \ldots, U_m$ be a finite collection of pairwise disjoint open and connected subsets of X such that $x_i \in U_i$ for each $i \in \{1, \ldots, m\}$. Take $\mathcal{V} \in \mathcal{B}$ such that $\mathcal{V} \subset \langle U_1, \ldots, U_m \rangle_n$ and a finite collection $V_1, \ldots, V_m$ of pairwise disjoint open and connected subsets of X such that $x_i \in V_i \subset U_i$ for each $i \in \{1, \ldots, m\}$, and $\langle V_1, \ldots, V_m \rangle_n \subset \mathcal{V}$. By Lemma 2.5, we can find a sequence of different points in $R(X) \cap V_1$ such that $\{r_k\}_{k \in \mathbb{N}}$ converges to x_1, $[r_k, r_{k+1}]$ is an internal arc in X for every $k \in \mathbb{N}$, and $r_{k+1} \in (r_k, r_{k+2}) \subset V_1$ for each $k \in \mathbb{N}$. Given $i \in \{2, \ldots, m\}$, fix an arc I_i which is either an external arc or an internal arc such that $I_i \cap V_i \neq \emptyset$. Let $J_i = int_{U_i}(I_i \cap U_i)$ for each $i \in \{2, \ldots, m\}$. For every $k \in \mathbb{N}$, let

$$\mathcal{W}_k = \langle J_2, J_3, \ldots, J_m, (r_k, r_{k+1}), (r_{k+1}, r_{k+2}), \ldots, (r_{k+n-m}, r_{k+n-m+1}) \rangle_n.$$

Notice that $\mathcal{W}_k$ is connected. We obtain $\mathcal{W}_k \subset \langle U_1, \ldots, U_m \rangle_n \cap \mathcal{E}_n(X)$ by Theorem 2.8. Let $\mathcal{C}$ be the component of $\Lambda_n(X)$ such that $\mathcal{W}_k \subset \mathcal{C}$. By Theorem 2.7, we have $\mathcal{C} \cap (\langle U_1, \ldots, U_m \rangle_n \cap \mathcal{E}_n(X)) = \mathcal{W}_k$; thus, $\mathcal{W}_k$ is the component of $\langle U_1, \ldots, U_m \rangle_n \cap \mathcal{E}_n(X)$. Notice that $\mathcal{W}_k \cap \mathcal{V} \neq \emptyset$ for each $k \in \mathbb{N}$. Since $\mathcal{W}_k \cap \mathcal{W}_l = \emptyset$, if $k \neq l$, the set $\mathcal{V} \cap \mathcal{E}_n(X)$ has infinitely many components. $\square$

The following result is the version of [5, Lemma 4.5] for dendrites in $\mathfrak{D}$.

Theorem 2.10. *Let $X \in \mathfrak{D}$ and $n \geq 4$. For each $A \in F_n(X)$, the following conditions are equivalent:*

(a) $A \in F_1(X) \cap \Lambda_n(X)$;
(b) *$A \notin \mathcal{E}_n(X)$ and there is a basis $\mathcal{B}$ of open neighborhoods of A in $F_n(X)$ such that, for each $\mathcal{V} \in \mathcal{B}$, the set $\mathcal{V} \cap \mathcal{E}_n(X)$ is arcwise connected.*

Proof: Let us assume (a). By Theorem 2.8(c), we obtain $A \notin \mathcal{E}_n(X)$. Since $A \notin EA_n(X)$, by Theorem 2.6, there is a tree T in X such that $A \subset int_X(T)$ and $T \cap E_a(X) = \emptyset$. Notice that $A \in F_1(T) - R_n(T)$ so, by [5, Lemma 4.5], there is a basis $\mathcal{B}$ of open neighborhoods of A in $F_n(T)$ such that, for each $\mathcal{V} \in \mathcal{B}$, the set $\mathcal{V} \cap \mathcal{E}_n(T)$ is arcwise connected. Since $A \subset int_X(T)$, we can assume that all the members of $\mathcal{B}$ are contained in $F_n(int_X(T)) = \langle int_X(T) \rangle_n$. Since $\langle int_X(T) \rangle_n$ is an open set in $F_n(X)$, then $\mathcal{B}$ is also a basis of open neighborhoods of A in $F_n(X)$. For each $\mathcal{V} \in \mathcal{B}$, we have that $\mathcal{V} \cap \mathcal{E}_n(X) = \mathcal{V} \cap \mathcal{E}_n(T)$ so this intersection is arcwise connected. This proves (b).

Now assume (b). Let $\mathcal{B}$ be a basis of open neighborhoods of A in $F_n(X)$ such that, for each $\mathcal{V} \in \mathcal{B}$, the set $\mathcal{V} \cap \mathcal{E}_n(X)$ is arcwise connected. Then $A \notin EA_n(X)$ by Theorem 2.9. Hence, by Theorem 2.6, there is a tree T in X such that $A \subset int_X(T)$ and $T \cap E_a(X) = \emptyset$. Since $A \in \langle int_X(T) \rangle_n$, we can suppose that all the elements of $\mathcal{B}$ are open subsets of $F_n(T)$ and thus, for each $\mathcal{V} \in \mathcal{B}$, $\mathcal{V} \cap \mathcal{E}_n(T) = \mathcal{V} \cap \mathcal{E}_n(X)$ is arcwise connected. Then, by [5, Lemma 4.5], $A \in F_1(T) - R_n(T)$. Therefore, $A \in F_1(X) - R_n(X) \subset F_1(X) \cap \Lambda_n(X)$, so (a) holds. □

Let Y be a continuum and let W be an open subset of Y. For any open subset U of Y, define $c(U, W, Y)$ as the number of components of $U \cap W$ if this number is finite and $c(U, W, X) = \infty$, otherwise. For each $p \in cl_Y(W)$, we define

$$v(p, W, Y) = min(\{m \in \mathbb{N}\colon \text{ there is a base } \mathcal{B} \text{ of open neighborhoods of } p \text{ in } Y \text{ such that } c(U, W, X) = m, \text{ for each } U \in \mathcal{B}\} \cup \{\infty\}).$$

We remark that $\mathcal{E}_3(X)$ is an open subset of $F_3(X)$ by Theorem 2.7 and Theorem 2.8(b).

Theorem 2.11. *Let $X \in \mathfrak{D}$ and let $p, q, r, x, y \in X$ be such that $ord_p X = n \geq 3$, $ord_q X = m \geq 3$, $ord_r X = k \geq 3$, and $x, y \in O(X) \cup (E(X) - E_a(X))$. Given $A \in F_3(X)$, the possible values for $v(A) = v(A, \mathcal{E}_3(X), F_3(X))$ are*

(a) *if $A = \{p\}$, then $v(A) = n + \binom{n}{2} + \binom{n}{3}$;*
(b) *if $A = \{p, x\}$, then $v(A) = n + \binom{n}{2}$;*
(c) *if $A = \{p, x, y\}$ and $x \neq y$, then $v(A) = n$;*
(d) *if $A = \{p, q\}$ and $p \neq q$, then $v(A) = n\binom{m}{2} + m\binom{n}{2} + nm$;*
(e) *if $A = \{p, q, x\}$ and $p \neq q$, then $v(A) = nm$;*
(f) *if $A = \{p, q, r\}$ and p, q, and r are all different, then $v(A) = nmk$;*
(g) *if $A \in \Lambda_3(X)$, then $v(A) = 1$;*
(h) *if $A \in EA_3(X)$, then $v(A) = \infty$.*

Proof: The proofs of assertions (a)–(g) follow from Theorem 2.6 and [5, Lemma 5.3], and the proof of (h) follows from Theorem 2.9. □

3. The main result

In this section, we show that any element $X \in \mathfrak{D}$ has unique hyperspace $F_n(X)$ for each $n \in \mathbb{N}$. In order to do this, we first present six results. We recall that $\mathcal{O}$ is the class of dendrites whose set of ordinary points is open.

Theorem 3.1 (see [11, Theorem 8])**.** *Let $X, Y \in \mathcal{O}$ be such that $F_2(X)$ is homeomorphic to $F_2(Y)$. Then X is homeomorphic to Y.*

Theorem 3.2. *Let Y and Z be continua and let $n \in \mathbb{N}$. If $h : F_n(Y) \to F_n(Z)$ is a homeomorphism, then $h(\mathcal{E}_n(Y)) = \mathcal{E}_n(Z)$.*

Theorem 3.3. *Let $X, Y \in \mathfrak{D}$. If $h\colon F_3(X) \to F_3(Y)$ is a homeomorphism, then*

(a) $h(R_3(X) \cup EA_3(X)) = R_3(Y) \cup EA_3(Y)$;
(b) *if* $A \in F_3(X)$, *then* $v(A, \mathcal{E}_3(X), F_3(X)) = v(h(A), \mathcal{E}_3(Y), F_3(Y))$;
(c) *if* $p \in R(X)$, *then* $h(\{p\}) = \{r\}$ *for some* $r \in R(Y)$;
(d) *if* $q \in E_a(X)$, *then* $h(\{q\}) = \{t\}$ *for some* $t \in E_a(Y)$.

Proof: To show (a), let $A \in R_3(X) \cup EA_3(X)$. If $h(A) \notin R_3(Y) \cup EA_3(Y)$, then $h(A) \in \Lambda_3(Y)$. By Theorem 2.8(b), we obtain $h(A) \in \mathcal{E}_3(Y)$; by Theorem 3.2, $A \in \mathcal{E}_3(X)$. Applying Theorem 2.8(b) again, we have $A \in \Lambda_3(X)$. Thus, $A \notin R_3(X) \cup EA_3(X)$. This contradiction shows that $h(A) \in R_3(Y) \cup EA_3(Y)$. Hence, $h(R_3(X) \cup EA_3(X)) \subset R_3(Y) \cup EA_3(Y)$. This shows (a).

To show (b), take $A \in F_3(X)$. Let $v(A) = v(A, \mathcal{E}_3(X), F_3(X))$ and $v(h(A)) = v(h(A), \mathcal{E}_3(Y), F_3(Y))$. Suppose that $v(A) = m \in \mathbb{N}$, and let a basis $\mathcal{B}$ of open neighborhoods of A in $F_3(X)$ be such that, for each $U \in \mathcal{B}$, $c(U, \mathcal{E}_3(X), F_3(X)) = m$. Then $\mathcal{C} = \{h(U) : U \in \mathcal{B}\}$ is a basis of open neighborhoods of $h(A)$ in $F_3(Y)$ such that, for each $\mathcal{V} \in \mathcal{C}$, $c(\mathcal{V}, \mathcal{E}_3(Y), F_3(Y)) = m$. This shows $v(h(A)) \leq v(A)$ when $v(A)$ is finite. By a similar argument, $v(A) \leq v(h(A))$, and thus, $v(A) = v(h(A))$ when one of them is finite. The remaining case is the one in which both sides are infinite, which is straightforward. This shows (b).

To show (c), take $p \in R(X)$. Then $\{p\} \in R_3(X) \subset R_3(X) \cup EA_3(X)$ so, by (a), $h(\{p\}) \in R_3(Y) \cup EA_3(Y)$. By Theorem 2.11(a), $v(\{p\}, \mathcal{E}_3(X), F_3(X))$ is a finite number greater than two. By (b), we obtain

$$v(h(\{p\}), \mathcal{E}_3(Y), F_3(Y)) = v(\{p\}, \mathcal{E}_3(X), F_3(X)).$$

Thus, $v(h(\{p\}), \mathcal{E}_3(Y), F_3(Y))$ is a finite number greater than two. Then, Theorem 2.11(g),(h), it follows that $h(\{p\}) \notin \Lambda_3(Y)$ and $h(\{p\}) \notin EA_3(Y)$, so $h(\{p\}) \subset R(Y) \cup O(Y) \cup (E(Y) - E_a(Y))$. Thus, $h(\{p\})$ is one of the forms described in Theorem 2.11(a)–(f); next, by the same argument as in [5, Lemma 5.5], it is not difficult to see that the only possibility is that $h(\{p\}) = \{r\}$ for some $r \in R(Y)$. This shows (c).

To show (d), take $q \in E_a(X)$. By Lemma 2.5, there is a sequence $\{q_i\}_{i=1}^{\infty}$ of distinct ramification points of X that converges to q. By (c), for each $i \in \mathbb{N}$, there is $r_i \in R(Y)$ such that $h(\{q_i\}) = \{r_i\}$. We can assume that the sequence $\{r_i\}_{i=1}^{\infty}$ converges to some $r \in Y$. Since $Y \in \mathfrak{D}$, by Theorem 2.3(b), $r \in E_a(Y)$. Since the sequence $\{h(\{q_i\})\}_{i=1}^{\infty}$ converges to $h(\{q\})$, we obtain that $h(\{q\}) = \{r\}$. This shows (d). □

Let $X, Y \in \mathfrak{D}$ such that there is a homeomorphism $h : F_3(X) \to F_3(Y)$. By Theorem 3.3(c), for every $p \in R(X)$, there exists $r \in R(Y)$ such that $h(\{p\}) = \{r\}$. Since h is a homeomorphism, the element r is unique. Since $h^{-1} : F_3(Y) \to F_3(X)$ is also a homeomorphism, for every $q \in R(Y)$, there is a unique $t \in R(X)$ such that $h^{-1}(\{q\}) = \{t\}$. Hence, there is a bijection between the sets $R(X)$ and $R(Y)$. Using Theorem 3.3(d), we obtain, in a similar way, that there is also a bijection between the sets $E_a(X)$ and $E_a(Y)$. Then we can define a bijection

$$k : R(X) \cup E_a(X) \to R(Y) \cup E_a(Y) \tag{3.1}$$

so that if $x \in R(X) \cup E_a(X)$, then $k(x)$ is the only element of $R(Y) \cup E_a(Y)$ such that $h(\{x\}) = \{k(x)\}$. Then

$$k(R(X)) = R(Y) \text{ and } k(E_a(X)) = E_a(Y).$$

Since h is continuous, k is also continuous. The next result will show that some properties of the ramification points of X and the corresponding ramification points of Y remain invariant.

Theorem 3.4. *Let $X, Y \in \mathfrak{D}$ such that there is a homeomorphism $h : F_3(X) \to F_3(Y)$. Let $k : R(X) \cup E_a(X) \to R(Y) \cup E_a(Y)$ be the continuous bijection defined in* (3.1)*. Then the following assertions hold.*

(a) *If $x, y \in R(X)$ are adjacent in X, then $k(x)$ and $k(y)$ are adjacent in Y and $h(\langle [x,y] \rangle_3) = \langle [k(x), k(y)] \rangle_3$;*
(b) *if $x \in R(X)$, then $ord_x X = ord_{k(x)} Y$;*
(c) *if $x \in R(X)$, then there is a bijection between the set of end points adjacent to x and the set of end points adjacent to $k(x)$;*
(d) *X is homeomorphic to Y.*

Proof: We first remark that $h(\mathcal{E}_3(X)) = \mathcal{E}_3(Y)$ by Theorem 3.2. Next, by Theorem 2.7 and Theorem 2.8(b), we obtain $\mathcal{E}_3(X)$ is an open subset of $F_3(X)$ whose components are exactly the sets of the form

$$\langle int_X(I_1), \ldots, int_X(I_m) \rangle_3,$$

where $m \leq 3$ and $I_1, \ldots, I_m$ is a finite collection of pairwise disjoint subsets of X such that $I_j \in \mathcal{A}(X)$ for every $j \in \{1, ..., m\}$. Since $Y \in \mathfrak{D}$, the components of $\mathcal{E}_3(Y)$ are described in a similar way.

To show (a), let $x, y \in R(X)$ such that x and y are adjacent in X. The fact that $k(x)$ and $k(y)$ are adjacent in Y follows from the above statements, together with an argument similar to that of Claim 1 of Theorem 5.6 in [5]. Since $\langle [x,y] \rangle_3$ is a component of $\mathcal{E}_3(X)$, the set $h(\langle [x,y] \rangle_3)$ is a component of $\mathcal{E}_3(X)$. Thus,

$$h(\langle [x,y] \rangle_3) = \langle int_Y(K_1), \ldots, int_Y(K_m) \rangle_3,$$

where $m \leq 3$ and $K_1, ..., K_m$ is a finite collection of pairwise disjoint subsets of Y such that $K_j \in \mathcal{A}(Y)$ for each $j \in \{1, ..., m\}$. Notice that $h(\{x\}) = \{k(x)\}$ and $h(\{y\}) = \{k(y)\}$. Since x and y are the only ramification points of X such that $\{x\}, \{y\} \in cl_{F_3(X)}(\langle [x,y] \rangle_3)$, it follows that $k(x)$ and $k(y)$ are the only ramification points of Y such that

$$\{k(x)\}, \{k(y)\} \in \mathrm{cl}_{F_3(Y)}(\langle int_Y(K_1), \ldots, int_Y(K_m) \rangle_3)\,.$$

This can happen only when $m = 1$ and $K_1 = [k(x), k(y)]$. Thus, (a) holds.

To show (b), let $x \in R(X)$. By Theorem 2.2, we obtain $ord_x X < \omega$. The fact that $ord_x X = ord_{k(x)} Y$ follows from a similar argument to that in Claim 3 of Theorem 5.6 in [5], using Theorem 2.11(a), and $h(\mathcal{E}_3(X)) = \mathcal{E}_3(Y)$.

To show (c), let $x \in R(X)$. Notice that the number of end points adjacent to x can be obtained from $\mathrm{ord}_x X$ by subtracting the ones that are ramification points. Since $ord_x X = ord_{k(x)} Y$, and the number of ramification points of X adjacent to x is the same as the number of ramification points of Y adjacent to $k(x)$, the number of end points in X adjacent to x is the same as the number of end points in Y adjacent to $k(x)$. This shows (c).

To show (d), we are going to extend k to a homeomorphism between X and Y. For each pair of adjacent ramification points x and y in X, let $f_{\{x,y\}} : [x, y] \to [k(x), k(y)]$ be a homeomorphism such that $f_{\{x,y\}}(x) = k(x)$ and $f_{\{x,y\}}(y) = k(y)$. For each $x \in R(X)$, consider the set G_x of end points of X adjacent to x and the set L_x of end points of Y adjacent to $k(x)$. By (c), there is a bijection $g_x : G_x \to L_x$. If $x \in R(X)$ and $e \in G_x$, we define $f_{\{x,e\}} : [x, e] \to [k(x), g_x(e)]$ to be a homeomorphism such that $f_{\{x,e\}}(x) = k(x)$ and $f_{\{x,e\}}(e) = g_x(e)$. Notice that a homeomorphism $f_{\{x,y\}}$ has been defined for every $x, y \in X$ such that $[x, y] \in \mathcal{A}(X)$. Since the family $\mathcal{A}(X)$ is a locally finite cover of the open set $X - E_a(X)$, there exists a unique continuous function $f^* : X - E_a(X) \to Y$, which is an extension of each $f_{\{x,y\}}$; that is, for each $[x, y] \in \mathcal{A}(X)$, we have $f^*|[x, y] = f_{\{x,y\}}$. Also, there exists a continuous function $g^* : cl(X - E_a(X)) \to Y$ such that $g^*|(X - E_a(X)) = f^*$ and $g^*(q) = k(q)$ if $q \in Bd(X - E_a(X))$. We now define $f : X \to Y$ in the following way

$$f(p) = \begin{cases} k(p) & \text{if } p \in R(X) \cup E_a(X), \\ g^*(p) & \text{if } p \in cl(X - E_a(X)). \end{cases}$$

Clearly, f is well defined. It is not difficult to prove that f is a bijection. Now we prove that f is continuous. We remark that it remains to show that f is continuous at each of the points of $E_a(X)$.

Let $p \in E_a(X)$ and let $\{p_i\}_{i=1}^{\infty}$ be a sequence converging to p. If there is a subsequence of $\{p_i\}_{i=1}^{\infty}$ contained in $E_a(X)$, by the continuity of k, the sequence $\{f(p_i)\}_{i=1}^{\infty}$ converges to $f(p)$. If not, we can assume that for each $i \in \mathbb{N}$, there is an $A_i \in \mathcal{A}(X)$ such that $p_i \in A_i$. We can also assume that the collection of the A_i is pairwise disjoint and thus, $\lim A_i = \{p\}$ because dendrites cannot contain a continuum of convergence. Choose $q_i \in A_i \cap R(X)$ for each $i \in \mathbb{N}$, so the sequence $\{q_i\}_{i=1}^{\infty}$ converges to p and $\{f(q_i)\}_{i=1}^{\infty}$ converges to $f(p)$ by the continuity of k. By (a), for each $i \in \mathbb{N}$, there exists $B_i \in \mathcal{A}(Y)$ such that $h(\langle A_i \rangle_3) = \langle B_i \rangle_3$. Thus, the collection of the sets B_i for each $i \in \mathbb{N}$ is pairwise disjoint and contains the sequence $\{f(q_i)\}_{i=1}^{\infty}$. Again, dendrites cannot contain continua of convergence; therefore, $\lim B_i = \{f(p)\}$. Since $f(p_i) \in B_i$ for each $i \in \mathbb{N}$, we have that the sequence $\{f(p_i)\}_{i=1}^{\infty}$ converges to $f(p)$ which is what we wanted. □

We are ready to prove that $X \in \mathfrak{D}$ has unique hyperspace $F_n(X)$ in $\mathfrak{D}$ for each $n \in \mathbb{N}$.

Theorem 3.5. *Let $X, Y \in \mathfrak{D}$ and let $n \in \mathbb{N}$. If $F_n(X)$ is homeomorphic to $F_n(Y)$, then X and Y are homeomorphic.*

Proof: Let $X, Y \in \mathfrak{D}$, let $n \in \mathbb{N}$, and let $h : F_n(X) \to F_n(Y)$ be a homeomorphism. In order to show that X is homeomorphic to Y, we consider four cases. Assume first that $n = 1$. Then $F_1(X)$ is homeomorphic to $F_1(Y)$, so X is homeomorphic to Y. Now assume that $n = 2$. Then $F_2(X)$ is homeomorphic to $F_2(Y)$. By Corollary 2.4, we have that $X, Y \in \mathcal{O}$ so, by Theorem 3.1, we obtain that X is homeomorphic to Y. Now assume that $n = 3$. Then, by Theorem 3.4(d), we have X is homeomorphic to Y.

Let us consider that $n \geq 4$. Let $A \in F_1(X) \cap \Lambda_n(X)$. By Theorem 2.10, there is a basis $\mathcal{B}$ of open neighborhoods of A in $F_n(X)$ such that, for each $\mathcal{V} \in \mathcal{B}$, the set $\mathcal{V} \cap \mathcal{E}_n(X)$ is arcwise connected. Let $\mathcal{C} = \{h(\mathcal{V}) : \mathcal{V} \in \mathcal{B}\}$. By Theorem 3.2, for each $\mathcal{V} \in \mathcal{B}$, we have that

$$h(\mathcal{V} \cap \mathcal{E}_n(X)) = h(\mathcal{V}) \cap h(\mathcal{E}_n(X)) = h(\mathcal{V}) \cap \mathcal{E}_n(Y).$$

Then $\mathcal{C}$ is a basis of neighborhoods of $h(A)$ in $F_n(Y)$ such that, for each $\mathcal{U} \in \mathcal{C}$, the set $\mathcal{U} \cap \mathcal{E}_n(Y)$ is arcwise connected. Hence, $h(A) \in F_1(Y) \cap \Lambda_n(Y)$ by Theorem 2.10. This shows that $h(F_1(X) \cap \Lambda_n(X)) \subset F_1(Y) \cap \Lambda_n(Y)$. Since $O(X)$ is dense in X and $F_1(O(X)) \subset F_1(X) \cap \Lambda_n(X)$, it follows that $cl_{F_n(X)}(F_1(X) \cap \Lambda_n(X)) = F_1(X)$. Similarly, $cl_{F_n(Y)}(F_1(Y) \cap \Lambda_n(Y)) = F_1(Y)$. Then

$$h(F_1(X)) = h(cl_{F_n(X)}(F_1(X) \cap \Lambda_n(X))) \subset cl_{F_n(Y)}(F_1(Y) \cap \Lambda_n(Y)) = F_1(Y).$$

Since h^{-1} is a homeomorphism, we also have that $h^{-1}(F_1(Y)) \subset F_1(X)$. Therefore, $h(F_1(X)) = F_1(Y)$, which proves that X and Y are homeomorphic. □

The following result was proved in [3, Theorem 5.2].

Theorem 3.6. *Let $X \in \mathfrak{D}$ and $n \in \mathbb{N}$. If Y is a continuum such that $F_n(X)$ is homeomorphic to $F_n(Y)$, then $Y \in \mathfrak{D}$.*

Combining Theorem 3.5 and Theorem 3.6, we have the main result of this paper.

Theorem 3.7. *If* $X \in \mathfrak{D}$ *and* $n \in \mathbb{N}$, *then* X *has unique hyperspace* $F_n(X)$.

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