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A NOTE ON DUALITY PROPERTIES

ZUOMING YU AND ZIQIU YUN*

Abstract. We prove that if a topological space X is discrete complete and the union of a countable family of dually-discrete spaces, then X is a compact space. We also study the properties of dually closed countable spaces. At the end of this paper, we prove that regular meta-Lindelöf star-compact spaces with G_δ-diagonals are metrizable.

1. Introduction

An *open neighborhood assignment (ONA)*, for a topological space (X, τ) is a function $\phi : X \to \tau$ such that $x \in \phi(x)$. A set $Y \subseteq X$ is a *kernel* of ϕ if $\phi(Y) = \{\phi(y) : y \in Y\}$ covers X [5]. If ϕ is an ONA on X and $\phi(x) = U(x)$, we will sometimes abuse the notation and write $\phi = \{U(x) : x \in X\}$. Given a property $\mathcal{P}$ that a subset of a topological space might have, the class $\mathcal{P}^*$ dual to $\mathcal{P}$ (with respect to an ONA) consists of spaces X such that for any ONA ϕ on X, there is $Y \subseteq X$ with property $\mathcal{P}$ and $\phi(Y) = \{\phi(y) : y \in Y\}$ covers X. It is a development of an idea Eric K. van Douwen and Washek F. Pfeffer [6] used to define D-spaces. Many dually-$\mathcal{P}$ classes have been studied in [15], [5], [2], [1].

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In this paper, we study some properties of dually discrete spaces and dually closed countable spaces. We also study the metrization theorems of star-compact spaces with G_δ-diagonals.

2. Notation and terminology

All spaces under consideration are assumed to be T_1.

A space X is *discretely complete* if every infinite discrete subspace has a complete accumulation point in X [3]. Clearly, discretely completeness is a property between compactness and countable compactness. A noncompact discretely complete space was constructed in [3].

A space X is *linearly Lindelöf* if every open cover of X, linearly ordered by the subset relation, has a countable subcover.

A subset S of space X is called *preopen* if $S \subseteq int(clS)$. A space is *strongly Lindelöf* if every preopen cover of X admits a countable subcover. A space is *d-Lindelöf* if every cover of X by dense subsets has a countable subcover. A space is said to be *monotonically-Lindelöf* if one can assign to every open cover $\mathcal{U}$ a countable open cover $r(\mathcal{U})$ refining $\mathcal{U}$ such that $r(\mathcal{U})$ refines $r(\mathcal{V})$ whenever $\mathcal{U}$ refines $\mathcal{V}$.

Let $\mathbf{m}$ be a cardinal number. A space X is said to be $\mathbf{m}$-*expandable* (*discretely* $\mathbf{m}$-*expandable*, respectively), if for every locally finite (discrete, respectively) collection $\{F_\lambda : \lambda \in \Lambda\}$ of a subset of X with $\Lambda \leq \mathbf{m}$, there exists a locally finite collection $\{G_\lambda : \lambda \in \Lambda\}$ of open subset of X such that $F_\lambda \subseteq G_\lambda$ for each $\lambda \in \Lambda$. A topological space is said to be *expandable*, if it is $\mathbf{m}$-*expandable* for every cardinal number $\mathbf{m}$ [12].

Given a class $\mathcal{P}$ of topological spaces, say that a space X is *star*-$\mathcal{P}$ if, for any open cover $\mathcal{U}$ of the space X, there is a subspace $Y \subset X$, such that $Y \in \mathcal{P}$ and $\{st(y,\mathcal{U}) : y \in Y\}$ covers X, where $st(y,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : y \in U\}$ [14]. A space X is said to be *star-compact* (*finite*, *countable*, respectively) if, for every open cover $\mathcal{U}$ of X, there is a compact (finite, countable, respectively) subspace Y such that $\{st(y,\mathcal{U}) : y \in Y\}$ covers X.

3. Union of dually discrete spaces

It is not difficult to prove that X is dually discrete if it is the union of finitely many discrete subspaces. In [5], the authors asked *whether X is dually discrete if it is a countable union of discrete subspaces* ([5, Problem 4.9]). It was proved in [1] that a space is compact if and only if it is dually discrete and discrete complete. The following theorem not only gives a partial answer to the above question, but also improves the above result in [1]. The proof of our result is a modification of that of [16, Theorem 2].

Theorem 3.1. *Suppose that X is discrete complete and the countable union of dually-discrete spaces, then X is a compact space.*

To prove Theorem 3.1, we need the following lemma.

Lemma 3.2 ([9])**.** *A space X is linearly Lindelöf if and only if whenever $\mathcal{U}$ is an open cover of X of cardinality κ and $\mathcal{U}$ has no subcover of cardinality $< \kappa$, then $cf(\kappa) \leq \omega$.*

Proof of Theorem 3.1: Let $X = \bigcup\{X_i : i \in N\}$ where X_i is a dually discrete space for each $i \in N$. It is well known that a linearly Lindelöf countably compact space is compact; hence, we need only to prove that X is linearly Lindelöf since X is discrete complete. Assume the contrary that X is not linearly Lindelöf. Then by Lemma 3.2, there is an open cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ of some cardinality κ with $cf(\kappa) > \omega$, such that $\mathcal{U}$ has no subcover of cardinality $< \kappa$. Without loss of generality, we can assume that κ is the minimal cardinality satisfying the condition.

Consider the ONA defined by $\phi(x) = U_{\alpha_x}$ for each $x \in X$, where α_x is the least such that $x \in U_{\alpha_x}$. For each $i \in N$, there is a discrete subset D_i of X_i such that $\phi(D_i)$ covers X_i. Since $\mathcal{U}$ has no subcover of cardinality $< \kappa$, there is some $i_0 \in N$ such that $|\{\alpha_d : d \in D_{i_0}\}| = \kappa$, and it follows from the definition of α_d that we can assume that if $y, z \in D_{i_0}$ and $y \neq z$, then $\alpha_y \neq \alpha_z$.

STEP 0:

Let H_0 be the subset of X consisting of all the complete accumulation points of D_{i_0}. It is easy to prove that H_0 is closed in X.

Claim 1. $H_0 \cap \bigcup \phi(D_{i_0}) = \emptyset$.

Assuming the contrary, we can pick $x \in H_0 \cap \bigcup \phi(D_{i_0})$. Then there is $d \in D_{i_0}$ such that $x \in \phi(d)$. Since x is a complete accumulation point, it follows that $|D_{i_0} \cap \phi(d)| = \kappa$. On the other hand, $\alpha_y \leq \alpha_d < \kappa$ for each $y \in D_{i_0} \cap \phi(d)$, which is a contradiction.

Claim 2. $\mathcal{U}$ has no subfamily $\mathcal{U}_0$ of cardinality $< \kappa$, such that $\mathcal{U}_0$ covers H_0.

Assuming the contrary, there is some $\beta < \kappa$, such that $H_0 \subseteq \bigcup\{U_\alpha : \alpha < \beta\} = V_1$. Then $|\{\alpha_d : \alpha_d \in D_{i_0} \cap V_1\}| = \kappa$. So there is some $d \in D_{i_0}$ such that $\alpha_d > \beta$. Since $d \in V_1$, there is some $\alpha_0 < \beta$ such that $d \in U_{\alpha_0}$, a contradiction to the definition of α_d.

Let $\mathcal{V}_0 = \{X_i : X_i \cap H_0 = \emptyset\}$.

Suppose that we have defined a closed subset H_β of X and $\mathcal{V}_\beta$ for each $\beta < \alpha < \omega_1$, such that

(1) $H_\beta \subseteq \cap_{\gamma<\beta} H_\gamma$;
(2) $\mathcal{U}$ has no subfamily $\mathcal{U}_\beta$ of cardinality $< \kappa$, such that $\mathcal{U}_\beta$ covers H_β;
(3) $H_\beta \cap \bigcup \mathcal{V}_\beta = \emptyset$ and $\mathcal{V}_{\gamma_1} \cap \mathcal{V}_{\gamma_2} = \emptyset$ whenever $\gamma_1, \gamma_2 \leq \beta$, and $\gamma_1 \neq \gamma_2$.

STEP α:

Case 1: $\alpha = \beta + 1$.

Note that $H_\beta = \cup\{H_\beta \cap X_i : X_i \in \{X_i : i \in N\} \setminus \{\mathcal{V}_\gamma : \gamma \leq \beta\}\}$, and $H_\beta \cap X_i$ is closed in X_i for each $X_i \in \{X_i : i \in N\} \setminus \{\mathcal{V}_\gamma : \gamma \leq \beta\}$. For each $X_i \in \{X_i : i \in N\} \setminus \{\mathcal{V}_\gamma : \gamma \leq \beta\}$, let $\phi(x) = U_{\alpha_x} \cap (H_\beta \cap X_i)$ for each $x \in H_\beta \cap X_i$. Then ϕ is an ONA on $H_\beta \cap X_i$; hence, there is a discrete subset D_i of $H_\beta \cap X_i$ such that $\phi(D_i)$ covers $H_\beta \cap X_i$. Since $\mathcal{U}$ has no subfamily of cardinality $< \kappa$ covering H_β, there is some $i_\beta \in N$ such that $|\{\alpha_d : d \in D_{i_\beta}\}| = \kappa$, and we can assume that if $y, z \in D_{i_\beta}$, then $\alpha_y \neq \alpha_z$. Let H' be the subset consisting of all the complete accumulation points of D_{i_β}, then $H' \subseteq H_\beta$. It is easy to see that H' is a closed subspace of H_β, hence closed in X. With a similar proof to Step 0, $H' \cap \bigcup \phi(D_{i_\beta}) = \emptyset$, and $\mathcal{U}$ has no subfamily $\mathcal{U}'$ of cardinality $< \kappa$, such that $\mathcal{U}'$ covers H'. Let $H_\alpha = H'$ and $\mathcal{V}_\alpha = \{X_i : X_i \cap H' = \emptyset\} \setminus \{\mathcal{V}_\gamma : \gamma \leq \beta\}$.

Case 2: α is a limit ordinal number.

Let $H_\alpha = \cap\{H_\beta : \beta < \alpha\}$. Since X is discrete complete, hence countably compact, H_α is a nonempty closed subset of X. $\mathcal{U}$ has no subfamily $\mathcal{U}_\alpha$ of cardinality $< \kappa$, such that $\mathcal{U}_\alpha$ covers H_α, for otherwise, there is some $\beta < \alpha$, such that $\mathcal{U}_\alpha$ covers H_β by countable compactness of X, which contradicts condition (2). Let $\mathcal{V}_\alpha = \{X_i : X_i \cap H_\alpha = \emptyset\} \setminus \{\mathcal{V}_\gamma : \gamma < \alpha\}$.

It is easy to prove that H_α and $\mathcal{V}_\alpha$ satisfy conditions (1)–(3).

Claim 3: There is some $\alpha < \omega_1$, such that $H_\alpha \cap (\bigcup(\{X_i : i \in N\} \setminus \bigcup\{\mathcal{V}_\gamma : \gamma \leq \alpha\}) = \emptyset$.

Otherwise, for each $\beta < \omega_1$, we can repeat the step above and get disjoint families $\{\mathcal{V}_\beta : \beta < \omega_1\}$. But this contradicts the fact that $|\{X_i : i \in N\}| = \omega$.

On the other hand, $H_\alpha \cap \bigcup\{\mathcal{V}_\gamma : \gamma < \alpha\} = \emptyset$ by condition (3), which means that $H_\alpha = \emptyset$, which contradicts condition (2). This contradiction implies that X is linearly Lindelöf. $\square$

Remark 3.3. It is proved in [15, Example 2.3] that ω_1 is dually discrete; hence, the condition"discrete complete" can not be weakened to "countably compact" in Theorem 3.1.

Proposition 3.4. *If X is a union of finitely many strongly Lindelöf subspaces, then X is a D-space.*

Before we prove this proposition, we need the following lemmas.

Lemma 3.5 ([8, Proposition 2.2])**.** *For a space X, the following are equivalent:*

(1) *X is d-Lindelöf;*
(2) *$X \setminus I_X$ is countable, where I_X is the set of isolated points of X.*

Lemma 3.6 ([8, Theorem 2.5])**.** *For a space X, the following are equivalent:*

(1) *X is strongly Lindelöf;*
(2) *X is Lindelöf and d-Lindelöf.*

Lemma 3.7. *d-Lindelöf is hereditary.*

Proof: Let X be a d-Lindelöf space and $A \subseteq X$. Suppose that $\{A_\alpha : \alpha < \beta\}$ is a cover of A by dense subsets of A, then $\{A_\alpha \cup (X \setminus A) : \alpha < \beta\}$ is a cover of X by dense subsets. By the definition of d-Lindelöf, there is a countable subcover $\{A_{\alpha_n} \cup (X \setminus A) : n < \omega\}$ of X. Then $\{A_{\alpha_n} : n < \omega\}$ covers A. $\square$

Lemma 3.8. *If $X = X_1 \cup X_2$, where X_1 is a D-space and a closed subspace of X such that for each open set $U \supseteq X_1$, $X \setminus U$ is countable, then X is a D-space.*

Proof: Suppose that ϕ is an ONA on X. Then there is a closed discrete subset C of X such that $\bigcup \phi(C) \supseteq X_1$. Since $X \setminus \bigcup \phi(C)$

is a countable closed subset of X, there is a closed discrete subset D of X such that $\bigcup \phi(D) \supseteq X \setminus X_1$. It is obvious that $C \cup D$ is a closed discrete set in X and $\bigcup \phi(C \cup D) = X$. □

Proof of Proposition 3.4: Suppose that $X = \bigcup_{i=1}^{k} X_i$, where X_i is strongly Lindelöf for each $i \leq k$, and ϕ is an ONA on X.

Let $F_{i,1} = X_i \setminus I_{X_i}$, where I_{X_i} is the set of isolated points of X_i. By Lemma 3.5 and Lemma 3.6, $F_{i,1}$ is a countable subset of X_i. Since X_i is strongly Lindelöf, for each open set $U \supseteq F_{i,1}$, $X_i \setminus U$ is countable. Hence, by Lemma 3.8, to prove that X is a D-space, we only need to prove that $\overline{F_{i,1}}$ is a D-space for each $i \leq k$.

Let $i_1 \in \{1, ..., k\}$, $K_{j,1} = X_j$, and $K_{j,2} = \overline{F_{i_1,1}} \cap K_{j,1}$, $j \in \{1, ..., k\} \setminus \{i_1\}$. Then $K_{j,2}$ is strongly d-Lindelöf by Lemma 3.7. Let $F_{j,2} = K_{j,2} \setminus I_{K_{j,2}}$, where $I_{K_{j,2}}$ is the set of isolated points of $K_{j,2}$. Since $\overline{F_{i_1,1}} \subseteq F_{i_1,1} \cup \bigcup\{K_{j,2} : j \in \{1, ..., k\} \setminus \{i_1\}\}$ and for each open set $U \supseteq F_{j,2}$, $K_{j,2} \setminus U$ is countable, by Lemma 3.8, to prove that $\overline{F_{i_1,1}}$ is a D-space, we need only to prove that $\overline{F_{j,2}}$ is a D-space for each $j \in \{1, ..., k\} \setminus \{i_1\}$.

Let $i_2 \in \{1, ..., k\} \setminus \{i_1\}$ and $K_{j,3} = \overline{F_{i_2,2}} \cap K_{j,2}$, $j \in \{1, ..., k\} \setminus \{i_1, i_2\}$. Let $F_{j,3} = K_{j,3} \setminus I_{K_{j,3}}$, where $I_{K_{j,3}}$ is the set of isolated points of $K_{j,3}$. Since $\overline{F_{i_2,2}} \subseteq F_{i_1,1} \cup F_{i_2,2} \cup \bigcup\{K_{j,3} : j \in \{1, ..., k\} \setminus \{i_1, i_2\}\}$ and for each open set $U \supseteq F_{j,3}$, $K_{j,3} \setminus U$ is countable, by Lemma 3.8, to prove that $\overline{F_{i_2,2}}$ is a D-space, we need only to prove that $\overline{F_{j,3}}$ is a D-space for each $j \in \{1, ..., k\} \setminus \{i_1, i_2\}$.

We continue in this way. Let $i_{k-1} \in \{1, ..., k\} \setminus \{i_1, ..., i_{k-2}\}$ and $K_{j,k} = \overline{F_{i_{k-1},k-1}} \cap K_{j,k-1}$, $j \in \{1, ..., k\} \setminus \{i_1, ..., i_{k-1}\}$. Let $F_{j,k} = K_{j,k} \setminus I_{K_{j,k}}$, where $I_{K_{j,k}}$ is the set of isolated points of $K_{j,k}$. Since $\overline{F_{i_{k-1},k-1}} \subseteq F_{i_1,1} \cup F_{i_2,2} \cup \cdots \cup F_{i_{k-1},k-1} \cup K_{j,k}$ for $j \in \{1, ..., k\} \setminus \{i_1, i_2, \cdots, i_{k-1}\}$, and for each open set $U \supseteq F_{j,k}$, $K_{j,k} \setminus U$ is countable, by Lemma 3.8, to prove that $\overline{F_{i_{k-1},k-1}}$ is a D-space, we need only to prove that $\overline{F_{j,k}}$ is a D-space for $j \in \{1, ..., k\} \setminus \{i_1, ..., i_{k-1}\}$.

Since $|F_{j,k}| \leq \omega$ and $\overline{F_{j,k}} \subseteq F_{i_1,1} \cup F_{i_2,2} \cup \cdots \cup F_{i_{k-1},k-1} \cup F_{j,k}$ for $j \in \{1, ..., k\} \setminus \{i_1, i_2, \cdots, i_{k-1}\}$, $\overline{F_{j,k}}$ is a countable D-space.

Therefore, X is a D-space. □

Question 3.9. If X is a union of countably many strongly Lindelöf subspaces, then is X a D-space (or dually discrete)?

4. DUALLY CLOSED COUNTABLE SPACES

In [15, Corollary 2.9], dually countable was proved to be equivalent to Lindelöf; hence, dually closed countable spaces are Lindelöf. One can see that Lindelöf D-spaces and strongly Lindelöf spaces are dually closed countable spaces, but dually closed countable spaces need not be monotone Lindelöf, since there is a countable space which is not monotone Lindelöf [13]. It is a long standing question whether a regular Lindelöf space is a D-space, and a negative answer to the following question also gives a negative one to the following question.

Question 4.1. Is every Lindelöf space dually closed countable?

It is obvious that closed subspaces of dually closed countable spaces are dually closed countable spaces. Open subspaces of dually closed countable spaces may fail to be dually closed countable. For example, the open subspace $[0, \omega_1)$ of the compact space $[0, \omega_1]$ is not dually closed countable.

Proposition 4.2. *If X is the countable union of closed dually closed countable spaces, then X is dually closed countable.*

Proof: Let $X = \bigcup_{n<\omega} F_n$, where F_n is a closed dually closed countable space for each $n < \omega$. Suppose that ϕ is an ONA on X. Pick a closed countable subset H_1 of F_1 such that $\phi(H_1)$ covers F_1. Since $F_2 \setminus \bigcup \phi(H_1)$ is closed in F_2, we can find a closed countable subset H_2 of $F_2 \setminus \bigcup \phi(H_1)$ such that $\phi(H_2)$ covers $F_2 \setminus \bigcup \phi(H_1)$. Note that ${H_2}^* = H_1 \cup H_2$ is a closed countable subset of $F_1 \cup F_2$ such that $\phi({H_2}^*)$ covers $F_1 \cup F_2$. Inductively, as the collection $\phi(F_n \setminus \bigcup \phi(H^*_{n-1}))$ consisting of open subsets of X covers $F_n \setminus \bigcup \phi(H^*_{n-1})$, we can pick a closed countable subset H_n of $F_n \setminus \bigcup \phi(H^*_{n-1})$ such that $\phi(H_n)$ covers $F_n \setminus \bigcup \phi(H^*_{n-1})$. Note that $H^*_n = H^*_{n-1} \cup H_n$ is closed countable subset of $\bigcup_{i=1}^n F_i$ such that $\phi(H^*_n)$ covers $\bigcup_{i=1}^n F_i$.

Let $H = \bigcup_{n<\omega} H_n$. Then H is a countable subset of X such that $\phi(H)$ covers X, and hence, we need only to prove that H is closed in X. For each $x \notin H$, take $n_x < \omega$ and $y \in H_{n_x}$ such that $x \in \phi(y)$. Then $\phi(y) \cap H_n = \emptyset$ whenever $n > n_x$. Since $H^*_{n_x}$ is closed, there is a neighborhood U of x such that $U \cap H^*_{n_x} = \emptyset$. Then $U \cap \phi(y)$ is a neighborhood of x and $(U \cap \phi(y)) \cap H = \emptyset$. Therefore, H is closed in X. □

Corollary 4.3. *F_σ subspaces of dually closed countable spaces are dually closed countable.*

Proposition 4.4. *Dually closed countable spaces are invariant under closed maps.*

Proof: Let $f : X \to Y$ be a closed map between X and Y, such that X is dually closed countable, and let $\phi = \{U(y) : y \in Y\}$ be an ONA on Y. Define an ONA ϕ' on X as follows: $\phi'(x) = f^{-1}(U(y))$, if $x \in f^{-1}(y)$ for each $x \in X$. Since X is dually closed countable, there is a closed countable subset $F \subseteq X$ such that $\phi'(F)$ covers X. Hence, $f(F)$ is a closed countable subset of Y such that $\phi(f(F))$ covers Y. □

Proposition 4.5. *Dually closed countable spaces are inversely invariant under countable-to-one closed maps.*

Proof: Let $f : X \to Y$ be a closed map between X and Y such that Y is dually closed countable. Assume that $\phi = \{U(x) : x \in X\}$ is an ONA on X. Since f is closed, for each $y \in Y$, we can take an open set $V(y)$ in Y such that $f^{-1}(y) \subseteq V(y) \subseteq \bigcup\{U(x) : x \in f^{-1}(y)\}$. Then $\{V(y) : y \in Y\}$ is an ONA on Y. Take a countable closed A of Y, such that $\{V(y) : y \in A\}$ covers Y. Then $f^{-1}(A)$ is a countable closed subset of X, since f is countable to one. It is obvious that $\{U(x) : x \in f^{-1}(A)\}$ covers X. Therefore, X is dually closed countable. □

Question 4.6. Is every dually closed countable space inversely preserved under perfect mappings?

Notice that a negative answer to the above question will also give us a Lindelöf space which is not a D-space, since Lindelöfness is inversely preserved under closed mappings with Lindelöf fibers.

Remark 4.7. For a dually closed countable space X, X^2 may fail to be Lindelöf. For example, let X be the Sorgenfrey line. Then X is a Lindelöf D-space, and hence X is dually closed countable. But it is well known that X^2 is not Lindelöf.

5. When are star-compact spaces with G_δ-diagonals metrizable?

Lemma 5.1. *If X is an $\aleph_0$-expandable (discretely $\aleph_0$-expandable, respectively) star-compact space, then X is countably compact.*

Proof: We only prove for the case of $\aleph_0$-expandable. Assume the contrary, let $D = \{x_n : n < \omega\}$ be an infinite closed discrete subset of X. Since X is $\aleph_0$-expandable, there is a locally finite collection of open subsets $\mathcal{U} = \{U_n : n < \omega\}$ such that $x_n \in U_n$. Let $\mathcal{U}' = \{(U_n \setminus D) \cup \{x_n\} : n < \omega\} = \{U'_n : n < \omega\}$. Clearly, each U' of $\mathcal{U}'$ contains only one element of D. Let $\mathcal{W} = \mathcal{U}' \bigcup \{X \setminus D\}$. Obviously, $\mathcal{W}$ is a point-finite open cover of X. Since X is star-compact, there is a compact subspace K of X such that $\{st(z, \mathcal{W}) : z \in K\}$ covers X. Since $st(a, \mathcal{W}) \cap D = \emptyset$ for each $a \notin \bigcup \mathcal{U}'$, we have $K \cap \bigcup \mathcal{U}' \neq \emptyset$. Because $st(x_n, \mathcal{W}) = U'_n$ for each $x_n \in D$, $K \cap U'_n \neq \emptyset$ for each $n < \omega$, else we have $x_n \notin \bigcup \{st(z, \mathcal{W}) : z \in K\}$. Pick $z_n \in K \cap U'_n$ for each $n < \omega$. Therefore, we get an infinite locally finite subset of K since $\mathcal{U}'$ is locally finite, which is a contradiction to the compactness of K. Hence, X is countably compact. □

Since normal spaces are discretely $\aleph_0$-expandable, we have the following corollary.

Corollary 5.2. *Normal star-compact spaces are countably compact.*

Remark 5.3. Indeed, the result that normal star-compact spaces are countably compact also follows from the fact that star-compact spaces are pseudocompact [7, Theorem 2.1.6]. With a proof similar to that of [7, Example 2.3.3], one can show that the Tychonoff plank is star-compact but not countably compact.

Proposition 5.4. *A Hausdorff star-compact space X with G_δ-diagonal is metrizable if it is a countably paracompact space or a wM-space.*

Proof: Since wM-spaces are expandable [17], and countably paracompact spaces are $\aleph_0$-expandable [12], the proposition above follows from Lemma 5.1. □

Lemma 5.5. *Let X be a star-compact space with G_δ-diagonal. Then X is star-countable.*

Proof: Assume that $\mathcal{U} = \{U_\alpha : \alpha \in \Gamma\}$ is an open cover of X. Since X is star-compact, there is a compact subspace K of X, such that $st(K, \mathcal{U}) = X$. Since K also has a G_δ-diagonal, K is a compact metrizable space, and hence K is separable. Let H be a countable dense subset of K, and let $\mathcal{U}' = \{U \in \mathcal{U} : U \cap K \neq \emptyset\}$. Obviously,

$H \cap U \neq \emptyset$ for each $U \in \mathcal{U}'$, and hence $\{st(y, \mathcal{U}) : y \in H\}$ covers X. So, X is star-countable. □

Definition 5.6 ([10, Definition 1.2])**.** A topological space X is said to have property ($\sharp$) if, for each (uncountable) closed discrete subset $Z \subset X$, there is an open neighborhood V_z for each $z \in Z$ such that

(1) $V_z \cap Z = \{z\}$;
(2) $\mathcal{V} = \{V_z : z \in Z\}$ is point-countable.

It is known that separable spaces with property ($\sharp$) are ω_1-compact [10, Theorem 1.1]. In fact, separable spaces are star-countable, but the converse is not true; for example, $[0, \omega_1)$ is star-countable but not separable. We can improve the result above by the following lemma.

Lemma 5.7. *Star-countable spaces with property ($\sharp$) are ω_1-compact.*

Proof: Let X be a space with property ($\sharp$) and let Z be an uncountable closed discrete subset of X. Pick an open neighborhood V_z for each $z \in Z$ such that

(1) $V_z \cap Z = \{z\}$;
(2) $\mathcal{V} = \{V_z : z \in Z\}$ is point-countable.

Let $\mathcal{U} = \mathcal{V} \bigcup \{X \backslash Z\}$. Notice that $st(x, \mathcal{U}) \cap Z$ is at most countable for each $x \in X$; hence, X is not star-countable, a contradiction. So, X is ω_1-compact. □

A space is an *aD-space* [4] if for each closed subset F of X and each open cover $\mathcal{U}$ of X, there exists a subset A of F, locally finite in F, and a mapping ϕ of A into $\mathcal{U}$ such that $a \in \phi(a)$ for each $a \in A$, and $\phi(A) = \{\phi(a) : a \in A\}$ covers F. Obviously, T_1 ω_1-compact aD-spaces are Lindelöf spaces.

Proposition 5.8. *Regular star-compact aD-spaces with G_δ-diagonals and property ($\sharp$) are metrizable.*

Proof: Let X be a regular star-compact aD-space with G_δ-diagonal and property ($\sharp$). Since X is an aD-space, X is a regular Lindelöf space by Lemma 5.5 and Lemma 5.7. Hence, X is normal and metrizable by Corollary 5.2. □

Corollary 5.9. *Regular meta-Lindelöf star-compact spaces with G_δ-diagonal are metrizable.*

Corollary 5.10. *Star-compact Moore spaces with property (♯) are metrizable.*

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