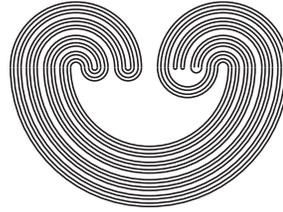


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## DOMAIN REPRESENTABLE SPACES AND COMPLETENESS

by

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## DOMAIN REPRESENTABLE SPACES AND COMPLETENESS

HAROLD BENNETT AND DAVID LUTZER

**ABSTRACT.** We survey a family of interrelated open questions that link classical completeness theories of John C. Oxtoby, Gustave Choquet, and Johannes de Groot to the newer area of domain-representable topological spaces.

### 1. INTRODUCTION

We will say that a topological property  $P$  is a *strong completeness property* provided any topological product of spaces each having property  $P$  will be a Baire space. The goal of this paper is to show how some open questions about the strong completeness properties studied by John C. Oxtoby, Gustave Choquet, and Johannes de Groot and his Amsterdam colleagues in the 1960s and 1970s are closely related to questions from the relatively new field of domain representability of topological spaces.

In section 2, we remind the reader of some classical completeness properties and questions. In section 3, we give a brief sketch of what topologists need to know about domain theory, and in section 4, we present the basic topology of domain-representable spaces. In section 5, we describe questions about domain representable spaces

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and show how they are linked to classical completeness questions from section 2. In section 6, we examine domain-representability and completeness questions in the special context of generalized ordered spaces constructed on sets of real numbers.

Throughout this paper, we will assume that all spaces are at least Hausdorff except where specifically noted. The symbols  $\mathbb{R}$ ,  $\mathbb{P}$ , and  $\mathbb{Q}$  are reserved for the usual sets of real, irrational, and rational numbers, respectively.

## 2. CLASSICAL COMPLETENESS PROPERTIES

Baire spaces are topological spaces in which every intersection of countably many dense open sets is dense. Complete metric spaces and locally compact Hausdorff spaces are the classical examples of Baire spaces. Unfortunately, the Baire space property is unstable under formation of topological products and formation of certain kinds of function spaces, even in good topological categories. For example, it is possible that  $X \times Y$  can fail to be a Baire space, even when both  $X$  and  $Y$  are metrizable Baire spaces [14], and research has shown that it is very difficult for the space  $C_p(X)$  of all continuous, real-valued functions on a space  $X$  to be a Baire space when endowed with the pointwise convergence topology [16], [24].

Positive results about products of Baire spaces can be obtained by imposing severe restrictions on all but one factor in a product, or by imposing certain milder, but still restrictive, hypotheses on all factors. As an example of the first, we have the following consequence of the main result in [1].

**Theorem 2.1.** *If  $Z$  is compact and  $Y$  is a Baire space, then  $Z \times Y$  is a Baire space. Therefore, if  $\{X_\alpha : \alpha \in A\}$  is a collection of Baire spaces in which all but one are compact, then  $\Pi\{X_\alpha : \alpha \in A\}$  is a Baire space.*

As an example of the second, we have the following extension of two classical theorems.

**Theorem 2.2.** *Let  $A$  be any index set. If each space  $X_\alpha$  is either a complete metric space or a locally compact Hausdorff space, then the product space  $\Pi\{X_\alpha : \alpha \in A\}$  is a Baire space.*

Theorem 2.2 is surprising because there is no cardinality restriction on the index set  $A$ , so that the product space is likely to be far

outside of the categories of locally compact spaces and completely metrizable spaces. Theorem 2.2 led topologists to look for structures strong enough to yield an analog of Theorem 2.2 for more general spaces. Here are examples of what was found.

**Theorem 2.3.** *Let  $A$  be any index set. The product space  $\prod\{X_\alpha : \alpha \in A\}$  will be a Baire space if any one of the following holds.*

- (1) *Each  $X_\alpha$  is pseudo-complete in the sense of Oxtoby [23].*
- (2) *Each  $X_\alpha$  is one of subcompact, base-compact, co-compact, or regularly co-compact in the sense of de Groot and his co-authors (see [2], for example).*
- (3) *Each  $X_\alpha$  is strongly complete in the sense of Choquet [10].*

Because the properties in Theorem 2.3 will be important in later parts of this paper, we reproduce their definitions here. Let  $X$  be a regular space.

- (a) The space  $X$  is *pseudo-complete* if  $X$  has a sequence  $\langle \mathcal{P}(n) \rangle$  of  $\pi$ -bases with the property that  $\bigcap\{P_n : n \geq 1\} \neq \emptyset$  provided  $P_n \in \mathcal{P}(n)$  and  $\text{cl}(P_{n+1}) \subseteq P_n$  for each  $n$  [23].
- (b) The space  $X$  is *subcompact* if it has a base  $\mathcal{B}$  of non-empty open sets with the property that  $\bigcap \mathcal{F} \neq \emptyset$  whenever  $\mathcal{F} \subseteq \mathcal{B}$  has the property that, given  $B_1, B_2 \in \mathcal{F}$ , some  $B_3 \in \mathcal{F}$  has  $\text{cl}(B_3) \subseteq B_1 \cap B_2$ . Such a collection  $\mathcal{F}$  is called a *regular filterbase*.
- (c) The space  $X$  is *base-compact* if it has a base  $\mathcal{B}$  of open sets with the property that  $\bigcap\{\text{cl}(C) : C \in \mathcal{C}\} \neq \emptyset$  whenever  $\mathcal{C} \subseteq \mathcal{B}$  is *centered*, i.e., has the finite intersection property.
- (d) The space  $X$  is *co-compact* provided  $X$  has a collection  $\mathcal{C}$  of closed subsets of  $X$  with the property that any centered subcollection of  $\mathcal{C}$  has a nonempty intersection, and the property that if  $U$  is open and  $x \in U$ , then some  $C \in \mathcal{C}$  has  $x \in \text{Int}(C) \subseteq C \subseteq U$ . A stronger property, called *regular co-compactness*, requires that each  $C \in \mathcal{C}$  is the closure of its interior, i.e., each  $C$  is a regularly closed set. (Clearly, any regularly co-compact space is co-compact, and the Sorgenfrey line is an example of a co-compact space that is not regularly co-compact [1].)
- (e) The *strong Choquet game*  $Ch(X)$  on the space  $X$  is a topological game played as follows. Player 1 selects a pair  $(x_1, U_1)$

where  $U_1$  is open and  $x_1 \in U_1$ . Player 2 responds with an open set  $U_2$  that must have  $x_1 \in U_2 \subseteq U_1$ . Player 1 specifies a pair  $(x_3, U_3)$  where  $U_3$  is open and  $x_3 \in U_3 \subseteq U_2$ . Player 2 responds by specifying an open set  $U_4$  with  $x_3 \in U_4 \subseteq U_3$ . This game continues and generates a sequence  $(U_1, x_1), U_2, (x_3, U_3), U_4, \dots$ . Player 2 wins if  $\bigcap\{U_n : n < \omega\} \neq \emptyset$  and Player 1 wins otherwise.

Because the literature is not consistent concerning the names of the players or the numbering of the moves, we will use the term *non-empty player* for the player who chooses the sets  $U_2, U_4, \dots$  and whose goal is a non-empty intersection  $\bigcap\{U_n : n \geq 1\}$ . A *strategy* for the non-empty player is a decision rule  $\rho$  that allows the non-empty player to specify  $U_{2n}$  given any sequence of previous moves  $(x_1, U_1), U_2, \dots, (x_{2n-1}, U_{2n-1})$ . The strategy  $\rho$  is a *winning strategy* for the non-empty player if the non-empty player wins using  $\rho$ , no matter what the other player does. A space  $X$  is *strongly Choquet complete* if the non-empty player has a winning strategy in the strong Choquet game on  $X$ .

- (f) We say that a strategy  $\rho$  in the strong Choquet game is *stationary* if  $\rho$  depends only on knowing a pair  $(x, U)$  with  $x \in U$ , i.e., does not depend on knowing all previous moves and does not depend on knowing how many moves have already been made. Because of its relationship with things to come, we mention that in subcompact spaces, and in Čech-complete spaces (see [26]), the nonempty player has a stationary winning strategy in the strong Choquet game on  $X$ .

As explained in the Introduction, we will use the term *strong completeness property* for any topological property  $P$  such that for any index set  $A$ , if  $X_\alpha$  has property  $P$  for each  $\alpha \in A$ , then  $\Pi\{X_\alpha : \alpha \in A\}$  is a Baire space. Theorems 2.2 and 2.3 above show that Čech completeness, Oxtoby's pseudo-completeness, and the Amsterdam properties of subcompactness, base-compactness, co-compactness, regular-co-compactness, and strong Choquet completeness are each strong completeness properties. Strong completeness properties were widely studied in the period 1965–1985,

and most of the relationships between them are now known. However, several classical questions were left open.

**Classical Question 1:** Suppose  $Y$  is a (dense)  $G_\delta$ -subspace of a subcompact space  $X$ . Must  $Y$  be subcompact? In particular, is every Čech-complete space subcompact?

**Classical Question 2:** Suppose that  $Y \neq \emptyset$  and  $X \times Y$  is subcompact. Does it follow that  $X$  is subcompact?

**Classical Question 3:** Under what conditions is the function space  $C_p(X)$  subcompact?

**Classical Question 4:** Suppose that  $(X, \tau)$  is subcompact and  $S \subseteq X$ . Form a new topology  $\tau^S$  on  $X$  by isolating all points of  $S$ , i.e., by using the collection  $\tau \cup \{\{x\} : x \in S\}$  as a base, isolating some set of points of  $X$ . Must  $(X, \tau^S)$  be subcompact?

Classical Question 3 has now been solved [17]; see Theorem 5.7, below. The other classical questions remain open. We mention these four questions here because, in addition to their intrinsic interest, they have important analogues in the new field of domain representations, as sections 3 and 4 will show.

### 3. DOMAINS AND DOMAIN REPRESENTATION

Dana Scott introduced a way to construct mathematical models of the foundations of computer science, in the study of the lambda calculus. Later researchers, e.g., Abbas Edalat, Martín Escardó, and Keye Martin, used Scott's construction in topology. They used the maximal elements of certain kinds of partially ordered sets (posets) to represent points of a space, with those maximal elements being approximated by the sub-maximal elements of the posets. Consider the following simple example. Let  $\mathcal{J}$  be the collection of all non-empty, closed, bounded intervals in the usual real line  $\mathbb{R}$ , including degenerate intervals of the form  $[a, a] = \{a\}$  where  $a \in \mathbb{R}$ . Partially order  $\mathcal{J}$  by reverse inclusion, i.e., for  $J, K \in \mathcal{J}$ , write  $J \sqsubseteq K$  to mean that  $K \subseteq J$ . Clearly, the maximal members of  $\mathcal{J}$  are the singleton sets, so that, in some sense, the set  $\max(\mathcal{J})$  of all maximal elements of  $\mathcal{J}$  is a copy of  $\mathbb{R}$ . There is a special

topology on  $\mathcal{J}$ , called the Scott topology (see below), and when  $\max(\mathcal{J})$  is topologized as a subspace of  $\mathcal{J}$  using the relative Scott topology, then  $\max(\mathcal{J})$  becomes homeomorphic to  $\mathbb{R}$ . We say that  $(\mathcal{J}, \sqsubseteq)$  represents  $\mathbb{R}$ .

The above example, where  $(\mathcal{J}, \sqsubseteq)$  represents  $\mathbb{R}$ , is a special case of a more general construction. Start with a poset  $(P, \sqsubseteq)$ . A subset  $D \subseteq P$  is *directed* if it is non-empty and has the property that for any  $a, b \in D$ , some  $c \in D$  has  $a, b \sqsubseteq c$ . To say that an element  $p \in P$  is the *supremum* of a set  $S \subseteq P$  means  $p$  is an upper bound for  $S$  and  $p \sqsubseteq q$  for every upper bound  $q$  for  $S$ . To say that  $(P, \sqsubseteq)$  is a *directed complete partial order* (dcpo) means that every directed subset of  $P$  has a supremum in  $P$ . Then Zorn's lemma provides maximal elements of a dcpo, and the set of all maximal points of a dcpo  $P$  is denoted by  $\max(P)$ .

There is an auxiliary relation  $\ll$  defined as follows: For  $a, b \in P$ , we say that  $a \ll b$  provided that for every directed set  $D$  with  $b \sqsubseteq \sup(D)$ , some  $d \in D$  has  $a \sqsubseteq d$ . The set  $\{b \in P : b \ll a\}$  is denoted by  $\Downarrow(a)$ , and we say that the poset  $(P, \sqsubseteq)$  is *continuous* provided each set  $\Downarrow(a)$  is directed and has  $a = \sup(\Downarrow(a))$ . A continuous dcpo is called a *domain*. If a domain  $P$  has the additional property that each non-empty bounded subset of  $P$  has a supremum in  $P$ , then we say that  $(P, \sqsubseteq)$  is a *Scott domain*.

Suppose  $(P, \sqsubseteq)$  is a domain. Then, with  $\Uparrow(a) := \{c \in P : a \ll c\}$ , the collection of all sets  $\Uparrow(a)$  for  $a \in P$  is a basis for a topology on  $P$  called the *Scott topology*. To say that a topological space  $(X, \tau)$  is *domain representable* means that there is a domain  $(P, \sqsubseteq)$  such that  $(X, \tau)$  is homeomorphic to  $\max(P)$  when the latter carries the relative Scott topology. If the poset  $(P, \sqsubseteq)$  is actually a Scott domain, rather than just a domain, we say that  $X$  is *Scott-domain representable*. For a readable survey of domain theory in topology, see [19].

Keye Martin's paper [18] made it clear that domain representability is a topological completeness property. He showed, for example, that any domain representable space is a Baire space, and, more generally, that domain-representable spaces are actually strongly Choquet complete. Consequently, any metrizable domain representable space is completely metrizable. See below for further examples of Martin's results.

4. THE BASIC TOPOLOGY OF DOMAIN REPRESENTABLE AND SCOTT-DOMAIN REPRESENTABLE SPACES

Suppose that  $(P, \sqsubseteq)$  is a domain. The space  $P$  with the Scott topology is not a good space in the traditional sense – for example, it is  $T_0$  but almost never  $T_1$ . However, its dense subspace  $\max(P)$  will always be  $T_1$  and might have other nice topological properties.

The most basic topology of domain representable spaces and Scott-domain representable spaces is now understood. A rule of thumb for formulating conjectures is that domain representability seems to behave like the Baire space property, or perhaps like subcompactness. It is not surprising that being domain representable and being Scott-domain representable are both open-hereditary properties. However, like the Baire space property, neither is closed-hereditary, as can be seen from an easy example-machine. Take any completely regular space  $X$  that is not domain representable (there are many such spaces, e.g.,  $\mathbb{Q}$ ). The space  $\beta X$  is Scott-domain representable, being compact Hausdorff, and, in [3], we showed that if one forms a new space by isolating each point of  $\beta X - X$ , the resulting space  $Y$  is also Scott-domain representable. Note that  $X$  is a closed subspace of  $Y$ . Thus, neither domain representability nor Scott-domain representability is a closed-hereditary property.

Classical Question 1 has an analog for domain representable spaces that asks whether  $G_\delta$ -subsets inherit domain representability. In [3] we gave an affirmative answer, showing that domain representable spaces behave differently from Baire spaces and perhaps also differently from subcompact spaces.

**Theorem 4.1.** *Suppose  $Y$  is a  $G_\delta$ -subset of a domain-representable space  $X$ . Then  $Y$  is also domain-representable. (Note that we do not assume that  $\text{cl}(Y) = X$ .)*

It is surprising that Scott-domain representability behaves still differently, as can be seen from an example in [6].

**Example 4.2.** *There is a Scott-domain representable Moore space with a (closed)  $G_\delta$ -subspace that is not Scott-domain-representable.*

Classical Question 2 has an analog for domain representability. It is known that the product of two (Scott-) domain-representable spaces is again (Scott-) domain representable. However, the next question is open.

**Domain Question 1:** Suppose the product space  $X \times Y$  is (Scott-) domain-representable, where  $Y \neq \emptyset$ . Is  $X$  (Scott-) domain representable? What if the factor  $Y$  has additional properties, such as compactness or metrizability?

Classical Question 3 (about the function space  $C_p(X)$ ) has an analog for domain representability that has been solved for normal spaces (see Theorem 5.8, below). The analog of Classical Question 4 (about the effect of isolating a set of points in a subcompact space) has been answered in [3] for domain representable spaces.

**Proposition 4.3.** *Suppose  $(X, \tau)$  is domain representable and  $S \subseteq X$ . With  $\tau^S$  as in Classical Question 4, the space  $(X, \tau^S)$  is domain representable.*

The preservation of domain representability (Scott-domain representability, respectively) by topological operations, such as perfect mappings, open-compact mappings, etc., remains to be clarified.

## 5. DOMAIN REPRESENTABILITY AND STRONG COMPLETENESS PROPERTIES

Martin [18] answered a question about domain representability of spaces in elementary analysis by proving that if  $(P, \sqsubseteq)$  is a domain, then in the subspace  $\max(P)$ , the non-empty player has a winning strategy in the strong Choquet game. Thus, any domain representable space is strongly Choquet complete (so that, for example,  $\mathbb{Q}$  is not domain representable), and domain representability is what we called a “strong completeness property” because any product of domain representable spaces is a Baire space.

**Domain Question 2:** What is the relationship between domain representability and the classical strong completeness properties mentioned above?

Some parts of the answer to Domain Question 2 are known. In [3], we proved the following.

**Proposition 5.1.** *Any subcompact  $T_3$  space is domain representable.*

How is domain representability related to the other strong completeness properties? Theorem 4.1 shows that any Čech-complete

space is domain-representable. The Sorgenfrey line is domain-representable but not Čech-complete [3], showing that domain-representability is strictly weaker than Čech-completeness. That a pseudo-complete space can fail to be domain representable is an easy consequence of the facts that (i) a metric space is pseudo-complete if and only if it has a dense completely metrizable subspace [2], while (ii) a metric space is domain representable if and only if it is completely metrizable [18]. The following example shows that domain representability is strictly stronger than strong Choquet completeness. It depends on Theorem 5.8 and Theorem 5.11.

**Example 5.2.** *Let  $X$  be the set  $[0, \omega_1]$  topologized in such a way that each countable ordinal is isolated and so that neighborhoods of  $\omega_1$  are co-countable. Then the function space  $C_p(X)$  is pseudo-complete and strongly Choquet complete (and the nonempty player has a stationary winning strategy in  $Ch(C_p(X))$ ), but  $C_p(X)$  is not domain representable.*

The most important part of Domain Question 2 grows out of the fact, noted above, that every subcompact  $T_3$ -space is domain representable.

**Domain Question 3:** Is it true that every domain representable space is subcompact?

We expect a negative answer to Domain Question 3, and  $G_\delta$ -subspaces of the cube  $X = [0, 1]^{\mathfrak{c}}$  (where  $\mathfrak{c} = 2^\omega$ ) are natural potential counterexamples.<sup>1</sup> Any counterexample to Domain Question 3 would also be a counterexample for Classical Question 1. Indeed, the linkage between Classical Question 1 and Domain Question 3 is very strong because we have the following proposition.

**Proposition 5.3.** *If every domain representable space is subcompact, then every  $G_\delta$ -subspace of a subcompact space is subcompact.*

Notice that Proposition 5.3 does not restrict its conclusion to dense  $G_\delta$ -subspaces. Therefore, if one could find any  $G_\delta$ -subspace of a subcompact space that is not subcompact, one would have

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<sup>1</sup>In an earlier version of this paper, we suggested letting  $D$  be a countable dense subset of  $X = [0, 1]^{\mathfrak{c}}$  and we asked whether  $X - D$  must be subcompact. After hearing a talk on this paper at the Milwaukee Topology Conference, William Fleissner sent us a clever proof that  $X - D$  must be subcompact. Other  $G_\delta$ -subspaces of  $X$  might still provide the expected counterexamples.

a solution for both Classical Question 1 and Domain Question 3. Whether this observation makes life easier is not yet clear.

There are many potential approaches to finding the expected counterexample in Domain Question 3 beyond the big-cube example mentioned above. One is presented here.

**Domain Question 4:** Must every domain-representable space be pseudo-complete in the sense of Oxtoby?

A negative answer to Domain Question 4 would also give an example of a domain-representable space that is not subcompact (because every subcompact space is pseudo-complete).

Classical Question 4 asked about the effect of isolating some set of points in a subcompact space – would the result remain subcompact? As noted in Proposition 4.3, isolating any set of points in a domain-representable space always produces a domain-representable space, so that a negative answer to Classical Question 4 would also provide a negative answer to Domain Question 3.

One particularly nice property of subcompactness (and also of Baire spaces in general) is that “local implies global.” More precisely, if  $X$  has an open cover by subcompact subspaces (by Baire subspaces, respectively), then  $X$  is subcompact (a Baire space, respectively) [2].

**Domain Question 5:** Suppose  $X$  is  $T_3$  and has an open cover by domain-representable subspaces. Is  $X$  domain-representable?

Domain Question 5 is interesting even when the open covers are finite. Note that a negative answer to Domain Question 5 would give a negative answer to Domain Question 3. (Also note that “local implies global” fails for Scott-domain representable spaces [9].)

There is a delicate linkage between domain representability and the strong Choquet game. Martin’s proof [18] that any domain representable space is strongly Choquet complete produced an interesting bit of extra information. If  $X$  is domain representable, then the non-empty player has a winning strategy  $\rho$  in the strong Choquet game on  $X$  where  $\rho$  depends only on the previous two moves in the game (rather than on the entire history of the game so far), and  $\rho$  does not need to know how far along the game is (i.e.,

whether the non-empty player is responding to move 17 or move 117.) This is unusual and raises the next question.

**Domain Question 6:** If  $X$  is domain representable, does the non-empty player have a winning strategy in the strong Choquet game that depends on knowing just one previous move? That is, does the non-empty player have a stationary winning strategy in the strong Choquet game on  $X$ ?

Because the non-empty player is known to have a stationary winning strategy in the strong Choquet game in a subcompact space, a negative answer to Domain Question 6 would give a negative answer to Domain Question 2.

Instead of asking about the relation between domain representability and classical completeness properties in general spaces (as in Domain Question 2 and Domain Question 3), one can restrict attention to some special class  $\mathcal{C}$  of spaces and show that subcompactness and domain representability are equivalent for members of  $\mathcal{C}$ .<sup>2</sup> The most spectacular result of this type focuses on the class  $\mathcal{C}$  of metrizable spaces. It combines classical theorems with the newer results of Martin [18] (who proved the equivalence of (a) and (d)) and of Ralph Kopperman, Hans-Peter A. Künzi, and Paweł Waszkiewicz [15] (who proved that (a) implies (e)), thereby linking domain representability with a family of properties long-known to be mutually equivalent in metric spaces.

**Theorem 5.4.** *For a metrizable space  $X$ , the following are equivalent.*

- (a)  $X$  is Čech complete and hence completely metrizable;
- (b)  $X$  has one of the Amsterdam properties (subcompactness, base compactness, regular co-compactness, co-compactness);
- (c)  $X$  is strongly Choquet complete;
- (d)  $X$  is domain representable;
- (e)  $X$  is Scott-domain representable.

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<sup>2</sup>A proposition that subcompactness and domain representability are equivalent notions in a certain class  $\mathcal{C}$  could have some utility, because it would follow, for example, that  $G_\delta$ -subspaces in  $\mathcal{C}$  of subcompact spaces inherit subcompactness, and that locally domain-representable spaces in  $\mathcal{C}$  are domain representable.

Theorem 5.4 is consistent with a rule of thumb that, among metric spaces, there is only one type of completeness; however, outside of the metrizable category, the equivalence described in it breaks down. For example, consider the broader class of Moore spaces.<sup>3</sup> In the following theorem, parts (ii) and (iii) are classical, and part (i) is a combination of results from [2] and [9].

**Theorem 5.5.** *Let  $X$  be a Moore space.*

- (i) *The following properties of  $X$  are equivalent: subcompactness, Rudin completeness, the nonempty player has a winning strategy in the strong Choquet game on  $X$ , and the nonempty player has a stationary winning strategy in the strong Choquet game on  $X$ .*
- (ii) *If  $X$  is also completely regular, the following properties of  $X$  are equivalent: Moore completeness, Čech completeness.*
- (iii) *There are completely regular Moore spaces with the properties in (i) that are neither Moore-complete nor Čech-complete.*

As it happens, most of the equivalences in part (i) of Theorem 5.5 hold in the wider class of BCO spaces (i.e., spaces with a base of countable order in the sense of J. M. Worrell, Jr., and H. H. Wicke [27]) and coincide with the property “ $X$  has a monotonically complete BCO.”

There appears to be another old rule of thumb that there are just two types of completeness among Moore spaces (namely Rudin completeness and Moore completeness), but that old rule of thumb is wrong. Franklin D. Tall [25] showed that the classical properties of base compactness and co-compactness are not equivalent to either Rudin or Moore completeness, and Example 4.5 in [9] is a Čech-complete Moore space that is not Scott-domain representable.

Scott-domain representability is a very interesting and delicate property in Moore spaces and the following question remains open.

**Domain Question 7:** Characterize Scott-domain representability in the category of Moore spaces.

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<sup>3</sup>A regular space  $X$  is a *Moore space* provided there is a sequence  $\langle \mathcal{G}(n) \rangle$  of open covers of  $X$  with the property that for each  $x \in X$ , the collection  $\{\text{St}(x, \mathcal{G}(n)) : n < \omega\}$  is a base of neighborhoods at  $x$ .

A theorem in [15] links Scott-domain representability with de Groot’s co-compactness in [13] and shows that, among completely regular spaces, Scott-domain representability is equivalent to co-compactness plus a bitopological condition called “pairwise complete regularity” with respect to a certain co-topology in the sense of de Groot. It is not clear how to apply that result in the Moore space context. This leads to the next question.

**Domain Question 8:** Is there a completely regular Moore space that is co-compact but not Scott-domain-representable?

V. Miškin’s characterizations of base-compact and regularly-co-compact Moore spaces [22] may be useful in studying Domain Question 8.

Spaces with a  $G_\delta$ -diagonal are a generalization of both metric spaces and Moore spaces. In [4], we proved the next proposition.

**Proposition 5.6.** *Suppose that  $X$  is a  $T_3$ -space with a  $G_\delta$ -diagonal. Then  $X$  is domain-representable provided the non-empty player has a stationary winning strategy in the strong Choquet game on  $X$ .*

Recall that a strongly Choquet complete space is one in which the non-empty player has a winning strategy in the strong Choquet game. However, in general, one does not know how much information that winning strategy requires in order to choose the next step of the game. The fact that Proposition 5.6 assumed the existence of a stationary winning strategy raises a natural question.

**Domain Question 9:** Suppose  $X$  is a  $T_3$ -space and has a  $G_\delta$ -diagonal, and suppose that the nonempty player has a not-necessarily-stationary winning strategy in the strong Choquet game on  $X$ . Must  $X$  be domain representable?

If that question has an affirmative answer, then domain representability is equivalent to strong Choquet completeness among regular spaces with a  $G_\delta$ -diagonal. That would be surprising. There are variations of Domain Question 9 that ask about regular spaces with a  $G_\delta$ -diagonal in which the nonempty player has a Markovian winning strategy, i.e., one that depends only on the opponent’s previous move and on the number of moves already made.

Next, recall Classical Question 3 concerning subcompactness in  $C_p(X)$ , the space of continuous real-valued functions on  $X$ , endowed with the topology of pointwise convergence. If  $X$  has the discrete topology, then  $C_p(X) = \mathbb{R}^X$ , which is subcompact because any topological product of subcompact spaces is subcompact. In several private communications and conferences during 1980s, Jan van Mill asked whether  $C_p(X)$  could be subcompact in any other situation. A recent paper by D. J. Lutzer, van Mill, and V. V. Tkačuk [17] used Tkačuk's techniques to prove the following theorem.

**Theorem 5.7.** *Suppose  $X$  is completely regular. Then  $C_p(X)$  is subcompact if and only if  $X$  is discrete.*

Theorem 5.7 suggests looking at function spaces of the type  $C_p(X)$  to see when they are domain representable. Either one might obtain a generalization of the previous theorem (because subcompact spaces are domain representable) or one might obtain a space  $C_p(X)$  that is domain representable but not subcompact. As it happens, in [5] we proved the following theorem.

**Theorem 5.8.** *Suppose  $X$  is a normal space. Then the following are equivalent.*

- (a)  $X$  is discrete;
- (b)  $C_p(X)$  is subcompact;
- (c)  $C_p(X)$  is domain representable;
- (d)  $C_p(X)$  is Scott-domain representable.

The proof in [5] uses normality in a crucial way, and this raises the next question.

**Domain Question 10:** Can Theorem 5.8 be proved if  $X$  is completely regular, but not necessarily normal? (If not, then there is a counterexample to Domain Question 2.)

Certain variations of Theorem 5.8 are available. Recall that a space  $X$  is *pseudo-normal* if any two disjoint closed sets have disjoint neighborhoods, provided one of the sets is countable. Clearly, pseudo-normality is weaker than normality, and sometimes it is enough.

**Theorem 5.9.** *Suppose  $X$  is completely regular and pseudo-normal. Then  $C_p(X)$  is Scott-domain-representable if and only if  $X$  is discrete.*

And immediately, a variation of Domain Question 9 is raised.

**Domain Question 11:** Suppose  $X$  is completely regular and  $C_p(X)$  is Scott-domain representable. Must  $X$  be discrete?

One way to weaken the hypothesis of normality in Theorem 5.8 is to put restrictions on the limit-point structure of  $X$ . For example, a space  $X$  is *pseudo-radial* provided that whenever  $S$  is a non-closed subset of  $X$ , there is some  $x \in X - S$ , some cardinal  $\kappa$ , and some net  $\{x(\alpha) : \alpha < \kappa\}$  of points of  $S$  that converges to  $x$ . Clearly, first-countable spaces and generalized ordered spaces (see below) are examples of pseudo-radial spaces. The following theorem is proved in [8].

**Theorem 5.10.** *Suppose that  $X$  is completely regular and pseudo-radial. Then the following are equivalent.*

- (a)  $C_p(X)$  is domain representable;
- (b)  $X$  is discrete;
- (c)  $C_p(X)$  is Scott-domain representable.

The previous four results have asked about domain-representability of function spaces with the pointwise convergence topology. As it happens, these same function spaces can be used to produce examples related to the other strong completeness properties mentioned above; see Example 5.2. In [8], we proved the following.

**Proposition 5.11.** *Suppose the space  $X$  is completely regular and pseudo-normal. Then the following are equivalent.*

- (a) every countable subset of  $X$  is closed;
- (b)  $C_p(X)$  is strongly Choquet complete and the non-empty player has a stationary strategy in the strong Choquet game played in  $C_p(X)$ ;
- (c)  $C_p(X)$  is strongly Choquet complete;
- (d)  $C_p(X)$  is pseudo-complete in the sense of Oxtoby [23].

**Domain Question 12:** Can the equivalence in Proposition 5.11 be proved without assuming that  $X$  is pseudo-normal?

## 6. DOMAIN-REPRESENTABILITY AND SUBCOMPACTNESS IN GO-SPACES

Recall that a *generalized ordered space* (GO-space) is a triple  $(X, <, \tau)$  where  $(X, \tau)$  is a Hausdorff space with a base of open sets that are convex with respect to the ordering  $<$ . Typically one constructs GO-spaces by specifying (i) which points of a linearly ordered set  $(X, <)$  are to be isolated, (ii) which points  $x \in X$  have basic neighborhoods of the form  $[x, y[$  with  $x < y$ , (iii) which points  $x \in X$  have basic neighborhoods of the form  $]w, x]$  with  $w < x$ , and (iv) which points  $x \in X$  have basic neighborhoods of the form  $]u, v[$  with  $u < x < v$ . All such GO-spaces are said to be *constructed on  $X$* . Even when  $X$  is the set of real numbers, interesting GO spaces, such as the Sorgenfrey and Michael lines, can be constructed on  $X$ .

Earlier sections surveyed known results on the relation between domain representability and other strong completeness properties in Moore spaces and function spaces. GO-spaces are another topological category in which that relationship can be investigated, and in this section, we present a family of results related to the following special case of Domain Question 2.

**Domain Question 13:** How are the other strong completeness properties related to domain representability among GO-spaces? In particular, is there a GO-space that is domain-representable but not subcompact? What about GO-spaces constructed on  $\mathbb{R}$  or on subsets of  $\mathbb{R}$ ?

We do not know the answer to those questions, and this section presents preliminary results. Proofs will appear in [7].

GO-spaces constructed on (subsets of) the set of real numbers have been useful counterexamples in product theory and in the study of the Amsterdam completeness properties, so it is natural to wonder whether they might have a role to play in studying the relationship between subcompactness and domain representability. All GO-spaces constructed on the entire set  $\mathbb{R}$  are known to be domain representable [12], and even more.

**Theorem 6.1.** *Any GO-space constructed on the space  $\mathbb{R}$  is domain representable by a Scott domain.*

Might there be such spaces that are not subcompact? The answer is “No,” as our next result shows.

**Proposition 6.2.** *Let  $\tau$  be any GO-topology defined on the set  $\mathbb{R}$ . Then  $(\mathbb{R}, \tau)$  is subcompact.*

Proposition 6.2 shows that no GO-space constructed on the entire set  $\mathbb{R}$  can be a counterexample to Domain Question 2, but perhaps there are GO-spaces constructed on subsets of  $\mathbb{R}$  that could provide the desired counterexample. As a start, this requires understanding which subsets of  $\mathbb{R}$  can support subcompact GO-topologies and which can support domain representable GO-topologies.

**Proposition 6.3.** *Let  $X \subseteq \mathbb{R}$  and let  $\sigma$  be a GO-topology on  $X$ . The following are equivalent.*

- (a)  $(X, \sigma)$  is subcompact;
- (b) there is some GO-topology  $\tau$  on  $\mathbb{R}$  such that  $X$  is a  $G_\delta$ -subset of  $(\mathbb{R}, \tau)$  and  $\sigma = \tau|_X$ ;
- (c) for every GO-topology  $\tau$  on  $\mathbb{R}$  with  $\sigma = \tau|_X$ ,  $X$  is a  $G_\delta$ -subset of  $(\mathbb{R}, \tau)$ .

As a consequence of Proposition 6.3, no GO-space constructed on a subset of  $\mathbb{R}$  can provide counterexamples to Classical Question 1 or Classical Question 4 because we have the following.

**Corollary 6.4.** *Suppose  $X \subseteq \mathbb{R}$  and suppose that  $\sigma$  is a GO-topology on  $X$  that is subcompact. Then*

- (a) any  $G_\delta$ -subspace of  $(X, \sigma)$  is subcompact;
- (b) if  $T \subseteq X$  and if  $\sigma^T$  denotes the topology on  $X$  having the collection  $\sigma \cup \{\{x\} : x \in T\}$  as a base, then  $(X, \sigma^T)$  is also subcompact.

Now we return to the theme of comparing subcompactness and domain representability in GO-spaces constructed on subsets of  $\mathbb{R}$ . We need to understand which GO-spaces  $(X, \sigma)$ , for  $X \subseteq \mathbb{R}$ , will be domain representable. The best result to date concerns GO-spaces that are dense-in-themselves, i.e., spaces with no isolated points.

**Proposition 6.5.** *Suppose  $X \subseteq \mathbb{R}$  and suppose that  $\sigma$  is a dense-in-itself GO-topology on  $X$  such that  $(X, \sigma)$  is domain representable. Then there is a subset  $Y \subseteq X$  that is dense in  $(X, \sigma)$  and is a  $G_\delta$ -subset of the usual space of real numbers. Consequently,  $(Y, \sigma|_Y)$  is a dense subcompact subspace of  $(X, \sigma)$ .*

The property given in Proposition 6.5 has independent interest because we can prove the following proposition.

**Proposition 6.6.** *Suppose  $(X, \sigma)$  is a dense-in-itself GO-space constructed on some subset  $X \subseteq \mathbb{R}$ . The following are equivalent.*

- (a)  $(X, \sigma)$  is pseudocomplete in the sense of Oxtoby.
- (b) There is a  $G_\delta$ -subset  $S$  of the usual real line  $(\mathbb{R}, \lambda)$  that is a dense subspace of  $(X, \sigma)$ .
- (c) The space  $(X, \sigma)$  has a dense subcompact subspace.

Proposition 6.5 eliminates many of the more pathological subsets of the real numbers from consideration in the search for domain representable spaces that are not subcompact. For example, no  $Q$ -set, no set with cardinality less than  $2^\omega$ , no totally non-meager subset, and no Bernstein set can carry a dense-in-itself GO-topology that is domain representable, because none of these spaces could contain a dense subspace that is a  $G_\delta$ -subset of the usual real line.

**Domain Question 14:** (a) For which subsets  $X \subseteq \mathbb{R}$  is there *some* dense-in-itself GO-topology  $\tau$  on  $X$  with the property that  $(X, \tau)$  is domain-representable?  
 (b) For which subsets  $X \subseteq \mathbb{R}$  will  $(X, \tau)$  be domain representable for *every* dense-in-itself GO-topology  $\tau$  on  $X$ ?

There are other questions that might be solved using GO-spaces constructed on subsets of  $\mathbb{R}$ . Recall  $Ch(X)$ , the strong Choquet game described in section 2. Martin showed in [18] that if  $X$  is domain representable, then the non-empty player has a winning strategy in  $Ch(X)$ . One can classify winning strategies based on how much of the history of the game is used by the strategy in designing its next move. Some strategies require *perfect information*, i.e., knowing the entire history of the game up to now. Others need to know only the opponent's move and how many moves have already been made. Still others need to know only the single preceding move of the opponent, and these are called *stationary strategies*. Martin observed that in a domain representable space  $X$ , the nonempty player has a winning strategy in  $Ch(X)$  that depends on at most the two preceding moves. In [4], we showed that if  $X$  is a regular space with a  $G_\delta$ -diagonal and if the nonempty player has a stationary winning strategy in  $Ch(X)$ , then  $X$  is domain representable. Any GO-topology on any subset of  $\mathbb{R}$  will have

a  $G_\delta$ -diagonal, and this suggests a way to explore the difference between various types of winning strategies in  $Ch(X)$ .

**Domain Question 15:** Suppose  $\sigma$  is a GO-topology on a subset  $X \subseteq \mathbb{R}$  and suppose that the nonempty player has a winning strategy in  $Ch(X, \sigma)$ . Is  $(X, \sigma)$  domain representable? Next, suppose  $(X, \sigma)$  is domain representable. Does the nonempty player have a stationary strategy in  $Ch(X, \sigma)$ . What if we restrict attention to dense-in-themselves GO-spaces on subsets of  $\mathbb{R}$ ?

### 7. OTHER DIRECTIONS

The referee pointed out that there are additional topics for investigation that are related to the topic of this paper. Recall the definition of the  $\ll$  relation in a domain  $P$ : We say that  $a \ll b$  if whenever  $D$  is a directed subset of  $P$  with  $b \sqsubseteq \sup(D)$ , then some  $d \in D$  has  $a \sqsubseteq d$ . Bob Coecke and Martin [11] used posets to model finite dimensional quantum states and were forced to introduce a weaker relation  $\ll_w$ , defined as follows:  $a \ll_w b$  provided if  $D \subseteq P$  is directed and  $b = \sup(D)$ , then some  $d \in D$  has  $a \sqsubseteq d$ . Analogous to  $\Downarrow(q)$  in a domain, the set  $\Downarrow_w(q) := \{p \in P : p \ll_w q\}$  and  $\Uparrow_w(p)$  is similarly defined. A poset  $P$  is *exact* if each  $\Downarrow_w(q)$  is directed and  $\sup(\Downarrow_w(q)) = q$  for all  $q \in P$ .

Another technical requirement is that the relation  $\ll_w$  must be *weakly increasing* (see [20] for the technical definition), and a *weak domain* is a dcpo that is exact and in which  $\ll_w$  is weakly increasing. A topological space is *weak domain representable* if there is a weak domain  $P$  such that  $X$  is homeomorphic to  $\max(P)$  with the relativized version of the topology induced on  $P$  by the collection  $\{\Uparrow_w(p) : p \in P\}$ . See [20] for a survey. Even though for a point  $q \in \max(P)$  there is no difference between  $p \ll_w q$  and  $p \ll q$ , there is a major difference between weak domain representability and the domain representability studied in earlier sections. Weak domain representability is strictly weaker than domain representability as can be seen from Joe Mashburn’s proof that the usual space  $\mathbb{Q}$  of rational numbers is a dense open subset of a weakly domain representable space  $X$  (showing that  $X$  is not even a Baire space),

so that weak domain representability is not a strong completeness property in the sense of this paper.

Many basic questions about weak domain representability remain open – see [21] for a listing. One of Mashburn’s questions in [21], asks, “If  $Y$  is a weak domain representable Baire space, must  $Y$  be domain representable?” We can use a result from [21] to answer that question in the negative. Mashburn proved the following.

**Proposition 7.1.** *Suppose  $X$  is a LOTS. Then  $X$  is homeomorphic to a dense open subset of a weak domain representable space.*

To apply Proposition 7.1, begin with a Bernstein set  $B \subseteq \mathbb{R}$ , (i.e., neither  $B$  nor  $\mathbb{R} - B$  contains an uncountable compact set). Note that  $B$  is a Baire space and that in its relative topology from  $\mathbb{R}$ ,  $B$  is a LOTS. Therefore, Proposition 7.1 shows that  $B$  is a dense open subset of a weak domain representable space  $Y$ . Because  $B$  is a Baire space, so is  $Y$ . We claim that  $Y$  is the required example: If  $Y$  were domain representable, then so would be its dense open subspace  $B$ , and  $B$  cannot be domain representable because  $B$  is metrizable but not completely metrizable (see Theorem 5.4).

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