TOPOLOGY PROCEEDINGS
Volume 34, 2009
Pages 283–301

http://topology.auburn.edu/tp/

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Electronically published on June 23, 2009

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

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Volume 34 (2009)
Pages 283-301

E-Published on June 23, 2009

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Abstract. In this paper we develop basic properties of some qualitative concepts of generalized flow theory, including the f-orbit closure and f-limit set operators, as well as some forms of invariance, minimality, and stability.

1. Introduction

The axioms for a generalized dynamical system or f-flow were introduced in [1]. The generalization results from the use of a continuous initial function in lieu of the identity mapping of the classical dynamical system identity axiom. Of course, this generalization is also reflected in the group axiom.

Historically, flows were generalized via the phase space from the plane of Poincaré's work to $\mathbb{R}^n$ to metric spaces and to Hausdorff spaces. Moreover, the groups $\mathbb{R}$ and $\mathbb{Z}$ were generalized to topological groups. In each case, the generalization kept the axioms intact; however, there was a generalization to phase sets which also dropped the continuity axiom. For a classical flow, the identity homeomorphism $\pi(x, 0) = x$ resulted from an initial condition for certain differential equations. Motivated by systems of differential equations which have a continuous initial condition $\pi(x, 0) = f(x)$, the author introduced the generalized axioms in [1]. As the author noted in [1], the use of $\pi(x, 0) = f(x)$, where $f : X \longrightarrow X$ is a

2000 *Mathematics Subject Classification.* Primary 54H99; Secondary 34C99, 34D99, 37B99, 37C99.

Key words and phrases. continuous flow, discrete flow, f-flow.

homeomorphism of X, is still just a classical flow. However, when a continuous (nonhomeomorphism of X) $f : X \longrightarrow X$ is used, the class of systems to which the axioms apply is expanded to a super class containing the classical flows as a special case. Hence, the structure of a generalized flow offers nothing new relative to classical flows. However, it gives some structure toward classifying and characterizing generalized nonclassical spaces. That is the direction under investigation now.

In this paper we shall continue the development of these systems by analyzing some properties of invariance, f-invariance, f-orbit closures, and f-limit sets. Although various notations we shall use are not standardized, the definitions and notations used herein are consistent with those used by many authors (including the present author who used them in several papers, including [1] through [4].) Before beginning our development of the concepts in this paper, we restate the definition of a generalized dynamical system or an f-flow given in [1].

Definition 1.1. Let f be a function defined on a space X. A *continuous (discrete) f-flow*, or simply an *f-flow* (X, π_f), consists of a topological space X and a function π_f from the product space $X \times \mathbb{R}$ $(X \times \mathbb{Z})$ into the space X satisfying the following axioms:

(1) Initial deformation: $\pi_f(x, 0) = f(x)$ for each $x \in X$.
(2) Group deformation action: $\pi_f\big(\pi_f(x,t), s\big) = \pi_f(f(x), t+s)$ for each $x \in X$ and each $t, s \in \mathbb{R}$ $(t, s \in \mathbb{Z})$.
(3) Continuity: $\pi_f : X \times \mathbb{R} \longrightarrow X$ $(\pi_f : X \times \mathbb{Z} \longrightarrow X)$ is continuous.

For brevity, whenever X is understood, we shall say that π_f is an f-flow when referring to an f-flow (X, π_f). If $\mathbb{R}$ $(\mathbb{Z})$ is replaced by $\mathbb{R}^+ = [0, \infty)$ or $\mathbb{R}^- = (-\infty, 0]$ ($\mathbb{Z}^+$ or $\mathbb{Z}^-$), then (X, π_f) is called a positive or negative (discrete) f-semiflow, respectively. Each type of f-flow π_f is the corresponding type of classical flow if $f = i_X$, the identity on X, in which case we use the usual notation $\pi = \pi_{i_X}$ and refer to it as a dynamical system or a flow.

A given f-flow (X, π_f) on a Hausdorff phase space X will be assumed throughout this paper. Despite the fact that corresponding results hold for discrete f-flows and f-semiflows, we state and prove results throughout this paper primarily for continuous f-flows. Note that whenever f is injective, the mappings f^n are defined on the subspaces $f^{|n|}(X)$ of X for each n in $\mathbb{Z}$.

The following examples of families of continuous and discrete generalized flows are given in [1].

The family of f-flows $\pi_f : (\mathbb{R} \times \mathbb{C}) \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{C}$, defined by $\pi_f(x, t) = \big(g(x), p_2(x)\exp(it)\big)$ for $x \in \mathbb{R} \times \mathbb{C}$ and $t \in \mathbb{R}$, are products of generalized flows where $f : \mathbb{R} \times \mathbb{C} \longrightarrow \mathbb{R} \times \mathbb{C}$ is defined by $f(x) = \big(g(x), p_2(x)\big)$ with $g : \mathbb{R} \times \mathbb{C} \longrightarrow \mathbb{R}$ continuous and $p_2 : \mathbb{R} \times \mathbb{C} \longrightarrow \mathbb{C}$ the second coordinate projection map. This type of f-flow is not induced by a classical flow and does not induce a classical flow for such maps as $g(x, y) = x^3 - x$. This family of f-flows arises from solutions to systems of partial differential equations. The system $u_t = 0$, $v_t = iv$ with initial conditions $u(x_o, 0) = g(x_o)$ and $v(x_o, 0) = p_2(x_o)$ for $x_o \in \mathbb{R} \times \mathbb{C}$ has solutions $u(x, t) = g(x)$ and $v(x, t) = p_2(x)\exp(it)$ for each $t \in \mathbb{R}$ and $x \in \mathbb{R} \times \mathbb{C}$ determining these f-flows for any continuous g.

For a continuous function $f : X \longrightarrow X$, let (X, π_f) be defined by $\pi_f(x, n) = f^{n+1}(x)$ on $X \times \mathbb{Z}^+$ or defined on $X \times \mathbb{Z}$ whenever f is an embedding. These discrete f-semiflows and f-flows are just translations of discrete semiflows and flows as long as $f(X) = X$. With $f(X) \subset X$, we have $\pi_f(X \times \mathbb{Z}^+) \subset f(X)$. In 1963 the author was intrigued by Gordon Thomas Whyburn's consideration of some dynamical properties of a flow generated by iterates of a continuous f with $f(X) \subset X$ [5, p. 239]. Also, a far as the author has ascertained, this was the first type of published discrete example of a continuous function initial condition in the sense of this paper.

2. f-invariant Sets

First, we turn our attention to invariance and f-invariance under the orbit operator. The corresponding classical notions have been especially fruitful relative to the concepts of stability and attraction. Our development makes use of the bilateral and unilateral f-orbits in [1] at a point x in X which are $C_f(x) = \pi_f(x, \mathbb{R})$, $C_f^+(x) = \pi_f(x, \mathbb{R}^+)$, and $C_f^-(x) = \pi_f(x, \mathbb{R}^-)$.

Definition 2.1. A subset M of X is called f-invariant (positively f-invariant, negatively f-invariant) if and only if $C_f(M) = f(M)$ ($C_f^+(M) = f(M)$, $C_f^-(M) = f(M)$). Also, M is called invariant (positively invariant, negatively invariant) if and only if $C_f(M) = M$ ($C_f^+(M) = M$, $C_f^-(M) = M$).

The following proposition and its dual are a direct result of Proposition 5.6 of [1].

Proposition 2.2. *For each point x in X, $C_f(x)$ ($C_f^+(x)$) is f-invariant and positively f-invariant (positively f-invariant).*

Obviously, $f(M) \subset C_f(M)$, $f(M) \subset C_f^+(M)$, and $f(M) \subset C_f^-(M)$ so that the properties $C_f(M) \subset f(M)$, $C_f^+(M) \subset f(M)$, and $C_f^-(M) \subset f(M)$ are equivalent to the respective equations of Definition 2.1. The unilateral (bilateral) invariance and f-invariance concepts coincide for a set M if and only if M is invariant relative to f. However, invariance relative to f need not mean that the set is unilaterally (bilaterally) invariant or f-invariant. For example, whenever π_f is a flow with $f = i_X$, $f(M) = M$ for every set M in X, including sets which are not unilaterally (bilaterally) invariant or f-invariant.

Proposition 2.3. *An invariant (positively invariant, negatively invariant) set M in X is f-invariant (positively f-invariant, negatively f-invariant) and $f^n(M) = M$ for each $n \geq 0$. If f is bijective, $f^n(M) = M$ for each integer n.*

Proof: Let M be an invariant subset of X. Then $f(M) \subset C_f(M) = M$. On the other hand, to show $M \subset f(M)$, let $x \in M$. There exists a $y \in M$ and a $t \in \mathbb{R}$ such that $x = \pi_f(y,t)$. Also, there exists a $z \in M$ and an $s \in \mathbb{R}$ such that $y = \pi_f(z,s)$. Consequently, $x = \pi_f(y,t) = \pi_f\big(\pi_f(z,s),t\big) = \pi_f\big(f(z), s+t\big) = f\big(\pi_f(z,s+t)\big) \in f\big(C_f(z)\big) \subset f(M)$, and hence, $f(M) = M$. Similarly, $f(M) = M$ follows from positive invariance and negative invariance. The equation $f^n(M) = M$ follows by induction. □

Proposition 2.4. *A subset M of X is f-invariant (invariant) if and only if it is positively and negatively f-invariant (invariant).*

Proof: Let $C_f^+(M) = f(M)$ and $C_f^-(M) = f(M)$. Then, $C_f(M) = C_f^+(M) \cup C_f^-(M) = f(M) \cup f(M) = f(M)$. Conversely, $C_f^+(M) \subset C_f(M) = f(M) \subset C_f^+(M)$, and hence, $C_f^+(M) = f(M)$. Similarly, $C_f^-(M) = f(M)$. The corresponding statement for invariance follows similarly now by Proposition 2.3 since $f(M) = M$. □

Proposition 2.5. *If f is injective, a subset M of $f(X)$ is positively f-invariant (invariant) if and only if $f(X) \setminus M$ is negatively f-invariant (invariant).*

Proof: First, let $C_f^+(M) = f(M)$. Since $f\big(f(X)\setminus M\big) \subset C_f^-\big(f(X)\setminus M\big)$, let $y \in C_f^-\big(f(X) \setminus M\big)$. There is an $x \in f(X) \setminus M$ and a $t < 0$ such that $y = \pi_f(x,t)$. Suppose that $y \in f(M)$. Then $y = f(z)$ for some $z \in M$ and $f^2(x) = \pi_f(y,-t) = \pi_f\big(f(z),-t\big)$ so that $f(x) = \pi_f(z,-t) \in C_f^+(M) = f(M)$; hence, $x \in M$ contradicting $x \in f(X) \setminus M$. Therefore, $y \in f\big(f(X) \setminus M\big)$ and we have $C_f^-\big(f(X) \setminus M\big) = f\big(f(X) \setminus M\big)$. The converse is the dual and follows similarly.

Finally, the corresponding invariance statement is now a result of Proposition 2.2 as follows. From $C_f^+(M) = M$, we have $C_f^+(M) = f(M)$ and $C_f^-\big(f(X)\setminus M\big) = f\big(f(X)\setminus M\big)$. Since $f(M) = M$ and f is injective, $f\big(f(X)\setminus M\big) = f(X)\setminus M$, and hence, $C_f^-\big(f(X)\setminus M\big) = f(X) \setminus M$. Again, the converse follows similarly. □

By parallel reasoning we have the following corollary.

Corollary 2.6. *If f is injective, a subset M of $f(X)$ is f-invariant (invariant) if and only if $f(X) \setminus M$ is f-invariant (invariant).*

Corollary 2.7. *If f is bijective, a subset M of X is positively f-invariant (invariant) if and only if $X \setminus M$ is negatively f-invariant (invariant).*

Proposition 2.8. *If a subset M of X is f-invariant (positively f-invariant, negatively f-invariant), then $f^n(M)$ is f-invariant (positively f-invariant, negatively f-invariant) for each integer $n \geq 0$. Whenever f is bijective, these statements hold for all integers n. Corresponding statements hold for invariance.*

Proof: The statements follow by induction from statements (11) and (12) of Proposition 5.6 of [1] and the definitions. The corresponding invariance statements follow from Proposition 2.3. □

Proposition 2.9. *Let $\mathcal{M}$ be a collection of f-invariant (positively f-invariant, negatively f-invariant) subsets of X. Then, $\cup\mathcal{M}$ is f-invariant (positively f-invariant, negatively f-invariant). Moreover, if f is an injection, $\cap\mathcal{M}$ is f-invariant (positively f-invariant,* *negatively f-invariant). Corresponding statements hold for invariance.*

Proof: Let $\mathcal{M}$ be a collection of f-invariant subsets of X. Then, $f(\cup\mathcal{M}) \subset C_f(\cup\mathcal{M}) = \cup_{x\in\cup\mathcal{M}}C_f(x) \subset \cup_{M\in\mathcal{M}}\big(\cup_{x\in M}C_f(x)\big) = \cup_{M\in\mathcal{M}}C_f(M) = \cup_{M\in\mathcal{M}}f(M) = f(\cup\mathcal{M})$. Thus, $\cup\mathcal{M}$ is f-invariant. Moreover, $f(\cap\mathcal{M}) \subset C_f(\cap\mathcal{M}) \subset \cap_{M\in\mathcal{M}}C_f(M) = \cap_{M\in\mathcal{M}}f(M) = f(\cap\mathcal{M})$ since f is injective. Thus, $\cap\mathcal{M}$ is f-invariant. The unilateral f-invariance statements follow similarly. Again, the corresponding invariance statements are now a result of Proposition 2.3. □

Proposition 2.10. *If a subset M of X is f-invariant (positively f-invariant, negatively f-invariant), then $C_f(\overline{M}) \subset \overline{f(M)}$ ($C_f^+(\overline{M}) \subset \overline{f(M)}$, $C_f^-(\overline{M}) \subset \overline{f(M)}$).*

Proof: We verify the statement first for a positively f-invariant set M in X. Let $x \in \overline{M}$, let (x_i) be a net in M converging to x, and let $t > 0$. Then, since $C_f^+(M) = f(M)$, $\pi_f(x_i,t) \in f(M)$ and $\pi_f(x_i,t) \to \pi_f(x,t)$ where $\pi_f(x,t) \in \overline{f(M)}$. Thus, $C_f^+(\overline{M}) \subset \overline{f(M)}$. The dual follows similarly. On the other hand, if M is an f-invariant subset of X, M is positively and negatively f-invariant by Proposition 2.4, and hence, the result follows from the first part of the proof. □

Corollary 2.11. *If a subset M of X is invariant (positively invariant, negatively invariant), then $C_f(\overline{M}) \subset \overline{M}$ ($C_f^+(\overline{M}) \subset \overline{M}$, $C_f^-(\overline{M}) \subset \overline{M}$).*

Corollary 2.12. *Let f be an embedding of X and let M be an f-invariant (positively f-invariant, negatively f-invariant) subset of X. Then, $\overline{M}$ is f-invariant (positively f-invariant, negatively f-invariant). Also, M° is f-invariant (positively f-invariant, negatively f-invariant) provided $M \subset f(X)$.*

Proof: Let M be a positively f-invariant subset of X. Letting $\overline{f(M)}$ denote the closure relative to $f(X)$, Proposition 2.7 and f an embedding yield $\overline{f(M)} \cap f(X) = f(\overline{M}) \subset C_f^+(\overline{M}) \subset \overline{f(M)} \cap f(X)$, and hence, $C_f^+(\overline{M}) = f(\overline{M})$. The other two statements for $\overline{M}$ follow similarly.

On the other hand, to verify the statements for M°, we note that positive f-invariance of M implies negative f-invariance of $f(X) \setminus M$. By the dual to the first part of the proof, we have the closure $\overline{f(X) \setminus M}$ relative to $f(X)$ negatively f-invariant. Whence, $M^\circ = f(X) \setminus \overline{f(X) \setminus M}$ is positively f-invariant. The other two statements for M° follow similarly. □

Corollary 2.13. *Let f be an embedding of X and let M be an invariant (positively invariant, negatively invariant) subset of X. Then $\overline{M}$ and M° are invariant (positively invariant, negatively invariant).*

Proposition 2.14. *Let f be an embedding in X and let M be an f-invariant (invariant) subset of X. Then the boundary ∂M of M is f-invariant (invariant). The converse holds provided M is either open or closed in $f(X)$.*

Proof: Let M be an f-invariant subset of X. Then $C_f(\partial M) = C_f(\overline{M} \cap \overline{X \setminus M}) = f(\overline{M} \cap \overline{X \setminus M}) = f(\partial M)$ by Proposition 2.9 and by Corollary 2.6 and Corollary 2.12.

Conversely, let ∂M be an f-invariant subset of X. Suppose that $C_f(M^\circ) \cap f(\partial M) \neq \emptyset$. Then there is an x in M° such that $C_f(x) \cap f(\partial M) \neq \emptyset$. Let $y \in C_f(x) \cap f(\partial M)$. There is a $t \in \mathbb{R}$ and a $z \in \partial M$ such that $y = \pi_f(x,t) = f(z)$. Thus, $f^2(x) = \pi_f(y, -t) = \pi_f\big(f(z), -t\big)$, and hence, $f(x) = \pi_f(z, -t) \in \partial M$ or $x \in \partial M$. This is impossible since $x \in M^\circ$, and therefore, we have $C_f(M^\circ) \cap f(\partial M) = \emptyset$.

Now, by Corollary 2.6, $f(X) \setminus \partial M$ is f-invariant. Either $M = \partial M$ and we are done, or else M° is a nonempty subset of $f(X)$. Assume that M° is nonempty. If we show first that $f(M^\circ) = \cup_{p \in M^\circ} C_f(p)$, the remainder of the proof will follow easily. Certainly, $f(M^\circ) = \cup_{p \in M^\circ} f(p) \subset \cup_{p \in M^\circ} C_f(p)$. Next, let $y \in \cup_{p \in M^\circ} C_f(p)$. Then $y \in C_f(p)$ for some $p \in M^\circ$. The set $C_f(M^\circ) \cap f(\partial M)$ is empty, the set $C_f(p)$ is a connected subset of $C_f(M^\circ)$, and the point $f(p)$ is in the set $C_f(p) \cap f(M^\circ)$, and so, we have y in $C_f(p) \subset f(M^\circ)$. Thus, our claim that $f(M^\circ) = \cup_{p \in M^\circ} C_f(p)$ is verified.

The set M° is f-invariant because $C_f(M^\circ) = \cup_{p \in M^\circ} C_f(p)$. By virtue of Proposition 2.9, $M^\circ \cup \partial M$ is also f-invariant. Whenever M is open or closed, $M = M^\circ$ or $M = M^\circ \cup \partial M$, respectively. Consequently, in either case, M is f-invariant. Again, in view of

Proposition 2.3, the corresponding invariance statements now follow. □

3. f-orbit Closures

Various forms of attraction and of stability, such as Lagrange and Poisson stability, depend on properties of orbit closures and limit sets. In this section, we not only address properties of f-orbit closures for the same purpose in generalized flows, but we also obtain some Lagrange f-stability and minimality results.

Definition 3.1. The f-orbit closure, positive f-orbit closure, and negative f-orbit closure for each point x in X are $\overline{C_f(x)}$, $\overline{C_f^+(x)}$, and $\overline{C_f^-(x)}$, respectively, and are denoted by $K_f(x)$, $K_f^+(x)$, and $K_f^-(x)$, respectively.

For each subset M of X, we shall denote the sets $\cup_{x\in M} K_f(x)$, $\cup_{x\in M} K_f^+(x)$, and $\cup_{x\in M} K_f^-(x)$ by $K_f(M)$, $K_f^+(M)$, and $K_f^-(M)$, respectively.

The following proposition is an immediate consequence of basic properties and definitions.

Proposition 3.2. *For each x in X,*

(1) $K_f(x) = \{y|\pi_f(x,t_i) \to y$ *for some net* (t_i) *in* $\mathbb{R}\}$,
(2) $K_f^+(x) = \{y|\pi_f(x,t_i) \to y$ *for some net* (t_i) *in* $\mathbb{R}^+\}$, *and*
(3) $K_f^-(x) = \{y|\pi_f(x,t_i) \to y$ *for some net* (t_i) *in* $\mathbb{R}^-\}$.

Proposition 3.3. *The following and their duals hold for each x in X and for t and s in $\mathbb{R}$. Whenever f is an embedding of X or X is compact, equality holds for each set inclusion in statements* (1)—(4), (9), *and* (10).

(1) $f\big(K_f(x)\big) \subset C_f\big(K_f(x)\big) \subset K_f\big(f(x)\big) = K_f\big(C_f(x)\big) = K_f\big(C_f^+(x)\big)$.
(2) $f\big(K_f^+(x)\big) \subset C_f^+\big(K_f^+(x)\big) \subset K_f^+\big(f(x)\big) = K_f^+\big(C_f^+(x)\big)$.
(3) $\pi_f\big(K_f(x),t\big) \subset K_f\big(\pi_f(x,t)\big)$.
(4) $\pi_f\big(K_f^+(x),t\big) \subset K_f^+\big(\pi_f(x,t)\big)$.
(5) $K_f\big(\pi_f(x,t)\big) = K_f\big(\pi_f(x,s)\big) = K_f\big(f(x)\big)$.
(6) $K_f^+\big(\pi_f(x,t)\big) \subset K_f^+\big(\pi_f(x,s)\big)$ *for* $s \le t$.
(7) $K_f(x) = K_f^+(x) \cup K_f^-(x)$.

(8) $K_f(x)$ $(K_f^+(x))$ *is connected.*

(9) $f^n\big(K_f(x)\big) \subset K_f\big(f^n(x)\big)$ *for each* $n \in \mathbb{Z}^+$ $(n \in \mathbb{Z}$ *if* f *is an embedding*$)$.

(10) $f^n\big(K_f^+(x)\big) \subset K_f^+\big(f^n(x)\big)$ *for each* $n \in \mathbb{Z}^+$ $(n \in \mathbb{Z}$ *if* f *is an embedding*$)$.

(11) $K(x) = K_f\big(f^{-1}(x)\big)$ *and* $K_f(x) = K(f(x))$ *whenever one of* π *or* π_f *induces the other.*

(12) $K^+(x) = K_f^+\big(f^{-1}(x)\big)$ *and* $K_f^+(x) = K^+\big(f(x)\big)$ *whenever one of* π *or* π_f *induces the other.*

Proof: To verify statement (1), we first note that $f\big(K_f(x)\big) \subset C_f\big(K_f(x)\big)$, since $f(M) \subset C_f(M)$ for each $M \subset X$. For $y \in C_f\big(K_f(x)\big)$, there is a value $t \in \mathbb{R}$, a point $z \in K_f(x)$, and a net t_i in $\mathbb{R}$ such that $y = \pi_f(z,t)$ and $\pi_f(x,t_i) \to z$. Thus, $\pi_f\big(f(x), t_i + t\big) = \pi_f\big(\pi_f(x,t_i), t\big) \to \pi_f(z,t) = y$, and hence, $y \in K_f\big(f(x)\big)$, and we have $C_f\big(K_f(x)\big) \subset K_f\big(f(x)\big)$. Next, since $f(x) \in C_f^+(x) \subset C_f(x)$, we have $K_f\big(f(x)\big) \subset K_f\big(C_f^+(x)\big) \subset K_f\big(C_f(x)\big)$. Finally, $K_f\big(\pi_f(x,t)\big) = \overline{C_f\big(\pi_f(x,t)\big)} = \overline{C_f\big(f(x)\big)} = K_f\big(f(x)\big)$ for each t in $\mathbb{R}$ so that $K_f\big(C_f(x)\big) \subset K_f\big(f(x)\big)$, completing the proof of statement (1). Statement (2) follows similarly.

Statements (3), (4), (5), (6), and (8) follow by the continuity of π_f and by taking closures of the sets in statements (3), (4), (5), (6), and (11), respectively, of Proposition 5.6 of [1].

Statement (7) follows immediately from Proposition 3.2.

Statement (9) follows from the continuity of f^n and from statement (11) of Proposition 5.6 of [1] for each $n \in \mathbb{Z}^+$ ($n \in \mathbb{Z}$ if f is an embedding) which imply that $f^n\big(K_f(x)\big) = f^n\big(\overline{C_f(x)}\big) \subset \overline{f^n\big(C_f(x)\big)} = \overline{C_f\big(f^n(x)\big)} = K_f\big(f^n(x)\big)$. Statement (10) follows similarly from statement (12) of Proposition 5.6 of [1].

Statements (11) and (12) follow from Definition 3.1 and statements (14) and (15) of Proposition 5.6 of [1].

If f is an embedding of X, then $K_f\big(f(x)\big) = \overline{C_f\big(f(x)\big)} = \overline{f\big(C_f(x)\big)} = f\big(\overline{C_f(x)}\big) = f\big(K_f(x)\big)$ so that equality holds for the set inclusions of statement (1) and, similarly, for those of statement (2).

In view of Corollary 4.3 of [1] and statement (3) of Proposition 5.6 of [1], whenever f is an embedding of X, equality holds in statement (3) since $\pi_f^t\big(K_f(x)\big) = \pi_f^t\big(\overline{C_f(x)}\big) = \overline{\pi_f^t\big(C_f(x)\big)} = \overline{\pi_f\big(C_f(x),t\big)} = \overline{C_f\big(\pi_f(x,t)\big)} = K_f\big(\pi_f(x,t)\big)$.

Statement (4) follows similarly from Corollary 4.3 of [1] and statement (4) of Proposition 5.6 of [1].

Next, let X be compact. In statement (1), let $y \in K_f\big(f(x)\big)$. Then there is a net (t_i) in $\mathbb{R}$ such that $\pi_f\big(f(x),t_i\big) \to y$, and hence, $f\big(\pi_f(x,t_i)\big) \to y$. Some subnet $\big(\pi_f(x,t_j)\big)$ of $\big(\pi_f(x,t_i)\big)$ converges to a point z of X and $z \in K_f(x)$. Thus, $\Big(f\big(\pi_f(x,t_j)\big)\Big)$ converges to $f(z)$ and y yielding $y = f(z) \in f\big(K_f(x)\big)$, and hence, $K_f\big(f(x)\big) \subset f\big(K_f(x)\big)$, establishing equality for the set inclusions of statement (1) and, similarly, for those of statement (2).

For statement (3) with X compact, let $y \in K_f\big(\pi_f(x,t)\big)$. There is a net (t_i) in $\mathbb{R}$ such that $\pi_f\big(\pi_f(x,t),t_i\big) \to y$. Since $\pi_f\big(\pi_f(x,t),t_i\big) = \pi_f\big(f(x),t+t_i\big)$ for each i, we have $y \in K_f\big(f(x)\big)$. There is a subnet (t_j) of (t_i) such that $\pi_f(x,t_j) \to z$ for some point z in X. Of course, $z \in K_f(x)$ and $\pi_f(z,t) \in \pi_f\big(K_f(x),t\big)$. Since $\Big(\pi_f\big(\pi_f(x,t_j),t\big)\Big) = \Big(\pi_f\big(\pi_f(x,t),t_j\big)\Big)$ converges to both $\pi_f(z,t)$ and y, we have $y = \pi_f(z,t) \in \pi_f\big(K_f(x),t\big)$; therefore, equality follows for the set inclusions of statement (3) and, similarly, for those of statement (4).

Finally, whenever f is an embedding of X or X is compact, equality for the set inclusions of statements (9) and (10) follows by induction from statements (1) and (2) which imply that $f\Big(K_f\big(f^n(x)\big)\Big) = K_f\big(f^{n+1}(x)\big)$ and $f\Big(K_f^+\big(f^n(x)\big)\Big) = K_f^+\big(f^{n+1}(x)\big)$ for each $n \in \mathbb{Z}^+$ ($n \in \mathbb{Z}$ if f is an embedding.)

The duals follow in a similar manner, completing the proof. □

Note that, in the verification of statements (1) through (4) of Proposition 3.3 above, we used the compactness of X in order to be sure that the net $\big(\pi_f(x,t_i)\big)$ has a convergent subnet $\big(\pi_f(x,t_j)\big)$. When X is locally compact and f is a local homeomorphism at x, this is also the case, and hence, taking Theorem 2.8 of [1] and

Corollary 2.9 of [1] into consideration, we have the following corollary.

Corollary 3.4. *If X is locally compact and f^* is continuous with $f^*(\infty) = \infty$, f injective, or f a local homeomorphism at x, then equality holds for each set inclusion in statements* (1)—(4), (9), *and* (10) *of Proposition 3.3.*

Corollary 3.5. *If X is compact, f is an embedding of X, or else f^* is continuous with $f^*(\infty) = \infty$, f injective, or f a local homeomorphism at x, then $K_f^+(x)$ ($K_f^-(x)$, $K_f(x)$) is a closed positively (negatively, bilaterally) f-invariant connected set for each x in X.*

In keeping with classical terminology, we give the following definition.

Definition 3.6. A point x in X is called Lagrange (positively Lagrange, negatively Lagrange) f-stable whenever $K_f(x)$ ($K_f^+(x)$, $K_f^-(x)$) is compact.

Theorem 3.7. *If there is a non-empty compact f-invariant or invariant subset M of X, then there exists a Lagrange (positively Lagrange, negatively Lagrange) f-stable point in X. The converse holds for f-invariance if X is compact, f is an embedding of X, or else f^* is continuous with $f^*(\infty) = \infty$, f injective, or f a local homeomorphism at x.*

Proof: In view of Proposition 2.3, we need only verify the statement for f-invariance. If M is a non-empty compact f-invariant subset of X, then we have $C_f^+(x) \subset f(M)$ for each x in M, and hence, $K_f^+(x)$ are compact subsets of $f(M)$. Conversely, let $K_f^+(x)$ be compact. There is a sequence $\{n_i\}$ in $\mathbb{Z}^+$ such that $\pi_f(x, n_i) \to y$ for some $y \in K_f^+(x)$. Note that for each $t \in \mathbb{R}$, $f\big(\pi_f(x, n_i + t)\big) = \pi_f\big(f(x), n_i + t\big) = \pi_f\big(\pi_f(x, n_i), t\big) \to \pi_f(y, t)$. For some i_o, we have $n_i + t > 0$, $\pi_f(x, n_i + t) \in C_f^+(x)$, and $f\big(\pi_f(x, n_i + t)\big) \in f\big(C_f^+(x)\big)$ for each $i \geq i_o$. Thus, $\pi_f(y, t) \in f\big(C_f^+(x)\big) \subset f\big(K_f^+(x)\big)$, and so, $C_f(y) \subset f\big(K_f^+(x)\big)$ which is compact since f is continuous. The set $K_f(y)$ is compact and, by statement (1) of Proposition 3.3 and Corollary 3.4, it is f-invariant provided X is compact or f is either an embedding of X or else a local homeomorphism at x.

The dual and bilateral versions are valid by similar reasoning completing the proof. □

Definition 3.8. A nonempty subset M of X is called minimal (f-minimal) provided M ($f(M)$) is a closed and invariant (f-invariant) set with no proper subset enjoying these properties.

Theorem 3.9. *A nonempty subset M of X is minimal (f-minimal) if and only if $K_f(x) = M$ ($K_f(x) = f(M)$) for each x in M.*

Proof: If M is f-minimal, then $K_f(x) \subset f(M)$ for each $x \in X$ since $C_f(M) \subset f(M)$ and $f(M)$ is closed. Since $K_f(x)$ is a nonempty closed f-invariant subset of $f(M)$ for each $x \in M$, we must have $K_f(x) = f(M)$ for each $x \in M$. Conversely, if M is not f-minimal, there must be a nonempty proper f-invariant subset M_o of M such that $f(M_o)$ is closed. But $f(M) = K_f(x) \subset \overline{C_f(x)} \subset \overline{C_f(M_o)} \subset \overline{f(M_o)} = f(M_o) \subset f(M)$ for any point $x \in M_o$. Thus, M is f-minimal. The proof for minimality follows similarly. □

4. f-limit Sets

As with f-orbit closures, we anticipate using properties of f-limit sets in order to characterize and classify various f-stable and f-attractive sets. Thus, in this section we shall introduce and consider certain properties of f-limit sets.

Definition 4.1. For each x in X, we define and denote the f-limit set, positive f-limit set, and negative f-limit set by

(1) $L_f(x) = \{y : \pi_f(x, t_i) \to y \text{ for some net } t_i \to \infty\}$,
(2) $L_f^+(x) = \{y : \pi_f(x, t_i) \to y \text{ for some net } t_i \to +\infty\}$, and
(3) $L_f^-(x) = \{y : \pi_f(x, t_i) \to y \text{ for some net } t_i \to -\infty\}$, respectively.

For each subset M of X, we shall denote $L_f(M)$, $L_f^+(M)$, and $L_f^-(M)$ by $\cup_{x\in M} L_f(x)$, $\cup_{x\in M} L_f^+(x)$, and $\cup_{x\in M} L_f^-(x)$, respectively.

Proposition 4.2. *For each x in X, we have the following and their duals. If f is an embedding of X or X is compact, equality holds for each set inclusion.*

(1) $f\big(L_f(x)\big) \subset C_f\big(L_f(x)\big) \subset L_f\big(f(x)\big) = L_f\big(C_f^+(x)\big) = L_f\big(C_f(x)\big)$.

(2) $f\big(L_f^+(x)\big) \subset C_f^+\big(L_f^+(x)\big) \subset L_f^+\big(f(x)\big) = L_f^+\big(C_f^+(x)\big) = L_f^+\big(C_f(x)\big)$.

(3) $\pi_f\big(L_f(x),t\big) \subset L_f\big(\pi_f(x,t)\big)$.

(4) $\pi_f\big(L_f^+(x),t\big) \subset L_f^+\big(\pi_f(x,t)\big)$.

(5) $L_f\big(\pi_f(x,t)\big) = L_f\big(\pi_f(x,s)\big) = L_f\big(f(x)\big)$.

(6) $L_f^+\big(\pi_f(x,t)\big) = L_f^+\big(\pi_f(x,s)\big) = L_f^+\big(f(x)\big)$.

(7) $L_f(x) = L_f^+(x) \cup L_f^-(x)$.

(8) $K_f(x) = C_f(x) \cup L_f(x)$.

(9) $K_f^+(x) = C_f^+(x) \cup L_f^+(x)$.

(10) $f^n\big(L_f(x)\big) \subset L_f\big(f^n(x)\big)$ *for each* $n \in \mathbb{Z}^+$ ($n \in \mathbb{Z}$ *if* f *is an embedding*).

(11) $f^n\big(L_f^+(x)\big) \subset L_f^+\big(f^n(x)\big)$ *for each* $n \in \mathbb{Z}^+$ ($n \in \mathbb{Z}$ *if* f *is an embedding*).

(12) $f\big(L_f^+(x)\big) \subset \cap_{t\in M} L_f^+\big(\pi_f(x,t)\big) \subset \cap_{t\in M} K_f^+\big(\pi_f(x,t)\big)$ *where* M *is* $\mathbb{R}^+$ *or* $\mathbb{R}$.

(13) $L(x) = L_f\big(f^{-1}(x)\big)$ *and* $L_f(x) = L\big(f(x)\big)$ *whenever one of* π *or* π_f *induces the other.*

(14) $L^+(x) = L_f^+\big(f^{-1}(x)\big)$ *and* $L_f^+(x) = L^+\big(f(x)\big)$ *whenever one of* π *or* π_f *induces the other.*

Proof: To prove statement (1), since $f\big(L_f(x)\big) \subset C_f\big(L_f(x)\big)$, let $y \in C_f\big(L_f(x)\big)$. There are $z \in L_f(x)$ and $t \in \mathbb{R}$ such that $y = \pi_f(z,t)$. Also, there is a net $\pi_f(x,t_i) \to z$ with $t_i \to \infty$. Now, $\pi_f\big(f(x), t_i + t\big) = \pi_f\big(\pi_f(x,t_i),t\big) \to \pi_f(z,t) = y$ so that $y \in L_f\big(f(x)\big)$, and hence, $C_f\big(L_f(x)\big) \subset L_f\big(f(x)\big)$. That $L_f\big(f(x)\big) \subset L_f\big(C_f^+(x)\big) \subset L_f\big(C_f(x)\big)$ follows since $f(x) \in C_f^+(x) \subset C_f(x)$. Finally, whenever y is in $L_f\big(C_f(x)\big)$, and in particular, we have $y \in L_f\big(\pi_f(x,t)\big)$ where $\pi_f\big(\pi_f(x,t),t_i\big) \to y$ as $t_i \to \infty$, then $\pi_f\big(f(x), t+t_i\big) \to y$ which is in $L_f\big(f(x)\big)$ yielding $L_f\big(C_f(x)\big) \subset L_f\big(f(x)\big)$, completing the proof of statement (1). Statement (2) follows similarly.

To see that statements (3) and (4) follow from the proofs of statements (1) and (2), respectively, we need only observe that $\pi_f\big(f(x),t+t_i\big) = \pi_f\big(\pi_f(x,t),t_i\big) = \pi_f\big(\pi_f(x,t_i),t\big) \to \pi_f(z,t) = y$.

To verify statement (5), note that for $y \in L_f\big(\pi_f(x,t)\big)$, there is a $t_i \to \infty$ such that $\pi_f\big(\pi_f(x,t),t_i\big) \to y$. Then $\pi_f\big(\pi_f(x,s), t -$

$s+t_i\big) = \pi_f\big(f(x), t+t_i\big) = \pi_f\big(\pi_f(x,t), t_i\big) \to y$, and hence, $y \in L_f\big(\pi_f(x,s)\big)$. Since t and s are arbitrary, statement (5) is now evident and statement (6) follows similarly.

Statements (7), (8), and (9) are direct consequences of the definitions.

Statements (10) and (11) follow from $f\big(L_f(x)\big) \subset L_f\big(f(x)\big)$ and $f\big(L_f^+(x)\big) \subset L_f^+\big(f(x)\big)$ of statements (1) and (2), respectively, by induction. Whenever f is an embedding of X, $y \in f^{-1}\big(L_f(x)\big)$ means $f(y) \in L_f(x)$ and there is a net $t_i \to \infty$ such that $\pi_f(x,t_i) \to f(y)$, and hence, $f^{-1}\big(\pi_f(x,t_i)\big) = \pi_f\big(f^{-1}(x), t_i\big) \to y$ so that $f^{-1}\big(L_f(x)\big) \subset L_f\big(f^{-1}(x)\big)$. Similarly, $f^{-1}\big(L_f^+(x)\big) \subset L_f^+\big(f^{-1}(x)\big)$ and statements (10) and (11) are valid for each $n \in \mathbb{Z}$ by induction.

For statement (12), let z be in $f\big(L_f^+(x)\big)$ and select $y \in L_f^+(x)$ such that $z = f(y)$. Choose $t_i \to +\infty$ where $\pi_f(x,t_i) \to y$. Since $f\big(\pi_f(x,t_i)\big) \to z$, we have $f\big(\pi_f(x,t_i)\big) = \pi_f\big(\pi_f(x,t), t_i - t\big)$ where $t_i - t \to +\infty$, and hence, $z \in L_f^+\big(\pi_f(x,t)\big) \subset K_f^+\big(\pi_f(x,t)\big)$ for each $t \in \mathbb{R}$. Thus, $f\big(L_f^+(x)\big) \subset L_f^+\big(\pi_f(x,t)\big) \subset K_f^+\big(\pi_f(x,t)\big)$ for each $t \in \mathbb{R}$, and statement (12) follows.

Statements (13) and (14) follow from Definition 4.1, continuity, and the equations $\pi = \pi_f \circ (f^{-1} \times i_{\mathbb{R}})$ and $\pi_f = \pi \circ (f \times i_{\mathbb{R}})$ of Corollary 2.3 and Corollary 2.4 of [1].

Now, we let f be an embedding of X and consider equality for the set inclusions of statement (1). For $y \in L_f\big(f(x)\big)$, there is a net $\pi_f\big(f(x), t_i\big) \to y$ where $t_i \to \infty$. Thus, $f\big(\pi_f(x,t_i)\big) \to y$ where $\pi_f(x,t_i) \to z$ for some $z \in L_f(x)$. Hence, $y = f(z) \in f\big(L_f(x)\big)$, and equality holds for the set inclusions in statement (1) and, similarly, in statement (2).

For f an embedding of X in statement (3), choose a point+ y in $L_f\big(\pi_f(x,t)\big)$ and $t_i \to \infty$ such that $\pi_f\big(\pi_f(x,t), t_i\big) \to y$. Then $\pi_f\big(\pi_f(x,t_i), t\big) \to y$, where $\pi_f(x,t_i) = f^{-2}\Big(\pi_f\big(f^2(x), t_i\big)\Big) =$

$$f^{-2}\Bigg(\pi_f\Big(\pi_f\big(f(x), t+t_i\big), -t\Big)\Bigg) \to f^{-2}\big(\pi_f(y,-t)\big) = \pi_f\big(f^{-2}(y), -t\big)$$

so that $\pi_f\big(f^{-2}(y), -t\big) \in L_f(x)$. Thus, $y = \pi_f\Big(\pi_f\big(f^{-2}(y), -t\big), t\Big) \in \pi_f\big(L_f(x), t\big)$, and hence, equality holds for the set inclusion in statement (3) and, similarly, in statement (4).

When f is an embedding of X in statements (10) and (11), equality holds for the set inclusions for each $n \in \mathbb{Z}^+$ by induction since equality holds for the set inclusions in statements (1) and (2). The only part of statement (10) that remains to be verified is $L_f\big(f^n(x)\big) \subset f^n\big(L_f(x)\big)$ for $n \in \mathbb{Z}^-$. Let $y \in L_f\big(f^{-1}(x)\big)$ and choose $t_i \to \infty$ such that $\pi_f\big(f^{-1}(x), t_i\big) \to y$. Then $\pi_f(x, t_i) \to f(y) \in L_f(x)$, and thus, $y \in f^{-1}\big(L_f(x)\big)$, verifying statement (10) for $n = -1$. The statement now follows by induction, and statement (11) for $n \in \mathbb{Z}^-$ follows similarly.

Finally, in order to show equality for the set inclusions in statement (12) whenever f is an embedding of X, let t_o be in $\mathbb{R}^+$ or $\mathbb{R}$ and let y be in $\cap_{t\geq 0} K_f^+\big(\pi_f(x,t)\big)$. There is a net $t_i \geq 0$ such that $\pi_f\big(\pi_f(x,t_o),t_i\big) \to y$. If no subnet (t_j) of (t_i) exists such that $t_j \to +\infty$, then (t_i) is bounded above by some $T > 0$, and hence, $y \notin K_f^+\big(\pi_f(x, T+1)\big)$. Consequently, choose (t_i) so that $t_i \to +\infty$. Now, $\pi_f\big(\pi_f(x,t_o),t_i\big) = \pi_f\big(f(x), t_o + t_i\big) \to y$ where $t_o + t_i \to +\infty$, so that from statement (2), $y \in L_f^+\big(f(x)\big) = f\big(L_f^+(x)\big)$.

Next, we let X be compact. To establish equality for the set inclusions of statement (1), let $y \in L_f\big(f(x)\big)$. There is a net $t_i \to \infty$ in $\mathbb{R}$ such that $\pi_f\big(f(x), t_i\big) \to y$, and so $f\big(\pi_f(x,t_i)\big) \to y$. Some subnet $\big(\pi_f(x,t_j)\big)$ of $\big(\pi_f(x,t_i)\big)$ converges to a point z in X and $z \in L_f(x)$. Also, $f\big(\pi_f(x,t_j)\big) \to f(z)$ and $y = f(z) \in f\big(L_f(x)\big)$. Thus, $L_f\big(f(x)\big) \subset f\big(L_f(x)\big)$, yielding equality for the set inclusions in statement (1) and, similarly, for those of statement (2).

For X compact in statement (3), let $y \in L_f\big(\pi_f(x,t)\big)$. There is a net $t_i \to \infty$ in $\mathbb{R}$ such that $\pi_f\big(\pi_f(x,t),t_i\big) \to y$, and hence, $\pi_f\big(f(x), t + t_i\big) \to y$ with $t + t_i \to \infty$. There exists a subnet (t_j) of (t_i) such that $\pi_f(x, t_j) \to z$ for some point z in X. Note that $z \in L_f(x)$ and $\pi_f(z,t) \in \pi_f\big(L_f(x), t\big)$. Because $\Big(\pi_f\big(\pi_f(x,t_j),t\big)\Big) = \Big(\pi_f\big(\pi_f(x,t),t_j\big)\Big)$ converges to $\pi_f(z,t)$ and y, we have $y = \pi_f(z,t) \in \pi_f\big(L_f(x), t\big)$, and equality for the set inclusion in statement (3) follows and, similarly, for that in statement (4).

When X is compact, equality follows in statements (10) and (11) by induction from statements (1) and (2) which imply that, since $f^n(x) \in X$ for $x \in X$, $f\Big(L_f\big(f^n(x)\big)\Big) = L_f\big(f^{n+1}(x)\big)$ and $f\Big(L_f^+\big(f^n(x)\big)\Big) = L_f^+\big(f^{n+1}(x)\big)$ for each $n \in \mathbb{Z}^+$.

Finally, in order to show that equality holds for the set inclusions of statement (12) when X is compact, the proof given above for f an embedding of X suffices.

The duals follow in a similar manner, completing the proof. □

The verification of statements (1) through (4) of Proposition 4.2 for X compact relies on the existence of a convergent subnet $\big(\pi_f(x,t_j)\big)$ of a given net $\big(\pi_f(x,t_i)\big)$. When X is locally compact and f is a local homeomorphism at x, the existence is also assured, and consequently, by taking Theorem 2.8 of [1] and its Corollary 2.9 into consideration, we have the following corollary.

Corollary 4.3. *If X is locally compact and f^* is continuous with $f^*(\infty) = \infty$, f injective, or f a local homeomorphism at x, then equality holds for each set inclusion in Proposition 4.2.*

Corollary 4.4. *If either f is an embedding, or else, if X is locally compact and f^* is continuous with $f^*(\infty) = \infty$, f injective, or f a local homeomorphism at x, then $L_f^+(x)$ ($L_f^-(x)$, $L_f(x)$) is closed and f-invariant.*

The dual and bilateral versions of the following proposition and its corollary are valid but not stated. They can be verified by the obvious parallel reasoning.

Proposition 4.5. *Let X be locally compact and f^* be continuous with $f^*(\infty) = \infty$, f injective, or f a local homeomorphism at x. Then*

$$L_{f^*}^{*+}(x) = \begin{cases} L_f^+(x) & \text{if } L_f^+(x) \text{ is compact} \\ L_f^+(x) \cup \{\infty\} & \text{if } L_f^+(x) \text{ is noncompact and } f^*(\infty) = \infty. \end{cases}$$

Proof: Let $f^*(\infty) = \infty$ and let $L_f^+(x)$ be noncompact. Note that $L_f^+(x) \subset L_{f^*}^{*+}(x)$. The only point of X^* that a net $(\pi_{f^*}(x,t_i))$ could converge to outside of $L_f^+(x)$ is ∞. The set $K_{f^*}^{*+}(x)$ is compact, and hence, $L_{f^*}^{*+}(x)$ is compact. Thus, $L_{f^*}^{*+}(x) = L_f^+(x) \cup \{\infty\}$.

Next, let $f^*(\infty) = \infty$ and let $L_f^+(x)$ be compact. If $L_f^+(x)$ is empty, then $L_{f^*}^{*+}(x) = \{\infty\} = L_f^+(x) \cup \{\infty\}$. On the other hand, if $L_f^+(x)$ is nonempty, choose a compact neighborhood U of $L_f^+(x)$. If there does not exist a $T > 0$ such that $\pi_f(x,t) \in U$ for each $t \geq T$, then, because $C_f^+(x)$ is connected, there is a net $(\pi_f(x,t_i))$ in the compact boundary of U which converges to a point z of ∂U. But this means that $L_f^+(x) \cap \partial U \neq \emptyset$, and hence, that U is not a neighborhood of $L_f^+(x)$ which is, of course, absurd. Thus, there is a $T > 0$ such that $\overline{\pi_f(x,[T,+\infty))} \subset U$. The sets $\overline{\pi_f(x,[T,+\infty))}$ and $\pi_f(x,[0,T])$ are compact, and consequently, $K_f^+(x) = \overline{\pi_f(x,[0,+\infty))} = \overline{\pi_f(x,[0,T]) \cup \pi_f(x,[T,+\infty))} = \pi_f(x,[0,T]) \cup \overline{\pi_f(x,[T,+\infty))}$ is compact. Since $K_f^+(x)$ is compact, no net $(\pi_f(x,t_i))$ converges to ∞, implying that $\infty \notin L_{f^*}^{*+}(x)$ so that $L_{f^*}^{*+}(x) = L_f^+(x)$.

Finally, we claim that if $f^*(\infty) \neq \infty$, then $L_f^+(x)$ is compact and $L_{f^*}^{*+}(x) = L_f^+(x)$. Let $f^*(\infty) \neq \infty$. Then $L_f^+(x)$ is a subset of $L_{f^*}^{*+}(x)$ which is a compact in X^*. Since $\infty \notin L_{f^*}^{*+}(x)$, we have $L_{f^*}^{*+}(x) \subset X$; therefore, for any point y in $L_{f^*}^{*+}(x) \setminus L_f^+(x)$, there is a net $(\pi_{f^*}^*(x,t_i))$ which converges to y. But, since $\pi_{f^*}(x,t_i) = \pi_f(x,t_i) \in X$ for each i, this also means that y would be in $L_f^+(x)$. Hence, $L_{f^*}^{*+}(x) = L_f^+(x)$, completing the proof. $\square$

Corollary 4.6. *Let X be locally compact and f^* be continuous with $f^*(\infty) = \infty$, f injective, or f a local homeomorphism at x. Then,*

$$K_{f^*}^{*+}(x) = \begin{cases} K_f^+(x) & \textit{if } K_f^+(x) \textit{ is compact} \\ K_f^+(x) \cup \{\infty\} & \textit{if } K_f^+(x) \textit{ is noncompact and } f^*(\infty) = \infty. \end{cases}$$

Theorem 4.7. *Let X be locally compact and f^* be continuous with $f^*(\infty) = \infty$, f injective, or f a local homeomorphism at x. Then, x is Lagrange (positively Lagrange, negatively Lagrange) f-stable if and only if $L_f(x)$ ($L_f^+(x)$, $L_f^-(x)$) is nonempty compact.*

Proof: Since $L_f^+(x) \subset K_f^+(x)$ and $L_f^+(x)$ closed, $L_f^+(x)$ is compact. The net $(\pi_f(x,i))$ for $i \in \mathbb{Z}^+$ is in $K_f^+(x)$, and hence, must have a convergent subnet. Consequently, $L_f^+(x)$ is nonempty. Conversely, let $L_f^+(x)$ be nonempty compact and suppose that x is not positively Lagrange f-stable. By Corollary 4.6, $K_{f^*}^{*+}(x) =$

$K_f^+(x) \cup \{\infty\}$. If (x_i) is a net in $K_f^+(x)$ which has no convergent subnet in $K_f^+(x)$, then some subnet (x_j) of (x_i) converges to ∞ in X^*. No subnet of (x_i) is in $L_f^+(x)$ since it is compact. Thus, (x_j) is ultimately in $C_f^+(x)$, and hence, $x_j = \pi_f(x, t_j)$ for some net (t_j) in $\mathbb{R}^+$. If (t_j) is bounded by some $T > 0$, then (x_j) can be selected so that $x_j \to \infty$ and $t_j \to T_o$ for some $T_o \leq T$; i.e., $x_j = \pi_f(x, t_j) \to \pi_f(x, T_o) \in C_f^+(x) \setminus \{\infty\}$, which is impossible. Thus, $t_j \to +\infty$ and $\infty \in L_{f^*}^{*+}(x)$, contrary to Proposition 4.2. Consequently, $K_f^+(x)$ is compact.

The dual and bilateral versions follow similarly, completing the proof. □

Theorem 4.8. *Let either f be an embedding, or else, let X be locally compact, f be a surjection, and f^* be continuous with $f^*(\infty) = \infty$, f injective, or f a local homeomorphism at x. Whenever x is positively (negatively) Lagrange f-stable, $L_f^+(x)$ ($L_f^-(x)$) is a nonempty compact connected set. Conversely, if X is locally compact and $L_f^+(x)$ ($L_f^-(x)$) is not connected, then $L_f^+(x)$ ($L_f^-(x)$) does not have a compact component.*

Proof: The set $L_f^+(x)$ is a compact connected set, since $L_f^+(x)$ is the intersection of closed connected subsets of the compact set $K_f^+(x)$ simply ordered by set inclusion according to statements (6) and (8) of Proposition 3.3 and statement (12) of Proposition 4.2. Specifically, when f is an embedding on X, we have $L_f^+(x) = f\Big(L_f^+\big(f^{-1}(x)\big)\Big) = \cap_{t \in \mathbb{R}^+} K_f^+\Big(\pi_f\big(f^{-1}(x), t\big)\Big) \subset K_f^+\Big(\pi_f\big(f^{-1}(x), 0\big)\Big) = K_f^+(x)$. On the other hand, when X is locally compact, f is a surjection. Let $f(y) = x$; then, again, we have $L_f^+(x) = L_f^+\big(f(y)\big) = f\big(L_f^+(y)\big) = \cap_{t \in \mathbb{R}^+} K_f^+\big(\pi_f(y, t)\big) \subset K_f^+\big(\pi_f(y, 0)\big) = K_f^+(x)$. Because $K_f^+(x)$ is compact, the net $(xt)_{t \in \mathbb{R}^+}$ must have a convergent subnet, and hence, $L_f^+(x)$ is not empty.

For the converse, consider the extended flow $(X^*, \pi_{f^*}^*)$. Then $L_{f^*}^{*+}(x) = L_f^+(x)$ if $L_f^+(x)$ is compact, and $L_{f^*}^{*+}(x) = L_f^+(x) \cup \{\infty\}$ if $f^*(\infty) = \infty$ and $L_f^+(x)$ is noncompact. Note that $K_f^+(x)$ is compact if and only if $L_f^+(x)$ is nonempty compact. Since $K_f^+(x)$ compact implies $L_f^+(x)$ is connected, $L_f^+(x)$ is noncompact. Thus,

$L^{*+}_{f^*}(x) = L^+_f(x) \cup \{\infty\}$ which is nonempty compact and connected. Each component of $L^+_f(x)$ has ∞ as a limit point in $L^{*+}_{f^*}(x)$ since $L^+_f(x)$ is open in $L^{*+}_{f^*}(x)$.

The dual follows similarly, completing the proof. □

Corollary 4.9. *Let X be compact and f be a surjection. Then, $L^+_f(x)$ ($L^-_f(x)$) is a nonempty compact connected set and $L_f(x)$ is a nonempty compact set for each x in X.*

5. Epilogue

Properties of f-prolongations, f-prolongational limit sets, f-stability, and f-attraction will be addressed in subsequent sequels to this paper.

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