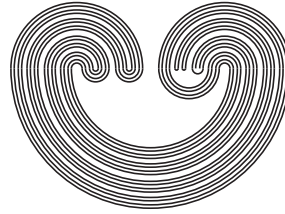

TOPOLOGY PROCEEDINGS



Volume 34, 2009

Pages 307–319

<http://topology.auburn.edu/tp/>

REFLECTIVE SUBCATEGORIES, TYCHONOFF SPACES, AND SPECTRAL SPACES

by

OTHMAN ECHI AND SAMI LAZAAR

Electronically published on July 16, 2009

Topology Proceedings

Web: <http://topology.auburn.edu/tp/>

Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

ISSN: 0146-4124

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**REFLECTIVE SUBCATEGORIES,
TYCHONOFF SPACES, AND
SPECTRAL SPACES**

OTHMAN ECHI AND SAMI LAZAAR

ABSTRACT. This paper deals with some reflective subcategories of the category of topological spaces **TOP**. The class of continuous functions in **TOP** orthogonal to all Tychonoff spaces is characterized. We also give a characterization of spaces X such that the Tychonoff reflection $\rho(X)$ is a spectral space.

INTRODUCTION

Let X be a topological space. The ring of all real valued continuous functions defined on X will be denoted by $C(X)$. Two subsets A and B are said to be *completely separated* in X if there exists a mapping f in $C(X)$ such that $f(a) = 0$ for all a in A and $f(b) = 1$ for all b in B . It will be convenient to say that $x, y \in X$ are *completely separated* if $\{x\}$ and $\{y\}$ are completely separated.

A space X is said to be *completely regular* if every closed set F of X is completely separated from any point x not in F . A completely regular T_1 -space is called a *Tychonoff space* [16]. Tychonoff spaces are also called $T_{3\frac{1}{2}}$ -spaces because they clearly sit between regular (or T_3 -) spaces and normal (or T_4 -) spaces.

Recall the standard notion of reflective subcategory \mathcal{A} of \mathcal{B} , that is, a full subcategory such that the embedding $\mathcal{A} \hookrightarrow \mathcal{B}$ has a left adjoint $F : \mathcal{B} \rightarrow \mathcal{A}$ (called reflection; in other words, one has

We thank the DGRST 03/UR/15 – 03 for its partial support.

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for each object B of \mathcal{B} a morphism $\tau_B : B \rightarrow F(B)$ such that for each object A of \mathcal{A} and $f : B \rightarrow A$ there is precisely one arrow $\bar{f} : F(B) \rightarrow A$ such that $\bar{f}\tau_B = f$. Further recall that for all $i = 0, 1, 2, 3, 3\frac{1}{2}$ the subcategories \mathbf{TOP}_i of T_i -spaces are reflective in \mathbf{TOP} , the category of topological spaces.

The Tychonoff reflection of X will be denoted by $\rho(X)$.

In [4], the authors have introduced the following separation axioms.

Definition 0.1. Let i and j be two integers such that $0 \leq i < j \leq 2$. Let us denote by \mathbf{T}_i the functor from \mathbf{TOP} to \mathbf{TOP} which takes each topological space X to its T_i -identification (the universal T_i -space associated with X). A topological space X is said to be a $T_{(i,j)}$ -space if $\mathbf{T}_i(X)$ is a T_j -space (thus, we have three new types of separation axioms, namely $T_{(0,1)}$, $T_{(0,2)}$, and $T_{(1,2)}$).

Definition 0.2. Let \mathbf{C} be a category and let F and G be two (covariant) functors from \mathbf{C} to itself.

- (1) An object X of \mathbf{C} is said to be a $T_{(F,G)}$ -object if $G(F(X))$ is isomorphic with $F(X)$.
- (2) Let \mathcal{P} be a topological property and F be a covariant functor from \mathbf{TOP} into itself. A topological space X is said to be a $T_{(F,\mathcal{P})}$ -space if $F(X)$ satisfies the property \mathcal{P} .

Following Definition 0.2, for a functor F from \mathbf{TOP} into itself, one may define another new separation axiom: A space X is said to be a $T_{(F,\rho)}$ -space if $F(X)$ is a Tychonoff space.

In the first section of this paper, we investigate $T_{(F,\rho)}$ -spaces, where F is the left adjoint functor of the embedding $\mathcal{A} \hookrightarrow \mathbf{TOP}$ of a reflective subcategory \mathcal{A} of the category of topological spaces.

A morphism $f : A \rightarrow B$ and an object X in a category \mathbf{C} are called *orthogonal* [10], if the mapping

$$\mathrm{hom}_{\mathbf{C}}(f, X) : \mathrm{hom}_{\mathbf{C}}(B, X) \rightarrow \mathrm{hom}_{\mathbf{C}}(A, X),$$

which takes g to gf , is bijective.

Let \mathbf{D} be a reflective subcategory of a category \mathbf{C} and let F be the left adjoint of the embedding $\mathbf{D} \hookrightarrow \mathbf{C}$. Then the orthogonality class of morphisms \mathbf{D}^\perp is the collection of all morphisms of \mathbf{C} rendered invertible by the functor F [1, Proposition 2.3].

The second section of this paper deals with some categorical properties of **TYCH**, the category of Tychonoff spaces. More precisely, a characterization of the orthogonality class of morphisms \mathbf{TYCH}^\perp is given.

In the third section of this work, we characterize topological spaces X such that $\rho(X)$ is a spectral space.

1. EXCEPTIONAL SUBCATEGORIES OF **TOP**

Recall that a continuous map $q : X \rightarrow Y$ is said to be a *quasi-homeomorphism* if $U \mapsto q^{-1}(U)$ defines a bijection $\mathcal{O}(Y) \rightarrow \mathcal{O}(X)$, where $\mathcal{O}(X)$ is the set of all open subsets of X [12].

Let $\mathbf{I} : \mathbf{TOP}_0 \hookrightarrow \mathbf{TOP}$ be the embedding functor, X be a space, and $\mathbf{T}_0(X)$ be its T_0 -identification. Let μ be the unit of the adjunction $(\mathbf{T}_0, \mathbf{I})$. Then $\mu_X : X \rightarrow \mathbf{T}_0(X)$ is a quasihomomorphism.

Recall that a space X is said to be *sober* if the open sets $X \setminus \overline{\{x\}}$ are the only meet-irreducible ones (an open set W is *meet-irreducible* if whenever U and V are open sets such that $W = U \cap V$, we have $W = U$ or $W = V$).

It is also worth noting that the subcategory **SOB** of sober spaces is reflective in **TOP**. Let $\mathbf{I} : \mathbf{SOB} \hookrightarrow \mathbf{TOP}$ be the embedding functor, \mathbf{S} be the sobrification functor, and η be the unit of the adjunction (\mathbf{S}, \mathbf{I}) . Then $\eta_X : X \rightarrow \mathbf{S}(X)$ is a quasihomomorphism.

In order to unify some separation axioms related to Tychonoff spaces, let us introduce the following concept.

Definition 1.1. A full subcategory \mathcal{A} of **TOP** is said to be an *exceptional subcategory* if it satisfies the following properties.

- (1) \mathcal{A} is reflective in **TOP**.
- (2) The real line \mathbb{R} is an object of \mathcal{A} .
- (3) Let F be the left adjoint functor of the embedding $\mathbf{I} : \mathcal{A} \hookrightarrow \mathbf{TOP}$ and let μ be the unit of the adjunction (F, \mathbf{I}) . Then for each space X in \mathcal{A} , $\mu_X : X \rightarrow F(X)$ is a quasihomomorphism.

Example 1.2. \mathbf{TOP}_0 and **SOB** are exceptional subcategories of **TOP**.

Before giving our first result, let us state a straightforward lemma.

Lemma 1.3. *Let Y be a topological space, let X be a set, and let $q: X \rightarrow Y$ be a map. We equip X with the topology inverse image of that of Y by q . If Y is completely regular, then so is X .*

Theorem 1.4. *Let \mathbf{B} be an exceptional subcategory of \mathbf{TOP} and let $F: \mathbf{TOP} \rightarrow \mathbf{B}$ be the left adjoint functor of the inclusion functor $I: \mathbf{B} \hookrightarrow \mathbf{TOP}$. Let X be a topological space. Then the following statements are equivalent.*

- (1) X is a $T_{(F,\rho)}$ -space.
- (2) X is a completely regular $T_{(F,1)}$ -space.

Proof: (1) \implies (2) That X is a $T_{(F,1)}$ -space is straightforward (since each Tychonoff space is a T_1 -space).

Since \mathbf{B} is an exceptional subcategory of \mathbf{TOP} , the unit μ of the adjunction (F, I) induces a quasihomomorphism $\mu_X: X \rightarrow F(X)$. In particular, the topology on X is the inverse image of that of $F(X)$. Hence, by Lemma 1.3, X is completely regular.

(2) \implies (1). It suffices to show that every closed set D of $F(X)$ is separated from any point $d \notin D$. Since μ_X is a quasihomomorphism and $F(X)$ is a T_1 -space, μ_X is an onto map. Let $C = \mu_X^{-1}(D)$ and $c \in X$ such that $d = \mu_X(c)$. Then C is a closed set of X not containing c . As X is completely regular, there exists a continuous map $f: X \rightarrow \mathbb{R}$ such that $f(c)=1$ and $f(C) = \{0\}$. Since \mathbb{R} is an object of \mathbf{B} , it is orthogonal to μ_X . Thus, $g: F(X) \rightarrow \mathbb{R}$ such that $g\mu_X = f$. Of course, g separates d and D , showing that X is a $T_{(F,\rho)}$ -space. \square

As an immediate consequence of the previous result, one can state the following.

Corollary 1.5. *Let X be a topological space. Then the following properties hold.*

- (1) X is a $T_{(\mathbf{S},\rho)}$ -space if and only if X is a completely regular $T_{(\mathbf{S},1)}$ -space.
- (2) X is a $T_{(0,\rho)}$ -space if and only if X is a completely regular $T_{(0,1)}$ -space.

Next, we shed light on $T_{(1,\rho)}$ -spaces. An easy remark is needed.

Remark 1.6. Let $q: X \rightarrow Y$ be an onto continuous map. If X is completely regular and $\mathbb{R} \perp q$, then Y is completely regular.

Proposition 1.7. *Let X be a completely regular space. Then X is a $T_{(1,\rho)}$ -space.*

Proof: Let X be a topological space and let $q: X \rightarrow \mathbf{T}_1(X)$ be the canonical onto map. According to the fact that \mathbb{R} is a T_1 -space, \mathbb{R} is orthogonal to q . Thus, by Remark 1.6, $\mathbf{T}_1(X)$ is completely regular. \square

Note 1.8. The converse of Proposition 1.7 obviously does not hold. Take a space X such that $X = \overline{\{a\}}$ for an $a \in X$ and such that there is a non-empty closed $A \subseteq X$, $a \notin A$. Then for any continuous $f: X \rightarrow \mathbb{R}$ with $f(a) = 1$, one has $X = \{a\} \subseteq f^{-1}\{1\}$.

Question 1.9. Give a characterization of $T_{(1,\rho)}$ -spaces.

Remark 1.10. Example 1.8 and Theorem 1.4 imply that the subcategory \mathbf{TOP}_1 is not exceptional in \mathbf{TOP} .

Let $q: X \rightarrow Y$ be a quasihomomorphism. Recall that it is well known that if X is a sober space and Y is a T_0 -space, then q is a homeomorphism (see, for instance, [12]). It is also well known that if X is a T_0 -space and Y is a T_1 -space, then q is a homeomorphism (see, for instance, [8]).

Proposition 1.11. *Let \mathbf{B} be an exceptional subcategory of \mathbf{TOP} and $F: \mathbf{TOP} \rightarrow \mathbf{B}$ be the left adjoint functor of the inclusion functor $I: \mathbf{B} \hookrightarrow \mathbf{TOP}$. Let $q: X \rightarrow Y$ be a quasihomomorphism. Then the following statements are equivalent.*

- (i) X is a $T_{(F,\rho)}$ -space.
- (ii) Y is a $T_{(F,\rho)}$ -space.

Proof: (i) \implies (ii) Of course, $F(q): F(X) \rightarrow F(Y)$ is a quasihomomorphism. On the other hand, since $F(X)$ is a Tychonoff space, it is a T_2 -space, therefore, a sober space. Since, in addition, $F(Y)$ is a T_0 -space, then $F(q)$ is a homeomorphism. Therefore, $F(Y)$ is completely regular, which means that Y is a $T_{(F,\rho)}$ -space.

(ii) \implies (i) The complete regularity of $F(Y)$ implies that it is a T_1 -space. Since, in addition, $F(X)$ is a T_0 -space, we conclude that $F(q)$ is a homeomorphism. Therefore, $F(X)$ is a Tychonoff space. This means that X is a $T_{(F,\rho)}$ -space. \square

2. THE CLASS OF CONTINUOUS MAPS
ORTHOGONAL TO ALL TYCHONOFF SPACES

This section is devoted to the study of the orthogonal class \mathbf{TYCH}^\perp ; hence, we will give a characterization of morphisms rendered invertible by the functor ρ .

The following concepts are needed.

Definition 2.1. Let $f: X \longrightarrow Y$ be a continuous map.

- (1) f is said to be ρ -injective (or ρ -one-to-one) if, for each $x, y \in X$, x and y are completely separated, then so are $f(x)$ and $f(y)$.
- (2) f is said to be ρ -surjective (or ρ -onto) if for each $y \in Y$, there exists $x \in X$ such that $f(x)$ and y are not completely separated.
- (3) f is said to be ρ -bijective if it is both ρ -injective and ρ -surjective.

Example 2.2. (1) Every onto continuous map is ρ -onto.

- (2) A ρ -onto continuous map need not be onto. Let $X = \{0\}$ and $Y = \{0, 1\}$ be equipped with the trivial topologies. Set $f: X \longrightarrow Y$ which takes 0 to 0.

Clearly, f is a ρ -onto continuous map which is not onto.

- (3) A ρ -injective continuous map need not be one-to-one. Let X be a topological space which is not a Tychonoff space. Of course, θ_X is a ρ -injective continuous map, but it is not one-to-one.
- (4) A one-to-one continuous map need not be ρ -injective. Let $X = \{0, 1\}$ be equipped with the discrete topology and $Y = \{0, 1\}$ be equipped with the trivial topology. Let $f = 1_X: X \longrightarrow Y$. Then f is a one-to-one continuous map. However, 0 and 1 are completely separated in X but $f(0)$ and $f(1)$ are not completely separated in Y .

Before giving the main result of this section, we need a lemma.

Lemma 2.3. *Let $f: X \longrightarrow Y$ be a continuous map. Then the following properties hold.*

- (1) f is ρ -injective if and only if $\rho(f)$ is injective.
- (2) f is ρ -surjective if and only if $\rho(f)$ is surjective.
- (3) f is ρ -bijective if and only if $\rho(f)$ is bijective.

- (4) Let $g: Y \longrightarrow Z$ be a continuous map. If two among f , g , and $g \circ f$ are ρ -bijective, then so is the third one.

Proof: (1) Suppose that $\rho(f)$ is injective. Let $x, y \in X$ be two completely separated points, that is, $\theta_X(x) \neq \theta_X(y)$. Since $\rho(f)$ is one-to-one, $\rho(f)(\theta_X(x)) \neq \rho(f)(\theta_X(y))$. Hence, $\theta_Y(f(x)) \neq \theta_Y(f(y))$. Thus, $f(x)$ and $f(y)$ are completely separated. Therefore, f is ρ -injective.

Conversely, suppose that f is ρ -injective. Let $x, y \in X$ be such that $\rho(f)(\theta_X(x)) = \rho(f)(\theta_X(y))$. Then $\theta_Y(f(x)) = \theta_Y(f(y))$. Hence, $f(x)$ and $f(y)$ are not completely separated. Since f is ρ -injective, we conclude that x and y are not completely separated; thus, $\theta_X(x) = \theta_X(y)$. Therefore, $\rho(f)$ is one-to-one.

(2) Suppose that $\rho(f)$ is surjective. Let $y \in Y$. Since $\rho(f)$ is onto, there exists $x \in X$ such that $\rho(f)(\theta_X(x)) = \theta_Y(y)$. Hence, $\theta_Y(f(x)) = \theta_Y(y)$; consequently, $f(x)$ and y are not completely separated. Therefore, f is ρ -onto.

Conversely, suppose that f is ρ -onto. Let $y \in Y$. Since f is ρ -onto, there exists $x \in X$ such that $f(x)$ and y are not completely separated. Hence, $\theta_Y(f(x)) = \theta_Y(y) = \rho(f)(\theta_X(x))$. Thus, $\rho(f)$ is onto.

(3) A direct consequence of (1) and (2).

(4) This is a direct consequence of (3) and the fact that ρ is a functor. \square

Now, we are in a position to state the main result of this section.

Theorem 2.4. *Let $f: X \longrightarrow Y$ be a continuous map. Then the following statements are equivalent.*

- (1) $\rho(f)$ is a homeomorphism.
- (2) f is ρ -bijective and $\mathbb{R} \perp f$.

Proof: (1) \implies (2) According to Lemma 2.3, f is ρ -bijective.

By [1, Proposition 2.6] and [1, Proposition 2.3], the morphism f is orthogonal to \mathbb{R} , since \mathbb{R} is a Tychonoff space.

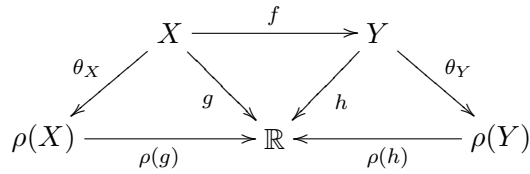
(2) \implies (1) By Lemma 2.3, $\rho(f)$ is bijective.

CLAIM. $\rho(f)$ is closed.

Proof of Claim: Let F' be a closed set of $\rho(X)$. Then there exists a collection $g_i: X \longrightarrow \mathbb{R}$ of continuous maps and a collection of closed sets F_i of \mathbb{R} such that $F' = \cap[\rho(g_i)^{-1}(F_i): i \in I]$. Let us

prove that $\rho(f)(F')$ is closed in $\rho(Y)$. Without loss of generality, one may suppose that $F' = \rho(g)^{-1}(F)$ where $g: X \rightarrow \mathbb{R}$ is a continuous map and F is a closed set of \mathbb{R} (because $\rho(f)$ is bijective). By (2), there exists a continuous map $h: X \rightarrow \mathbb{R}$ such that $g = h \circ f$.

The following diagram



is commutative. Then we have

$$\begin{aligned}
 \rho(f)(\rho(g)^{-1}(F)) &= \rho(f)(\theta_X(g^{-1}(F))) \\
 &= (\theta_Y \circ f)(g^{-1}(F)) \\
 &= (\theta_Y \circ f)(f^{-1}(h^{-1}(F))) \\
 &= (\theta_Y \circ f)[(\theta_Y \circ f)^{-1}(\rho(h)^{-1}(F))].
 \end{aligned}$$

Of course, $\theta_Y \circ f$ is onto (since f is ρ -onto). Hence, $\rho(f)(\rho(g)^{-1}(F)) = \rho(h)^{-1}(F)$; consequently, $\rho(f)$ is closed.

Now, $\rho(f)$ is a bijective continuous closed map, so it is a homeomorphism. □

Let us make some comments on the previous result. It is not difficult to prove that if **SOB** is the full subcategory of **TOP** whose objects are sober spaces and \mathcal{S} is the Sierpiński space, then the class of continuous maps orthogonal to **SOB** is the class of all quasihomomorphisms. This class coincides with the class of morphisms in **TOP** which are orthogonal to the Sierpiński space \mathcal{S} . In Theorem 2.4, we have said almost the same thing, just replace Sierpiński space with the real line \mathbb{R} . Thus, we close this section by the following question.

Question 2.5. Let $f: X \rightarrow Y$ be a continuous map such that $\mathbb{R} \perp f$. Is f ρ -bijective?

3. TYCHONOFF SPECTRAL SPACES

The main result of this section is the characterization of topological spaces X such that $\rho(X)$ is a spectral space.

Let us first recall that a topological space X is said to be *spectral* [14] if the following axioms hold.

- (i) X is a sober space.
- (ii) X is compact and has a basis of compact open sets.
- (iii) The family of compact open sets of X is closed under finite intersections.

Let $\text{Spec}(R)$ denote the set of prime ideals of a commutative ring R with identity. Recall that the *Zariski topology* or the *hull-kernel topology* for $\text{Spec}(R)$ is defined by letting $C \subseteq \text{Spec}(R)$ be closed if and only if there exists an ideal \mathcal{A} of R such that $C = \{\mathcal{P} \in \text{Spec}(R) : \mathcal{P} \supseteq \mathcal{A}\}$. M. Hochster [14] has proved that a topological space is homeomorphic to the prime spectrum of some ring equipped with the Zariski topology if and only if it is spectral. In lattice theory, a spectral space is characterized by the fact that it is homeomorphic to the prime spectrum of a bounded (with a 0 and a 1) distributive lattice.

Note that spectral spaces are of interest not only in (topological) ring and lattice theory, but also in computer science, particularly in domain theory.

In order to motivate the reader, we give some links between the previous axioms (i) and (ii) and functional analysis.

Ola Bratteli and George A. Elliott showed in [7] that a topological space X is homeomorphic to the primitive spectrum of an approximately finite-dimensional C^* -algebra (called *AF C^* -algebra*) equipped with the Jacobson topology if and only if it has a countable basis and it satisfies the above axioms (i) and (ii). By the way, C^* -algebras and foliation theory are strongly linked. Thus, there must be some link between spectral spaces and foliation theory; this was done by the authors of [5] and [6].

Recently, it was shown that spectral spaces are also related to the relatively new research topic combinatorics on words, an area in discrete mathematics motivated in part by computer science (see [9]).

Some authors (see, for example, [2] and [3]) have been interested in a particular type of spectral spaces constructed from some compactifications (namely, the one point-compactification, see [3], the Walman compactification, and the T_0 -compactification (of Horst Herrlich [13]), see [2]).

Pursuing these kinds of investigations for spectral spaces, we are interested, here, in topological spaces such that the Tychonoff reflection is a spectral space.

Definition 3.1. A topological space X is said to be *Tychonoff spectral*, if $\rho(X)$ is a spectral space.

Before giving the main result of this section, some concepts have to be introduced.

Definition 3.2. Let X be a topological space and let H be a subset of $C(X)$.

We say that H has the *finite intersection property (FIP)* if for each finite subset J of H , we have $\cap[f^{-1}(\{0\}) : f \in J] \neq \emptyset$.

Definition 3.3. Let X be a topological space and let U be an open subset of X .

- (1) U is said to be *zero-closed (z-closed)* if there exists a subset H of $C(X)$ such that $U = \cap[f^{-1}(\{0\}) : f \in H]$.
- (2) U is said to be *zero-clopen (z-clopen)* if U and $X \setminus U$ are both z-closed subsets of X .

Let us recall an interesting result which characterizes completely regular spaces in terms of zero sets. Let X be a topological space and $A \subseteq X$. A is called a *zero set* if there exists $f \in C(X)$ such that $A = f^{-1}(\{0\})$.

Proposition 3.4 ([18, Proposition 1.7]). *A space is Tychonoff if and only if the family of zero sets of the space is a base for the closed sets.*

Let us state a useful remark.

Remark 3.5. A closed set of $\rho(X)$ is of the form $\cap[\rho(f)^{-1}(\{0\}) : f \in H]$, where H is a collection of continuous maps $f: X \rightarrow \mathbb{R}$.

Indeed, $\rho(X)$ is a Tychonoff space. Then the collection

$$\{g^{-1}\{0\} \mid g: \rho(X) \rightarrow \mathbb{R} \text{ continuous}\}$$

is a basis of closed sets of $\rho(X)$.

According to the universal property of $\rho(X)$, each continuous map $g: \rho(X) \rightarrow \mathbb{R}$ may be written as $g = \rho(f)$ with $f = g \circ \theta_X$.

Now, we are in a position to give the main result of this section.

Theorem 3.6. *Let X be a topological space. Then the following statements are equivalent.*

- (1) $\rho(X)$ is spectral.
- (2) X satisfies the following properties.
 - (i) For each subset H of $C(X)$ satisfying the FIP, we have

$$\bigcap [f^{-1}(\{0\}) : f \in H] \neq \emptyset.$$

- (ii) For each completely separated points $x, y \in X$, there exists a z -clopen subset U of X containing one of x or y but not containing the other.

Proof: (1) \implies (2).

(i) Suppose that $\bigcap [f^{-1}(\{0\}) : f \in H] = \emptyset$. Then

$$\begin{aligned} \bigcap [\theta_X^{-1}(\rho(f)^{-1}(\{0\})) : f \in H] = \\ \theta_X^{-1}(\bigcap [(\rho(f))^{-1}(\{0\}) : f \in H]) = \emptyset. \end{aligned}$$

Since θ_X is onto, we have $\bigcap [\rho(f)^{-1}(\{0\}) : f \in H] = \emptyset$. Now since each spectral space is compact, then $\rho(X)$ is compact and consequently there exists a finite subset J of H such that $\bigcap [\rho(f)^{-1}(\{0\}) : f \in J] = \emptyset$. Thus, $\bigcap [f^{-1}(\{0\}) : f \in J] = \emptyset$, which contradicts the FIP. Therefore, $\bigcap [f^{-1}(\{0\}) : f \in H] \neq \emptyset$.

(ii) Let $x, y \in X$ be completely separated points. Then $\theta_X(x) \neq \theta_X(y)$. Since $\rho(X)$ is a T_1 -spectral space, $\rho(X)$ is totally disconnected by [14]. Now according to [19, Lemma 29.6], there exists a clopen set \tilde{U} of $\rho(X)$ containing $\theta_X(x)$ and not containing $\theta_X(y)$. By Remark 3.5, there exists a subset H of $C(X)$ such that $\tilde{U} = \bigcap [\rho(f)^{-1}(\{0\}) : f \in H]$. Let $U := \theta_X^{-1}(\tilde{U}) = \bigcap [f^{-1}(\{0\}) : f \in H]$. Thus, U is a z -closed subset of X .

On the other hand, we have $X \setminus U = \theta_X^{-1}(\rho(X) \setminus \tilde{U})$. Since $\rho(X) \setminus \tilde{U}$ is closed in $\rho(X)$, there exists a subset H_1 of $C(X)$ such that $\rho(X) \setminus \tilde{U} = \bigcap [\rho(f)^{-1}(\{0\}) : f \in H_1]$ (see Remark 3.5). Hence, $X \setminus U = \bigcap [f^{-1}(\{0\}) : f \in H_1]$ is a z -closed subset of X . Therefore, U is a z -clopen subset of X containing x and not containing y .

(2) \implies (1). First, let us remark that (i) means that $\rho(X)$ is compact. Indeed, it suffices to show that for each subset H of $C(X)$, with the property $\bigcap [\rho(f)^{-1}(\{0\}) : f \in H] = \emptyset$, there exists

a finite subset J of H such that $\cap[\rho(f)^{-1}(\{0\}) : f \in J] = \emptyset$ (see Remark 3.5).

In fact, $\cap[\rho(f)^{-1}(\{0\}) : f \in H] = \emptyset$ implies that $\cap[f^{-1}(\{0\}) : f \in H] = \emptyset$. It follows that H does not satisfy the FIP, which shows that there is a finite subset J of H such that $\cap[f^{-1}(\{0\}) : f \in J] = \emptyset$; consequently, $\cap[\rho(f)^{-1}(\{0\}) : f \in J] = \emptyset$. Therefore, $\rho(X)$ is compact.

But $\rho(X)$ is a T_1 -space; thus, to prove that it is spectral, it is enough to show that $\rho(X)$ is totally disconnected (see [14]). To do this, we use [19, Lemma 29.6] (since $\rho(X)$ is a compact T_2 -space).

Let $x, y \in X$ be such that $\theta_X(x) \neq \theta_X(y)$. Then x and y are completely separated. By (ii), there exists a z-clopen subset $U = \cap[f^{-1}(\{0\}) : f \in H]$ of X containing x not containing y . Set $\tilde{U} = \cap[\rho(f)^{-1}(\{0\}) : f \in H]$. Thus, \tilde{U} is a closed set of $\rho(X)$ containing $\theta_X(x)$ and not containing $\theta_X(y)$.

On the other hand, we have $\rho(X) \setminus \tilde{U} = \theta_X(X \setminus U)$. And since $X \setminus U$ is a z-closed subset of X , there exists a subset H' of $C(X)$ such that $X \setminus U = \cap[f^{-1}(\{0\}) : f \in H']$. Consequently, $\rho(X) \setminus \tilde{U} = \theta_X(\cap[f^{-1}(\{0\}) : f \in H']) = \theta_X(\cap[\theta_X^{-1}(\rho(f)^{-1}(\{0\})) : f \in H']) = \cap[\rho(f)^{-1}(\{0\}) : f \in H']$ is a closed set of $\rho(X)$, so that \tilde{U} is a clopen subset of $\rho(X)$ containing $\theta_X(x)$ and not containing $\theta_X(y)$. Therefore, $\rho(X)$ is totally disconnected. \square

Acknowledgment: We thank the referee for his/her concrete suggestions that helped to improve the presentation of the paper.

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(ECHI) DEPARTMENT OF MATHEMATICS; FACULTY OF SCIENCES OF TUNIS; UNIVERSITY OF TUNIS EL-MANAR; “CAMPUS UNIVERSITAIRE” 2092 EL-MANAR II, TUNIS, TUNISIA.

E-mail address: othechi@yahoo.com othechi@math.com

(LAZAAR) DEPARTMENT OF MATHEMATICS; FACULTY OF SCIENCES OF TUNIS; UNIVERSITY OF TUNIS EL-MANAR; “CAMPUS UNIVERSITAIRE” 2092 EL-MANAR II, TUNIS, TUNISIA.

E-mail address: salazaar72@yahoo.fr