COINCIDENCE VALUES OF COMMUTING FUNCTIONS

by

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ABSTRACT. Coincidence values of commuting functions from a topological space into itself have been investigated for more than the past 60 years. We survey some of the key results of these investigations in the context of questions of current interest.

1. Introduction

The author first became interested in the study of coincidence values of commuting functions when he read the following question on a web site that is maintained by Janusz R. Prajs and Włodzimierz J. Charatonik and which is dedicated to open problems in continuum theory [12].

Question 1. Does every pair of commuting self-maps of the simple triod have a coincidence value?

The origin of this question is unknown. According to our research, Question 1 appeared for the first time in the literature in 1983 as a question asked by David P. Bellamy [37]; however, at the 42nd Annual Spring Topology and Dynamics Conference in 2008,

Bellamy noted that the question probably dates back to the early 1970s. We will show in section 5 that a negative answer to Question 1 would allow for the construction of a (simple triod)-like continuum without the fixed point property; such an example would contribute significantly to the collection of tree-like continua without the fixed point property that began with the celebrated example by Bellamy in 1978 [4].

As Question 1 might suggest, there is a strong relationship between the notions of coincidence values and commutativity of functions; in particular, an interrelationship between the concepts of commuting functions and common fixed points of functions will be demonstrated in section 3. The focus of the most intensive investigations of common fixed points of commuting mappings has been on the unit interval; much of this work is surveyed in section 4. In section 5, we will consider results regarding coincidence values of mappings of more general spaces and show how several partial answers to Question 1 follow from results in the literature.

2. Definitions and notation

A *mapping* is a continuous function and *self-map* of a set S is a mapping $f : S \to S$. For a pair of self-maps $f, g : S \to S$, we will use fg to denote the composition $f \circ g$. For any $x \in S$, we define $f^0(x)$ to be x (so that f^0 is the identity self-map of S), and for any positive integer k we define f^k to be ff^{k-1}. A *fixed point* of a self-map $f : S \to S$ is any $x \in S$ for which $f(x) = x$; we will use $FP(f)$ to denote the set of fixed points of f. A point $x \in S$ is said to be a *periodic point* of f provided that $f^k(x) = x$ for some positive integer k; we will use $P(f)$ to denote the set of all periodic points of f. If f and g are self-maps of S and $f(x) = g(x)$ for some $x \in S$, we will call $f(x)$ a *coincidence value* of f and g; also, x will be called a *coincidence point* of f and g.

A pair of self-maps $f, g : S \to S$ is said to *commute* provided $fg(x) = gf(x)$ for all $x \in S$. The concept of commuting mappings was extended by Gerald Jungck [32] in the setting of metric spaces as follows: Self-maps $f, g : X \to X$ of a metric space (X, d) are *compatible* provided $\lim_{n \to \infty} d(fg(x_n), gf(x_n)) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = t$ for some $t \in X$. Jungck proves in [33] that in the setting of compact

metric spaces, $f, g : X \to X$ are compatible if and only if they commute on the set $A = \{x \in X : f(x) = g(x)\}$ of coincidence points of f and g; compatible self-maps of compacta for which $A \neq \emptyset$ are said to be *nontrivially compatible*. Examples of compatible mappings that are not commutative (or even weakly commutative–defined in [47]) are readily available (see, for example, [32]).

A metric space X is said to have the *fixed point property* (*FPP*) provided every self-map of X has a fixed point in X. John Philip Huneke and Henry H. Glover [24] defined X to have the *common fixed point property* (*CFP*) provided that for every pair of commuting self-maps $f, g : X \to X$, there exists some $x \in X$ for which $f(x) = x = g(x)$. We define X to have the *coincidence value property* (*CVP*) provided that for every pair of commuting self-maps $f, g : X \to X$ there is some $x \in X$ for which $f(x) = g(x)$.

3. Interdependence between common fixed points and commuting mappings

The hypothesis of commutativity in the definitions of CFP and CVP appears at first to be arbitrary. However, the following observation by Jungck [29] illustrates an interdependence between the concepts of common fixed points and commuting mappings.

Proposition 1 (Jungck [29], 1976)**.** *Let X be any set and let $f : X \to X$ be a function (not necessarily continuous). Then f has a fixed point if and only if there exists a constant map $g : X \to X$ that commutes with f. Moreover, f and g have a unique common fixed point if this condition holds.*

Proof: Suppose that a is a fixed point of f and define $g : X \to X$ by $g(x) = a$ for all $x \in X$. Then $f(g(x)) = f(a) = a$ and $g(f(x)) = a$ for all $x \in X$. Thus, g is a constant map that commutes with f.

Conversely, suppose that $g : X \to X$ is given by $g(x) = a$ for all $x \in X$ and that g commutes with f. Then $f(a) = f(g(a)) = g(f(a)) = a$. Thus, a is a fixed point of f. (Clearly, a is a fixed point of g as well.) □

The following result by Jungck further elucidates the interrelationship between fixed points and commuting mappings. (As a corollary, Jungck obtains M. Edelstein's theorem [17] that every contractive self-map of a compactum has a unique fixed point.)

Proposition 2 (Jungck [35], 2005). *A self-map f of a compact metric space (X, d) has a fixed point if and only if there exists a mapping $g : X \to f(X)$ which commutes with f and satisfies*

$$(*) \quad d(g(x), g(y)) < d(f(x), f(y)), \qquad f(x) \neq f(y).$$

Moreover, f and g have a unique common fixed point if (*) holds.

Solomon Leader [36, Proposition 3] shows that condition (*) in Proposition 2 can be replaced with

$$(*') \quad d(g(x), g(y)) < d(x, y), \qquad x \neq y.$$

We remark that in addition to appearing in [35], Proposition 2 also appears without proof in [29]. Moreover, Jungck shows that Proposition 2 can be obtained as a corollary to a similar result regarding periodic points of f [31, Theorem 2.1].

By strengthening the condition given in Proposition 2, Jungck proves a similar result in the setting of complete metric spaces which yields a generalization of the Banach contraction principle as a corollary; we state this result as Proposition 3. Both Proposition 2 and Proposition 3 should be compared to a result of Raul Machuca [38] which guarantees a coincidence value for self-maps f and g (not necessarily commutative) of a compactum (X, d) provided that $g(X) \subseteq f(X)$ and there exists some $\alpha < 1$ for which $d(g(x), g(y)) \leq \alpha d(f(x), f(y))$.

Proposition 3 (Jungck [29], 1976). *A self-map f of a complete metric space has a fixed point if and only if there exists $0 < \alpha < 1$ and $g : X \to X$ that commutes with f such that*

$$(*) \quad g(X) \subseteq f(X) \text{ and } d(g(x), g(y)) \leq \alpha d(f(x), f(y))$$
$$\text{for all } x, y \in X.$$

Moreover, f and g have a common fixed point if (*) holds.

In the same spirit as the above results, W. F. Pfeffer proves in [44, Proposition 1] that an involution σ of a circle has a fixed point if and only if it commutes with a fixed-point free involution different from σ. (Recall that a mapping $\sigma : S^1 \to S^1$ is an *involution* if $\sigma^2 = \sigma$.)

4. Common fixed points of
commuting self-maps of the unit interval

The earliest known paper which contributed to the study of co-incidence values of commutative mappings was published by J. F. Ritt in 1923 [45]. In his article, Ritt proved that if f and g are commutative polynomials, then three cases are possible:

(i) $f(x) = \pm x^n$ and $g(x) = \pm x^m$ for positive integers m, n;
(ii) f and g are both iterates of a third polynomial h (up to a linear transformation); or
(iii) f and g are both Tchebycheff polynomials (that is, both are of the form $\cos(n \arccos x)$ for some positive integer n).

While Ritt's paper did not specifically address the topic of coincidence values, it can be shown that common fixed points exist for each of the three cases identified above when the polynomials under investigation are considered on the unit interval. The first articles in which coincidence values of functions are deliberately studied in connection with commutativity appeared in 1946 [49], [50]; specifically, A. G. Walker provides a classification of the collection of pairs of certain commutative monotone increasing self-maps of an interval by considering coincidence values of iterates of such functions.

The paper credited for inspiring the flurry of research in the 1960s and beyond concerning coincidence values of commuting mappings was published by H. D. Block and H. P. Thielman in 1951 [5]. In their article, Block and Thielman define an *entire set of commutative polynomials* to be a set of polynomials which contains at least one polynomial of each positive degree, and such that the polynomials in the set are pairwise commutative. They show that $\mathcal{F}$ is an entire set of commutative polynomials if and only if either $\mathcal{F} = \{\lambda^{-1} p_n \lambda : n = 1, 2, \cdots\}$ (where $p_n(x) = x^n$) or $\mathcal{F} = \{\lambda^{-1} \cos(n \arccos)\lambda : n = 1, 2, \cdots\}$ where, in both cases, λ is any linear function. Moreover, they remark that for $\lambda(x) = ax + b$, the polynomials in the second case (that is, the linear transforms of the Tchebycheff polynomials) all have fixed point $(1 - b)/a$.

It has been reported by multiple sources that in 1954, Eldon Dyer was the first to conjecture that the unit interval I has the common fixed point property, and that this conjecture was independently raised by Allen Shields in 1955 and by Lester Dubins in 1956 (see [1], for example). However, reference to this conjecture first appeared

in the literature in 1957 when J. R. Isbell [25] asked the following more general question:

> Let T be a tree, i.e. a compact locally connected space in which every two points are joined by a unique arc. It can be seen that every commutative group Γ of homeomorphisms of T has a common fixed point. [By Zorn's lemma, it suffices to show that there is a proper subcontinuum which is mapped onto itself by every element of Γ. Observe that for $f, g \in \Gamma$, g maps the set S of fixed points of f into itself. So does g^{-1}; hence $g(S) = S$. Similarly g leaves invariant the least subcontinuum containing S. But this is all of T only if f leaves every end point of T fixed.] Is this true for commutative semigroups of continuous mappings? It is not known even for a semigroup generated by two mappings on an arc.

The Dyer/Shields/Dubins/Isbell conjecture (hereafter referred to as the common fixed-point conjecture) was independently settled in the negative by William M. Boyce [7] and Huneke [22] in 1967 (both papers appearing in the literature in 1969). Interestingly, the work settling this conjecture served as doctoral dissertations for both Boyce (advisor Gail Young, Jr.) and Huneke (advisor Walter Gottschalk); both men were awarded their degrees in 1967 [40]. In 1970, Huneke [23] provided an alternative proof that the interval fails to have CFP. Moreover, Huneke and Glover published a paper in 1971 [24] proving that no completely regular Hausdorff space containing an arc has the common fixed-point property, thereby completely settling the Isbell query discussed above.

When f and g are commuting self-maps of a tree, Isbell notes above that g maps the set of fixed-points of f into itself; indeed, this is true for commuting self-maps of any set. It is also easy to show that f and g permute the fixed-point set of fg; in fact, the restrictions of f and g to $FP(fg)$ are inverse permutations. Whenever f and g are self-maps of I, even more is true. Glen Baxter [1] and Baxter and J. T. Joichi [2] show that when $FP(fg)$ is partitioned into three classes according to whether the graph of fg crosses upward through the diagonal, crosses downward, or changes directions at the values contained in $FP(fg)$, f and g also permute each one

of these three equivalence classes. In [2], they show that these permutations satisfy certain conditions which qualify them as being *s-admissible* in their terminology. When f and g are restricted to only the crossing points of fg, the resulting permutations are said to be *w-admissible*. Permutations that are *w*-admissible were renamed *Baxter permutations* by Boyce [6] who subsequently used them as his main tool toward settling the common fixed point conjecture; a very readable account of Boyce's approach to the problem appears in [6]. Baxter permutations have since become an object of study in their own right (see, for example, [9], [14], and [39]).

The examples of Huneke and Boyce prompt the following natural question.

Question 2. Let f and g be commuting self-maps of I. What additional conditions guarantee that f and g have a common fixed point?

We note that Proposition 1 provides the following partial answer to Question 2.

Proposition 4 (Jungck [29], 1976). *If f and g are commuting self-maps of I and f is constant, then f and g have a common fixed point.*

Until otherwise noted, we will always assume that f and g are commuting self-maps of I.

Question 2 is tacitly suggested by the common fixed point conjecture, and most of the work relating to the conjecture that appeared prior to 1967 was focused on providing answers to it. Even after the examples of Boyce and Huneke were announced, Question 2 remained a question of interest and continues to attract attention still. Some of the most important answers to Question 2—from either a mathematical or an historical perspective—are stated below. Proofs are provided when they are short and illustrative.

The earliest result addressing Question 2 is due to Ritt who published several papers in the 1920s regarding the algebraic properties of functional composition as a binary operation on the set of rational complex functions. As discussed above, the following proposition can be inferred from his results.

Proposition 5 (Ritt [45], 1923). *If f and g are polynomials, then f and g have a common fixed point.*

The first result to be published regarding Question 2 following the appearance of Isbell's question in 1957 appears to be the following proposition due to Ralph DeMarr. (When DeMarr refers to the common fixed-point conjecture as an "old question" that "remains unanswered except in certain special cases," we assume that he has only Proposition 5 in mind.)

Proposition 6 (DeMarr [16], 1963). *If $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in I$, then f and g have a common fixed point.*

Actually, DeMarr proves a more general result: he shows that for any $\alpha \geq 1$ and any $0 \leq \beta < \frac{\alpha+1}{\alpha-1}$, f and g have a common fixed point provided that $|f(x) - f(y)| \leq \alpha|x - y|$ and $|g(x) - g(y)| \leq \beta$ for all $x, y \in I$.

Jungck improved Proposition 6 with the following result.

Proposition 7 (Jungck [28], 1966). *If there exists $\alpha > 0$ such that*

$$|f(x) - f(y)| \leq |x - y| + \alpha|gf(x) - gf(y)| \qquad x, y \in I,$$

then f and g have a common fixed point.

Jungck also offers the following two propositions.

Proposition 8 (Jungck [28], 1966). *If $|f(x) - f(y)| \leq |g(x) - g(y)| + |x - y|$ for all $x, y \in I$ such that $f(x) = g(x)$ or $f(y) = g(y)$, then f and g have a common fixed point.*

Proposition 9 (Jungck [28], 1966). *If there exists $\alpha > 0$ such that $|x - g(x)| \leq \alpha|gf(x) - f(x)| + |x - f(x)|$ for all $x, y \in I$, then f and g have a common fixed point.*

The following proposition is due to J. E. Maxfield and W. J. Mourant, as well as to S. C. Chu and R. D. Moyer. (Despite the different publication years of these articles, both were received in 1965—in April and July, respectively—by the journals in which they appeared.) In the process of developing their results, both sets of authors prove and use the fact that if $FP(f) = FP(f^2)$, then $FP(f) = FP(f^n)$ for all positive integers n. In fact, Chu and Moyer prove that these conditions are both equivalent to f having a fixed-point in every nonempty closed subset of I that is mapped

into itself by f. Recalling from the statement of the Isbell query that f maps the fixed-point set of g into itself, the proposition below follows immediately.

Proposition 10 (Maxfield and Mourant [41], 1965 / Chu and Moyer [13], 1966). *If $FP(f) = FP(f^2)$, then f and g have a common fixed point.*

Haskell Cohen [15] defined a self-map of I to be *full* if the interval may be subdivided into a finite number of subintervals on each of which the function is a homeomorphism onto I. This condition is equivalent to a self-map of I being open. Cohen proved the following important result.

Proposition 11 (Cohen [15], 1964). *If f and g are both open, then f and g have a common fixed point.*

Joichi [27] and Jon H. Folkman [18] independently strengthened Cohen's result.

Proposition 12 (Joichi [27], 1966 / Folkman [18], 1966). *If f is open, then f and g have a common fixed point.*

The following observation is made by both Joichi and Folkman in their proofs of Proposition 12. The proof provided below is from Folkman [18].

Proposition 13 (Joichi [27], 1966 / Folkman [18], 1966). *If f is monotone, then f and g have a common fixed point.*

Proof: Let x be a fixed point of g. Then $f^n(x) \in FP(g)$ for all $n = 1, 2, \cdots$. It follows easily that $\lim_{n\to\infty} f^n(x)$ belongs to $FP(f) \cap FP(g)$. $\qquad\qquad\square$

Joichi makes the following observation in his proof of Proposition 12. While this result is clearly implied by Proposition 12, we include it to highlight an elegant argument in common fixed point theory.

Proposition 14 (Joichi [27], 1966). *If f is open and $g(I) \neq I$, then f and g have a common fixed point.*

Proof: Let $J = g(I)$. Then we have that $f(J) = fg(I) = gf(I)$. But since open maps are surjective, this gives that $f(J) = g(I) = J$. So $f|J : J \to J$ is surjective. Since f is open and $J \neq I$, this implies that $f|J$ is monotone. Clearly, we have that $g(J) \subseteq J$. It

now follows from Proposition 13 that $f|J$ and $g|J$ have a common fixed point p; therefore, $p \in FP(f) \cap FP(g)$. $\square$

Recall that a family of mappings $\{\phi_n : n \in 1, 2, \cdots\}$ between metric spaces is said to be *equicontinuous* when for a given $\epsilon > 0$ the choice of δ (in the traditional definition of continuity of ϕ_n) is independent of n. In 1971, Boyce provided the following answer to Question 2.

Proposition 15 (Boyce [8], 1971). *If the family $\{f^n : n = 1, 2, \cdots\}$ of iterates of f is equicontinuous, then f and g have a common fixed point.*

Let $B = \{\phi : I \to I : \{\phi^n : n = 1, 2, \cdots\}$ is equicontinuous on $I\}$ and let $\mathcal{G}$ be a subset of B whose members are pairwise commuting. Theodore Mitchell [42, Theorem 3] improved Boyce's result by showing that if f is a self-map of I that commutes with each member of $\mathcal{G}$, then f and $\mathcal{G}$ have a common fixed point.

Julio Cano proved the next two results in 1982.

Proposition 16 (Cano [11], 1982). *If $FP(f)$ is a closed interval then f and g have a common fixed point.*

Proposition 17 (Cano [11], 1982). *If $FP(f) = P(f)$, then f and g have a common fixed point.*

For organizational purposes, we summarize most of the results presented thus far in this section as the following theorem.

Theorem 3. *Let f and g be commuting self-maps of the unit interval I. Then f and g have a common fixed-point if any of the following conditions hold:*

- *f is constant (Proposition 4);*
- *f and g are polynomials (Proposition 5);*
- *$|f(x) - f(y)| \le |x - y|$ for all $x, y \in I$ (Proposition 6);*
- *there exists $\alpha > 0$ such that $|f(x) - f(y)| \le |x - y| + \alpha|gf(x) - gf(y)|$ for all $x, y \in I$ (Proposition 7);*
- *$|f(x) - f(y)| \le |g(x) - g(y)| + |x - y|$ for all $x, y \in I$ such that $f(x) = g(x)$ or $f(y) = g(y)$ (Proposition 8);*
- *there exists $\alpha > 0$ such that $|x - g(x)| \le \alpha|gf(x) - f(x)| + |x - f(x)|$ for all $x, y \in I$ (Proposition 9);*
- *f is open (Proposition 12);*

- *f is monotone* (Proposition 13);
- *the family* $\{f^n : n = 1, 2, \cdots\}$ *of iterates of f is equicontinuous* (Proposition 15);
- $FP(f)$ *is a closed interval* (Proposition 16);
- $FP(f) = P(f)$ (Proposition 17).

For the remainder of this section, f and g will continue to denote self-maps of I; however, we will no longer assume that f and g commute on all of I. By restricting the set of values on which f and g are assumed to commute, the results of Boyce (Proposition 15) and of Cano (propositions 16 and 17) can be extended to several stunningly elegant characterizations.

Recall from section 2 that f and g are said to be nontrivially compatible provided that $A = \{x : f(x) = g(x)\}$ is nonempty and f and g commute on A. For a given self-map $f : I \to I$, let K_f denote the collection of all self-maps of I that are nontrivially compatible with f. Jungck proved the following extension of Proposition 17.

Proposition 18 (Jungck [34], 1992). *f has a common fixed point with every* $g \in K_f$ *if and only if* $FP(f) = P(f)$.

Using the Mean Value Theorem, Jungck observes that if f is differentiable on $(0, 1)$ with $f'(x) \neq -1$ for all $x \in (0, 1)$, then $P(f) = FP(f)$; thus, as an immediate corollary to Proposition 18, we have that such a map $f : I \to I$ shares a fixed point with all self-maps of I that are nontrivially compatible with f. Jungck also gives an example to show that families of nontrivially compatible self-maps of I, each of whose periodic and fixed points coincide, need not have a common fixed point.

The remaining characterizations that we include are due to Jacek R. Jachymski; together, these propositions rank as being amongst the most beautiful set of results in the study of common fixed points of the unit interval. Jachymski credits Proposition 18 as being his inspiration for their discovery.

Proposition 19 (Jachymski [26], 1996). *Let f be a self-map of I. Then the following conditions are equivalent.*

(1) $FP(f) = P(f)$;
(2) *the sequence* $\{f^n\}_{n=1}^{\infty}$ *is pointwise convergent on* I;

(3) *f has a common fixed point with every self-map $g : I \to I$ that commutes with f on $FP(g)$.*

Proposition 20 (Jachymski [26], 1996). *Let f be a self-map of I. Then the following conditions are equivalent.*

(1) *$FP(f)$ is a closed interval;*
(2) *the family $\{f^n : n = 1, 2, \cdots\}$ is equicontinuous on $FP(f)$, or $FP(f)$ is a singleton;*
(3) *f has a common fixed point with every self-map $g : I \to I$ that commutes with f on $FP(f)$.*

Proposition 21 (Jachymski [26], 1996). *Let f be a continuous self-map of I such that $FP(f)$ is not a singleton. Then the following conditions are equivalent.*

(1) *$\{f^n : n = 1, 2, \cdots\}$ is equicontinuous;*
(2) *the sequence $\{f^n\}_{n=1}^{\infty}$ is uniformly convergent on I;*
(3) *f has a common fixed point with every self-map $g : I \to I$ that commutes with f either on $FP(f)$, or on $FP(g)$.*

It is clear that propositions 19, 20, and 21 are extensions of propositions 17, 16, and 15, respectively. As Jachymski points out, Proposition 19 implies the sufficiency part of Proposition 18: Assume that $FP(f) = P(f)$ and let g commute with f on the set of their coincidence points. Choose $a \in I$ with $f(a) = g(a)$. By Proposition 19, $\{f^n(a)\}_{n=1}^{\infty}$ converges to some b. Nontrivial compatibility easily implies that $f^n(a) = g^n(a)$; thus, $\{f^n(a)\}_{n=1}^{\infty}$ converges to b as well. Continuity gives that $f(b) = b = g(b)$.

5. Coincidence values of commuting self-maps of compact metric spaces

As mentioned in the introduction, the author's interest in the study of coincidence values of commuting mappings was inspired by the question of whether the simple triod has the coincidence value property. As we will show in Proposition 23, a negative answer to this question would allow for the construction of a (simple triod)-like continuum that admits a fixed-point free map. The proof that we present for Proposition 23 will use the following proposition that expresses an interesting relationship between the coincidence value property (CVP) and the fixed point property (FPP) in the setting of continua. To our knowledge, the result is unattributed.

Proposition 22. *If every inverse limit of a continuum X with a single bonding map has FPP, then X has CVP. In particular, if f and g are commuting self-maps of X with no coincidence values and P is the inverse limit of X with bonding map f, then P fails to have FPP.*

Proof: Suppose that X does not have CVP, and let f and g be commuting self-maps of X with no coincidence values. Let P be the limit of the inverse sequence for which each factor space is X and each bonding map is f. Observe that since $fg = gf$, g induces the map $G : P \to P$ given by $G(x_0, x_1, \cdots) = (g(x_0), g(x_1), \cdots)$. Let $F : P \to P$ denote the shift homeomorphism $(x_0, x_1, x_2, \cdots) \mapsto (f(x_0), x_0, x_1, \cdots)$. Then it is easily seen that $F^{-1}G : P \to P$ is fixed-point free.   □

We remark that since every arc-like continuum has the fixed point property, Proposition 22 can be used to show that the unit interval has CVP; a more elementary proof of this result can be found in [28, Lemma 1].

As mentioned in the introduction, the existence of a pair of commuting self-maps of the simple triod would allow for the construction of a (simple triod)-like continuum without the fixed point property; this assertion is justified by the proof of the following proposition.

Proposition 23. *If the simple triod fails to have CVP, then there exists a (simple triod)-like continuum that admits a fixed point free mapping.*

Proof: Let T denote the simple triod and assume that there exist self-maps $f, g : T \to T$ with no coincidence values. If P is the inverse sequence of T with each bonding map f, then, by Proposition 22, we have that P does not belong to FPP; thus, P is not arc-like. It follows that some of the images of the projections of P into T cannot be contained in an arc; thus, these images must be simple triods. Therefore, P is (simple triod)-like.   □

Many of the results regarding commuting self-maps of compacta depend upon the following lemma. While several authors justify the statement below with a casual appeal to Zorn's lemma, we choose to present a more illustrative argument; the proof we provide contains

the most elegant portions of the proofs of Proposition 4.1 in [33] and Proposition 2 in [36].

Lemma 4. *If f and g are commuting self-maps of a compact metric space X, then there exists a compact subspace $K \subseteq X$ such that $f(K) = g(K) = K$. Moreover, if X is a continuum, then K can be chosen to be a continuum.*

Proof: Let ϕ be any self-map of X and let $K = \bigcap_{n=1}^{\infty} \phi^n(X)$. Since $\{\phi^n(X)\}$ is a nested sequence of nonempty compacta, K is nonempty and compact; moreover, if X is a continuum, then so is K.

We now show that $K = \phi(K)$. Clearly, we have that $K \subseteq \phi^n(X)$ for each n; thus, $\phi(K) \subseteq \phi^{n+1}(X)$ for each n. It follows that $\phi(K) \subseteq \bigcap_{n=1}^{\infty} \phi^{n+1}(X) = K$. Conversely, if $y \in K \subseteq \phi^{n+1}(X)$, then $y = \phi(x_n)$ for some $x_n \in \phi^n(X)$. By compactness, some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges to a point $p \in K$; thus, $\{\phi(x_{n_i})\}$ converges to $\phi(p)$. Since $\phi(x_{n_i}) = y$ for each i, it follows that $\phi(p) = y$. Thus, $y \in \phi(K)$ and so $K \subseteq \phi(K)$.

Observe that if h is any self-map of X that commutes with ϕ, then

$$
h(K) = h\left[\bigcap_{n=1}^{\infty} \phi^n(X)\right] \subseteq \bigcap_{n=1}^{\infty} h\phi^n(X)
$$
$$
= \bigcap_{n=1}^{\infty} \phi^n h(X) \subseteq \bigcap_{n=1}^{\infty} \phi^n(X) = K.
$$

In particular, if $\phi = fg$, then $f(K) \subseteq K$ and $g(K) \subseteq K$ since f and g both commute with ϕ. It follows that $f(K) \subseteq K = \phi(K) = f[g(K)] \subseteq f(K)$, and so $K = f(K)$. Similarly, $K = g(K)$. This proves the lemma. $\square$

Let X be a compact metric space and let f and g be commuting self-maps of X. We have already mentioned that f and g necessarily have a coincidence value in the case that X is the unit interval. While the literature is rich with results regarding common fixed points of the unit interval, surprisingly little work been published regarding the following question.

Question 5. What additional conditions on X, f, and g will guarantee that f and g have a coincidence value?

In 1938, O. H. Hamilton [21, Theorem II] showed that every hereditarily unicoherent, hereditarily decomposable metric continuum has the fixed-point property for homeomorphisms. William J. Gray significantly extended Hamilton's result with the following partial answer to Question 5.

Proposition 24 (Gray [19], 1969)**.** *If X is hereditarily unicoherent and hereditarily decomposable, then any abelian semigroup of monotone self-maps of X has a common fixed point.*

Gray and Carol M. Smith published the following result; (we note that a *dendroid* is an hereditarily unicoherent, arcwise connected continuum).

Proposition 25 (Gray and Smith [20], 1975)**.** *If X is a dendroid and $\mathcal{G}$ is an abelian semigroup of monotone self-maps of X, then every self-map of X that commutes with each element of $\mathcal{G}$ has a common fixed point with $\mathcal{G}$.*

With reference to Cohen's result (see the discussion preceding Proposition 12), Gray and Smith asked the following question (recall that a *dendrite* is a locally connected, hereditarily unicoherent continuum).

Question 6 (Gray and Smith [20], 1975)**.** If X is a dendrite and $\mathcal{G}$ is an abelian semigroup of continuous open self-maps onto X, must $\mathcal{G}$ have a fixed point?

At this point we turn our attention again to Question 1 and observe that in the case that X is a tree (in particular, a triod), either one of the above two propositions can be used to argue that every pair of monotone commuting self-maps of X has a common fixed point. As we will see, several additional partial answers to Question 1 follow quickly from results in the literature. We first recall several definitions: If Z_1 and Z_2 are compact metric spaces, then $f : Z_1 \to Z_2$ is said to be *confluent* provided that for any subcontinuum B of Z_2 and any component A of $f^{-1}(B)$, we have that $f(A) = B$. For continua X and Y, a surjective mapping $f : X \to Y$ is said to be *weakly monotone* provided that for any subcontinuum B of Y having nonempty interior in Y and any component A of

$f^{-1}(B)$, we have that $f(A) = B$. If f is weakly monotone and $f^{-1}(B)$ has at most finitely many components for every subcontinuum B of Y having nonempty interior, then f is said to be *quasi-monotone*.

For any two topological spaces C and D, a mapping $f : C \to D$ is called *universal* provided that it has a coincidence value with every mapping $g : C \to D$. In 1967, Helga Schirmer [46, Theorem 1] proved that weakly monotone mappings from continua onto trees are always universal. The following proposition follows immediately.

Proposition 26. *If f and g are surjective self-maps of a tree and f is weakly monotone, then f and g have a coincidence value.*

Surjective mappings which are also open, monotone, quasi-monotone or confluent are also weakly monotone [43, 13.18]. Therefore, Proposition 26 immediately implies the following partial answers to Question 1.

Corollary 7. *If f and g are surjective self-maps of a tree, then f and g have a coincidence value if f is also*

   (i) *monotone;*
  (ii) *weakly monotone;*
 (iii) *quasi-monotone;*
 (iv) *confluent;*
  (v) *open.*

When considering Question 1, the condition that f and g be surjective in the preceding result is less restrictive than it might appear. In the first place, surjectivity is a condition of the definitions of open, weakly monotone, and quasi-monotone mappings. Moreover, the answer to Question 1 is affirmative if and only if an affirmative answer exists under the additional assumption that f and g are surjective. Consider the following two statements regarding self-maps of a compact metric space X.

S_1 : Every pair of commuting self-maps of X has a coincidence value.

S_2 : Every pair of commuting surjective self-maps of X has a coincidence value.

Assume that S_2 is true and let f and g be any two commuting self-maps of X. Using Lemma 4, we can find a compact subset $K \subseteq X$ on which the restrictions of f and g to K are both surjective. By S_2, these restrictions have a coincidence value; thus, f and g have a coincidence value. Therefore, S_2 implies S_1. Since S_1 clearly implies S_2, we have that these statements are equivalent.

We conclude this section with three partial answers to Question 5 for which the spaces under consideration are not necessarily compact metric. The first of these provides a sufficient condition for commuting self-maps of a metric space to have a common fixed point.

Proposition 27 (Cano [10], 1968). *Let X be any metric space and let $f, g : X \to X$ be commuting mappings. If* (i) *$FP(g)$ is compact and nonempty, and* (ii) *$d(f^2(x), f(x)) < d(f(x), x)$ for all $x \notin FP(f)$, then f and g have a common fixed point.*

A *nested space* is an arcwise connected Hausdorff space in which the union of any nest of arcs is contained in an arc; dendrites and dendroids are examples of nested spaces. R. E. Smithson offers the following partial answer to Question 5. Recall that a map is *strongly monotone* if point inverses are arcwise connected.

Proposition 28 (Smithson [48], 1979). *Any two strongly monotone self-maps of a nested space have a common fixed point.*

The final proposition that we present offers a remarkable and insightful answer to Question 5. It follows immediately from [35, Corollary 3.7] and [35, Theorem 3.8] and was motivated by the result that we highlighted as Proposition 18 in section 4; the author is grateful to the referee for making us aware of the following result and for enhancing the present article by recommending its inclusion.

For a continuous self-map g of a topological space X, let K_g denote the set of continuous maps $f : X \to X$ such that $M = \{x \in X : f(x) = g(x)\} \neq \emptyset$ and $fg = gf$ on M. (This is the condition that was used to define nontrivial compatibility in the setting of compact metric spaces in section 2.) Also, recall that $x \in X$ is said to be a *recurrent* point of g if x is an accumulation point of $\{g^n(x) : n = 1, 2, \cdots\}$.

Proposition 29 (Jungck [35], 2005)**.** *Let g be a continuous self-map of a compact Hausdorff space X. If g has no recurrent points and $FP(g) = P(g)$, then g has a common fixed point with each $f \in K_g$. (Specifically, g has a fixed point.)*

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