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by

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# ON NON-NORMALITY POINTS IN ČECH-STONE REMAINDERS OF METRIZABLE CROWDED SPACES

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Abstract. Let $X$ be a realcompact locally compact metrizable crowded space and $p \in X^*$. If $p$ is not a $P$-point in $X^*$, then $X^* \setminus \{p\}$ is not normal.

What open subsets of compact spaces are normal?

In the theory of Čech-Stone compactification $\beta X$, this common question has usually the following form:

Is $\beta X \setminus \{p\}$ (or $X^* \setminus \{p\}$) not normal for any point $p$ of Čech-Stone remainder $X^* = \beta X - X$?

If so, then $p$ is called a *non-normality point* of $\beta X$ $(X^*)$.

As is well known, $\omega_1 = \beta\omega_1 \setminus \{\omega_1\}$ is normal.

What about realcompact spaces? Despite strained efforts, for $X = \omega$ these questions have been solved only by assuming either additional axioms, like $\mathsf{CH}$ [9], [10], or some special properties of $p$. If either $p$ is an accumulation point of some countable discrete subset of $\omega^*$ [1], or there is a discrete set $D$ in $\omega^*$ such that $|D| = \omega_1$ and $|D \setminus O| \leq \omega$ for any neighborhood $O$ of $p$ (Eric K. van Douwen, unpublished), then $\omega^* \setminus \{p\}$ is not normal in $\mathsf{ZFC}$. Non-normality points of another type are *strong* $\mathbb{R}$-*points* [3], having a rather technical definition and the following property: There is

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open $\sigma$-compact $U \subset \omega^* \setminus \{p\}$ such that $p \in [U]$, but $p \notin [V]$ for any $V \subset U$ with $|V| < C$.

Is $\beta X \setminus \{p\}$ non-normal whenever $X$ is realcompact and crowded and $p \in X^*$? Probably, but we unaware of any counterexample. First, an affirmative answer was obtained by the author for the real line $\mathbb{R}$ [4]. Locally compact Lindelöf separable crowded spaces with $\pi w(X) \leq \omega_1$, assuming that $p$ is remote, and some other spaces as well, were considered in [5], [6], [7], and [14]. In particular, the following result was obtained independently in [8] and [15].

**Theorem 1.** *Let $X$ be a metrizable crowded space. Then any point $p \in X^*$ is a butterfly-point in $\beta X$. Hence, $\beta X - \{p\}$ is not normal.*

Jun Terasawa [14], [15] asked whether $X^* \setminus \{p\}$ is non-normal for metrizable crowded locally compact spaces. We give a partial answer to this question, which remains open for $P$-points.

**Theorem 2.** *Let $X$ be a metrizable crowded space and $p \in X^*$. Let the following hold for some zero-set $Z$ in $\beta X$ and a set $K \subset X^* \setminus Z$: $p \in [K]_{\beta X} \setminus K \subset Z \subset X^*$. Then $X^* \setminus \{p\}$ is not normal.*

**Corollary 3.** *Let $X$ be a realcompact locally compact metrizable crowded space and $p \in X^*$. If $p$ is not a $P$-point in $X^*$, then $X^* \setminus \{p\}$ is not normal.*

**Corollary 4.** *Let $\mathbb{R}$ be the real line and $p \in \mathbb{R}^*$. If $p$ is not a $P$-point in $\mathbb{R}^*$, then $\mathbb{R}^* \setminus \{p\}$ is not normal.*

Recall that a point $p$ is called a $P$-point in $X$ if any intersection of countably many neighborhoods of $p$ in $X$ contains a neighborhood of $p$. As is well known, $P$-points do exist in $\omega^*$ under some additional axioms, like $\mathsf{CH}$ [11], but not in $\mathsf{ZFC}$ [13]. Moreover, there are $P$-points in $\omega^*$ if and only if there are $P$-points in $X^*$ for a variety of spaces, e.g., $X$ is any locally compact nonpseudocompact space [2]. A point $p$ is called a $b$-point or a butterfly-point in $X$ if $\{p\} = [A] \cap [B]$ for some $A, B \subset X \setminus \{p\}$ [12]. We say that $p \in X^*$ is a $b$-point in $\beta X$ if $\{p\} = [A]_{\beta X} \cap [B]_{\beta X}$ for some $A, B \subset X^* \setminus \{p\}$ with $[A \cup B]_{\beta X} \subset X^*$.

## 0.1. Proofs.

We follow [8] in notations, terminology, and some facts, including the proofs for completeness (lemmas 8 and 9). From now on a space $X$ is non-compact, metrizable, and crowded; i.e., $X$ has no isolated points. By $Ox \subset X$, we denote any open neighborhood of $x$ in $X$ by $[\ ]$ and $[\ ]_{\beta X}$ – the closure operators in $X$ and $\beta X$, respectively, $3 = \{0, 1, 2\}$. If $U \subset X$ is open, then $U^{\varepsilon} = \beta X \setminus [X \setminus U]_{\beta X}$ is the maximal open set in $\beta X$, whose trace on $X$ is $U$. A space $X$ is realcompact if, for any point $p \in X^*$, $p \in Z \subset X^*$ for some zero-set $Z$ in $\beta X$.

Let $\pi$ and $\sigma$ be arbitrary families. A set $U \in \pi$ is called a *maximal member* of $\pi$ if $U$ is not a proper subset of any other member of $\pi$. If members of $\pi$ are mutually disjoint (with closure), then $\pi$ is called *(strongly) cellular*. We write $\pi \succeq \sigma$ ($\pi \succ \sigma$) if $U \in \pi$ is a (proper) subset of $V \in \sigma$ whenever $U \cap V \neq \emptyset$. By $\exp \pi$, we denote all subfamilies $\{F : F \subset \pi\}$ of $\pi$, and $f^{\pi}_{\sigma}$ – the map from $\exp \pi$ to $\exp \sigma$ – is defined as $f^{\pi}_{\sigma} F = \{V \in \sigma : \bigcup F \cap V \neq \emptyset\}$ for every $F \in \exp \pi$.

A maximal cellular locally finite family of open sets is called *nice*. As introduced in [7], the *cellular refinement*

$$Cel\ (\pi) = \{\bigcap \phi - [\bigcup(\pi - \phi)]_{\beta X} : \phi \subset \pi\}$$

of $\pi$ is nice, if $\pi$ is an open locally finite cover of $X$ (Lemma 6).

Let $\pi$ and $\sigma$ be nice families, and $\mathcal{F} \subset \exp \pi$. For a point $p \in X^*$, we say that $\mathcal{F}$ is a $p$-filter on $\pi$ if $p \in [\bigcup(F_0 \cap ... \cap F_n)]_{\beta X}$ for any finite subcollection $\{F_0, ..., F_n\} \subset \mathcal{F}$. If so, then we denote $\bigcap \mathcal{F}^* = \bigcap\{[\bigcup F]_{\beta X} : F \in \mathcal{F}\}$. We write $\pi \prec_{\mathcal{F}} \sigma$ if $F \prec \sigma$ for some $F \in \mathcal{F}$. The image $f^{\pi}_{\sigma}\mathcal{F} = \{f^{\pi}_{\sigma} F : F \in \mathcal{F}\}$ of $\mathcal{F}$ is a $p$-filter on $\sigma$. If $\mathcal{F}$ is the union of any increasing sequence of $p$-filters, then $\mathcal{F}$ is a $p$-filter as well. So, by the Kuratowski-Zorn lemma, every $p$-filter $\mathcal{F}$ is contained in some $p$-ultrafilter $\hat{\mathcal{F}}$ on $\pi$, that is, $\hat{\mathcal{F}} = \mathcal{G}$ whenever $\mathcal{G}$ is a $p$-filter and $\hat{\mathcal{F}} \subseteq \mathcal{G}$. If $p$ is not a remote point, distinct $p$-ultrafilters may exist. But each of them contains $\pi(Op) = \{U \in \pi : U \cap Op \neq \emptyset\}$ for any neighborhood $Op \subset \beta X$.

We construct a sequence $\{\mathcal{P}_k\}_{k \in N}$ of open locally finite covers $\mathcal{P}_k$ of $X$ so that diam $U \leq \frac{1}{k}$ for any $U \in \mathcal{P}_k$ and any two different members of the family $\mathcal{P} = \bigcup_{k \in N} \mathcal{P}_k$ are different sets. Then it is well known and easy to see that the family of maximal members of any cover $\pi \subset \mathcal{P}$ of $X$ is a locally finite subcover of $X$. Moreover, $\mathcal{P}$ is a regular base as defined by A. V. Arhangel'skii, i.e., for any point $x \in X$ and, for any of its neighborhood $O \subset X$, there is another neighborhood $\hat{O} \subset X$ of $x$ with the following properties: $\hat{O} \subset O$ and at most finitely many members of $\mathcal{P}$ meet both $\hat{O}$ and $X \setminus O$ simultaneously.

By induction, we define the families of non-empty open sets $\mathcal{D}_k$ and $\mathcal{W}_k \subset \mathcal{P}$ for all $k \in N$ as

$$\mathcal{D}_1 = \ Cel\ (\mathcal{P}_1).$$

If a nice family $\mathcal{D}_k = \{U\}$ has been constructed, then

$$\mathcal{W}_k = \{U(\nu) : U \in \mathcal{D}_k \text{ and } \nu \in 3\}$$

is strongly cellular with $[U(\nu)] \subset U$ for any of its members and

$$\mathcal{D}_{k+1} = \ Cel\ (\mathcal{D}_k \cup \mathcal{W}_k \cup \mathcal{P}_{k+1}).$$

Then $\mathcal{D}_k \prec \mathcal{D}_{k+1}$ for all $k \in N$. For any $U \in \mathcal{P}$, there is a unique $k_0 \in N$ with $U \in \mathcal{P}_{k_0}$. We put

$$\hat{U} = \{V \in \mathcal{D}_{k_0} : V \subset U\} = \{V \in \mathcal{D}_{k_0} : V \cap U \neq \emptyset\}.$$

Then $\hat{U}$ is locally finite cellular and everywhere dense in $U$.

For any locally finite cover $\pi \subset \mathcal{P}$ of $X$, we denote by $\sigma(\pi)$ all maximal members of the family $\bigcup\{\hat{U} : U \in \pi\}$. Then $\sigma(\pi)$ is nice. Define

$$\Sigma = \{\sigma(\pi) : \pi \subset \mathcal{P} \text{ is a locally finite cover of } X\}$$

and put $\sigma(\nu) = \{U(\nu) : U \in \sigma\}$ for any $\sigma \in \Sigma$ and $\nu \in 3$. Notice that $\{[\bigcup \sigma(\nu)]_{\beta X} : \nu \in 3\}$ is cellular.

With the notations of Theorem 2, for each $n \in \omega$, we choose open $O_n \subset \beta X$ so that $[O_{n+1}]_{\beta X} \subset O_n$ and $Z = \bigcap_{n\in\omega} O_n$. Denote $\hat{O}_n = [O_n]_{\beta X} \setminus O_{n+1}$. We say that a family $\mathcal{R} = \{U\}$ is $Z$-attracted if $\{U \in \mathcal{R} : U \not\subset O_n\}$ is finite for all $n \in \omega$. Obviously, any finite union of $Z$-attracted families is $Z$-attracted.

Define $\pi_0$ to be all maximal members of the cover

$$\{U \in \mathcal{P} : [U]_{\beta X} \cap [K]_{\beta X} = \emptyset \text{ and } \forall n \in \omega (U \cap \hat{O}_{n+1} \neq \emptyset \Rightarrow U \subset O_n \setminus [O_{n+3}]_{\beta X})\}.$$

Let $T = X^* \setminus Z \setminus \bigcup\{U^\varepsilon : U \in \pi_0\}$ and $Y = X \cup T$.

**Proposition 5.** *Let $X$ be a normal space and let $Z$ be a non-empty closed $G_\delta$-subset of $\beta X$ contained in $X^*$. If $T$ is a closed subset of $\beta X \setminus Z$ contained in $X^*$, then the subspace $X \cup T$ is normal and $[T]_{\beta X} = \beta T$. In particular, if $p \in [T]_{\beta X} \setminus T$ and $p$ is a b-point of $[T]_{\beta X}$, then $p$ is a non-normality point of $X^*$.*

*Proof:* Let $A$ and $B$ be any closed disjoint subsets of $X \cup T$. Since $X$ is normal, $A \cap X$ and $B \cap X$ have disjoint open neighborhoods in $\beta X$. Since $T$ is $\sigma$-compact, $\hat{A} = A \cap T$ and $B \cap T$ have disjoint open neighborhoods in $\beta X$ .

Now it is enough to separate $\hat{A}$ and $B \cap X$. For every point $x \in \hat{A}$, we choose a neighborhood $Ox \subset \beta X$ so that $x \in \hat{A} \cap \hat{O}_{n+1}$ implies $[Ox]_{\beta X} \subset O_n \setminus [B]_{\beta X}$ for each $n \in \omega$. Since every $\hat{A} \cap \hat{O}_{n+1}$ is compact, there is a $Z$-attracted family $\mathcal{R} \subset \{Ox : x \in \hat{A}\}$ with $\hat{A} \subset \bigcup \mathcal{R}$. But then $\hat{\mathcal{R}} = \{O \cap X : O \in \mathcal{R}\}$ is locally finite in $X$. So $\hat{A} \subset (\bigcup \hat{\mathcal{R}})^\varepsilon$ and $B \cap X \cap [\bigcup \hat{\mathcal{R}}]_{\beta X} = \emptyset$.

Let $f : T \to [0,1]$ be any continuous map. Since $T$ is closed in normal $Y = X \cup T$, $f \subset g$ for some continuous map $g : Y \to [0,1]$. Since $X \subset Y \subset \beta X$, $g \subset h$ for some continuous map $h : \beta X \to [0,1]$. Then its restriction $\hat{f} = h/_{[T]_{\beta X}}$ is a continuous extension of $f$ over $[T]_{\beta X}$. So $[T]_{\beta X}$ is a Čech-Stone compactification of $T$ as well as $\hat{T} = [T]_{\beta X} \setminus \{p\}$. If $\{p\} = [A]_{\beta X} \cap [B]_{\beta X}$ for some closed disjoint $A, B \subset \hat{T}$, then $\hat{T}$ is non-normal. Since $\hat{T}$ is closed in $X^* \setminus \{p\}$, our proof is complete. □

Our task from now on is to construct $A$ and $B$.

**Lemma 6.** *If $\pi$ is an open locally finite cover of $X$, then $Cel(\pi)$ is nice.*

*Proof:* If $\varphi \subset \pi$ has non-empty intersection, then $\varphi$ is finite. So $\cap\varphi \setminus [\bigcup(\pi \setminus \varphi)]$ is open.

If $U \in \varphi \setminus \hat{\varphi}$, then $\bigcap \varphi \subset U$ and $\bigcap \hat{\varphi} \cap U = \emptyset$.

If $\psi = \{U \in \pi : U \cap Ox \neq \emptyset\}$ is finite for some $Ox \subset X$, then $\{\varphi \subset \pi : \bigcap \varphi \cap Ox \neq \emptyset\} \subset \exp \psi$ is finite as well.

Let $x \notin [U] \setminus U$ for any $U \in \pi$ and $\varphi = \{U \in \pi : x \in U\}$. Then $x \in \bigcap \varphi \setminus [\bigcup(\pi \setminus \varphi)]$ and $Cel(\pi)$ is maximal. □

**Lemma 7.** *There is a sequence $\{\sigma_\alpha : \alpha < \lambda\} \subset \Sigma$ and $p$-ultrafilters $\mathcal{F}_\alpha$ on $\sigma_\alpha$ with the following properties for all $\alpha < \beta < \lambda$ and $f^\alpha_\beta = f^{\sigma_\alpha}_{\sigma_\beta}$ :*

(1) $\bigcap \mathcal{F}_0 \subset Z$;
(2) $\sigma_\alpha \prec_{\mathcal{F}_\alpha} \sigma_\beta$;
(3) $f^\alpha_\beta \mathcal{F}_\alpha \subset \mathcal{F}_\beta$;
(4) *for any* $\sigma \in \Sigma \setminus \{\sigma_\alpha : \alpha < \lambda\}$, *there is* $\alpha < \lambda$ *with* $\neg(\sigma_\alpha \prec_{\mathcal{F}_\alpha} \sigma)$.

*Proof:* Let $\mathcal{F}_0$ be any $p$-ultrafilter on $\sigma_0 = \sigma(\pi_0)$. Then, for each $n \in \omega$, $\sigma_0(O_n) \in \mathcal{F}_0$ and $\bigcap \mathcal{F}_0^* \subset [\bigcup \sigma_0(O_{n+1})]_{\beta X} \subset [O_n]_{\beta X}$.

Assume that a sequence $\{\sigma_\alpha : \alpha < \lambda\} \subset \Sigma$ and $p$-ultrafilters $\mathcal{F}_\alpha$ on $\sigma_\alpha$ have been constructed for some ordinal $\lambda$ so that conditions (1)–(3) hold. If some $\sigma \in \Sigma \setminus \{\sigma_\alpha : \alpha < \lambda\}$ satisfies the condition $\sigma_\alpha \prec_{\mathcal{F}_\alpha} \sigma$ for all $\alpha < \lambda$, then we put $\sigma_\lambda = \sigma$ and embed the $p$-filter $\bigcup_{\alpha<\lambda} f^\alpha_\lambda \mathcal{F}_\alpha$ into some $p$-ultrafilter $\mathcal{F}_\lambda$ on $\sigma_\lambda$. Otherwise, our construction is complete. □

**Lemma 8.** *If* $\alpha < \beta < \lambda$, *then* $\bigcap \mathcal{F}^*_\beta \subset \bigcap \mathcal{F}^*_\alpha$.

*Proof:* There is $F \in \mathcal{F}_\alpha$ with $F \prec \sigma_\beta$ by (2). For any $G \in \mathcal{F}_\alpha$, we have $G \cap F \in \mathcal{F}_\alpha$ and $G \cap F \prec \sigma_\beta$. But then

$$\bigcap \mathcal{F}^*_\beta \subset [\bigcup f^\alpha_\beta (G \cap F)]_{\beta X} \subset [\bigcup (G \cap F)]_{\beta X} \subset [G]_{\beta X}. \qquad \square$$

**Lemma 9.** *For any neighborhood* $O$ *of* $p$ *in* $\beta X$, *there is* $\alpha < \lambda$ *with* $\bigcap \mathcal{F}^*_\alpha \subset O$.

*Proof:* Let $[\hat{O}]_{\beta X} \subset O$ for a neighborhood $\hat{O}$ of $p$ and let $\pi$ be all maximal members of the cover $\{U \in \mathcal{P} : U \cap \hat{O} \neq \emptyset \Rightarrow U \subset O\}$. For $\sigma = \sigma(\pi)$, there is $\alpha < \lambda$ with $\neg(\sigma_\alpha \prec_{\mathcal{F}_\alpha} \sigma)$ by (2) or (4). Since $\sigma_\alpha(\hat{O}) \in \mathcal{F}_\alpha$, $F = \{V \in \sigma_\alpha(\hat{O}) : V \subseteq U \text{ for some } U \in \sigma\}$ belongs to $\mathcal{F}_\alpha$ as well. So

$$\bigcap \mathcal{F}^*_\alpha \subset [\bigcup F]_{\beta X} \subset [\bigcup \sigma(\hat{O})]_{\beta X} \subset [O]_{\beta X}. \qquad \square$$

**Lemma 10.** *If* $\mathcal{R} \subset \pi_0$ *is* $Z$*-attracted, then* $p \notin [\bigcup \mathcal{R}]_{\beta X}$.

*Proof:* By our construction, $K$ and $[\bigcup \mathcal{R}]_Y$ are closed in normal $Y = X \cup T$ disjoint sets and $p \in [K]_{\beta X}$. □

**Lemma 11.** *For any* $\alpha < \lambda$ *and* $\nu \in 3$,

$$\bigcap_{\beta \in \lambda - \alpha} [\bigcup \sigma_\beta(\nu)]_{\beta X} \cap (\bigcap \mathcal{F}^*_\alpha) \setminus (\bigcup \{(\bigcup \mathcal{R})^\varepsilon : \mathcal{R} \subset \pi_0 \textit{ is } Z\textit{-attracted }\})$$

*is non-empty.*

*Proof:* For any finite sequence $\alpha < \beta_0 < \ldots < \beta_n < \lambda$, $F \in \mathcal{F}_\alpha$ and $Z$-attracted $\mathcal{R} \subset \pi_0$, we shall show by induction that

$$L = \bigcap_{i \leq n} (\bigcup \sigma_{\beta_i}(\nu)) \cap (\bigcup F) \setminus (\bigcup \mathcal{R})$$

is non-empty.

Since $\sigma(\pi_0) \preceq_{\mathcal{F}_0} \sigma_\alpha \prec_{\mathcal{F}_\alpha} \sigma_{\beta_0}$, we may assume $\mathcal{R} \preceq F \prec \sigma_{\beta_0}$. Since $\sigma_{\beta_i} \prec_{\mathcal{F}_{\beta_i}} \sigma_{\beta_{i+1}}$ for each $i < n$, $G_i \prec \sigma_{\beta_{i+1}}$ for some $G_i \in \mathcal{F}_{\beta_i}$. Define $F_0 = f^\alpha_{\beta_0} F \cap G_0$ and $F_{i+1} = f^{\beta_i}_{\beta_{i+1}} F_i \cap G_{i+1}$. Then $F_i \in \mathcal{F}_{\beta_i}$, $F_i \prec F_{i+1}$, and $\bigcup F_i \supseteq \bigcup F_{i+1}$. For $F_n \in \mathcal{F}_{\beta_n}$, we denote

$$\hat{F} = \{U \in F : V \subset U \text{ for some } V \in F_n\}.$$

Then $\bigcup \hat{F} \supseteq \bigcup F_n$ implies $\hat{F} \in \mathcal{F}_\alpha$ and $p \in [\bigcup \hat{F}]_{\beta X}$. Since $\mathcal{R} \preceq \hat{F}$ and $p \notin [\bigcup \mathcal{R}]_{\beta X}$ by Lemma 10, $\bigcup \mathcal{R} \cap U = \emptyset$ for some $U \in \hat{F}$. There are pairwise different $U_{\beta_i} \in F_i$ with

$$U_{\beta_n} \subset \ldots \subset U_{\beta_1} \subset U_{\beta_0} \subset U \subset X \setminus \bigcup \mathcal{R}.$$

Consider the initial segment $\{U_{\beta_1}, \ldots, U_{\beta_n}\}$. For $i = 1, ..., n$, we can replace $U_{\beta_i} \in \sigma_{\beta_i}$ with the same or another member $U^0_{\beta_i} \in \sigma_{\beta_i}$ of the same $\sigma_{\beta_i}$ so that

$$\bigcap_{i=1}^{n} U^0_{\beta_i} \cap U_{\beta_0}(\nu) \neq \emptyset$$

because $\sigma_{\beta_i}$ is nice. By our construction, since $U_{\beta_i}$ is a proper subset of $U_{\beta_0}$, $U^0_{\beta_i} \subset U_{\beta_0}(\nu)$. Possibly, $\{U^0_{\beta_1}, ..., U^0_{\beta_n}\}$ enjoys a new embedded order, directed by subscript $\beta^0$ and having some coinciding sets:

$$U^0_{\beta^0_n} \subset \ldots \subset U^0_{\beta^0_{k+1}} \subset U^0_{\beta^0_k} = \ldots =$$
$$U^0_{\beta^0_1} \subset U_{\beta_0}(\nu) \subset U_{\beta_0} \subset U \subset X \setminus \bigcup \mathcal{R}.$$

Let $U^0_{\beta^0_1}$ be a maximal member of $\{U^0_{\beta^0_1}, ..., U^0_{\beta^0_n}\}$ and let $U^0_{\beta^0_{k+1}}$ be a maximal proper subset of $U^0_{\beta^0_1}$. Consider $\{U^0_{\beta^0_{k+1}}, ..., U^0_{\beta^0_n}\}$. For $i = k+1, ..., n$, we replace $U^0_{\beta^0_i} \in \sigma_{\beta^0_i}$ with the same or another member $U^1_{\beta^0_i} \in \sigma_{\beta^0_i}$ of the same $\sigma_{\beta^0_i}$ so that

$$\bigcap_{i=k+1}^{n} U^1_{\beta^0_i} \cap U^0_{\beta^0_1}(\nu) \neq \emptyset.$$

Possibly, $\{U^1_{\beta^0_{k+1}}, ..., U^1_{\beta^0_n}\}$ enjoys a new embedded order, directed by subscript $\beta^1$:

$$U^1_{\beta^1_n} \subset \ldots \subset U^1_{\beta^1_{k_0+1}} \subset U^1_{\beta^1_{k_0}} = \ldots = \\ U^1_{\beta^1_{k+1}} \subset U^0_{\beta^0_k}(\nu) \subset U^0_{\beta^0_k} = \ldots = U^0_{\beta^0_1} \ldots .$$

Consider $\{U^1_{\beta^1_{k_0+1}}, \ldots, U^1_{\beta^1_n}\}$. For $i = k_0 + 1, ..., n$, we replace $U^1_{\beta^1_i} \in \sigma_{\beta^1_i}$ with the same or another $U^2_{\beta^1_i} \in \sigma_{\beta^1_i}$ so that

$$\bigcap_{i=k_0+1}^{n} U^2_{\beta^1_i} \cap U^1_{\beta^1_{k+1}}(\nu) \neq \emptyset$$

and obtain

$$U^2_{\beta^2_n} \subset \ldots \subset U^2_{\beta^2_{k_1+1}} \subset U^2_{\beta^2_{k_1}} = \ldots = \\ U^2_{\beta^2_{k_0+1}} \subset U^1_{\beta^1_{k_0}}(\nu) \subset U^1_{\beta^1_{k_0}} = \ldots = U^1_{\beta^1_{k+1}} \ldots$$

and so on until, for some $m \leq n$, the initial segment is empty, i.e., $k_m = n$. Then $U^{m+1}_{\sigma_{\beta^{m+1}_{k_m}}}(\nu) \subset L$ is not empty and our proof is complete. □

Define from now on $p_\alpha(\nu)$ to be any point of the set in Lemma 11. Then $p_\alpha(\nu) \in Z$ by lemmas 8 and 9.

**Lemma 12.** *If $q \in Z$ and $q \notin (\bigcup \mathcal{R})^\varepsilon$ for any $Z$-attracted $\mathcal{R} \subset \pi_0$, then $q \in [T]_{\beta X}$.*

*Proof:* Let $[Oq]_{\beta X} \cap T = \emptyset$ for some neighborhood $Oq \subset \beta X$. Then, by our construction,

$$[Oq]_{\beta X} \setminus Z \subset \bigcup \{U^\varepsilon : U \in \pi_0\}.$$

Since every $K_n = [Oq]_{\beta X} \cap \hat{O}_n$ is compact, $K_n \subset (\bigcup \pi_{0n})^\varepsilon$ for some finite $\pi_{0n} \subset \{U \in \pi_0 : U^\varepsilon \cap K_n \neq \emptyset\}$. Then $\mathcal{R} = \bigcup_{n \in \omega} \pi_{0n}$ is $Z$-attracted and $q \in (\bigcup \mathcal{R})^\varepsilon$. □

**Lemma 13.** *The point $p$ is a butterfly-point in $[T]_{\beta X}$.*

*Proof:* Consider $F_\nu = \{p_\alpha(\nu) : \alpha < \lambda\}$ for each $\nu \in 3$. By the previous lemma, $F_\nu \subset [T]_{\beta X}$.

If $\gamma < \lambda$, then $\{p_\alpha(\nu) : \alpha < \gamma\} \subset [\bigcup \sigma_\gamma(\nu)]_{\beta X}$. As the last sets are disjoint for $\nu \in 3$, then $\{F_\nu : \nu \in 3\}$ is cellular. So $p$ belongs to $F_\nu$ for no more then one unique $F_\nu$.

For any neighborhood $Op \subset \beta X$, there is $\gamma < \lambda$ with $\bigcap \mathcal{F}^*_\gamma \subset Op$ by Lemma 9. For all $\alpha \in \lambda \setminus \gamma$, we have $p_\alpha(\nu) \in \bigcap \mathcal{F}^*_\alpha \subset \bigcap \mathcal{F}^*_\gamma \subset Op$ by Lemma 8. So $\{p_\alpha(\nu) : \alpha \in \lambda - \gamma\} \subset Op$. As $\{p_\alpha(\nu) : \alpha < \gamma\} \subset [\bigcup \sigma_\gamma(\nu)]_{\beta X}$, the sets $[F_\nu \setminus Op]_{\beta X}$ are disjoint for $\nu \in 3$. But then $[F_\nu]_{\beta X} \setminus \{p\}$ are mutually disjoint and at least two of them ensure that $p$ is a $b$-point in $\beta X$.

By Proposition 5, our proof is complete. □

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