TOPOLOGY PROCEEDINGS
Volume 35, 2010
Pages 1–7
http://topology.auburn.edu/tp/

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Electronically published on April 8, 2009

Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
TOPOLOGY PROCEEDINGS
Volume 35 (2010)
Pages 1-7
E-Published on April 8, 2009

DECOMPOSITIONS OF TOPOLOGICAL SPACES AND TOTAL RECURRENCE

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Abstract. A topological space X is called totally (bijectively) recurrent if every, not necessarily continuous, mapping (bijection) $f : X \to X$ has a recurrent point. Earlier we proved that a Hausdorff space X is totally (bijectively) recurrent if and only if X is either finite or X is a one-point compactification of an infinite discrete space. In this paper we extend this result to some classes of non-Hausdorff spaces using decomposition of an arbitrary topological space into appropriate "good" subspaces.

Let X be a topological space, $f : X \to X$. A point $x \in X$ is called *recurrent* if x is a limit point of the orbit $(f^n(x))_{n\in\omega}$. A topological space X is called *totally recurrent* [2] (resp. *bijectively recurrent* [3]) if every, not necessarily continuous, mapping (resp. bijection) $f : X \to X$ has a recurrent point. By [3], a Hausdorff space X is totally (bijectively) recurrent if and only if X is either finite or X is a one-point compactification of an infinite discrete space. This result was extracted from the following statement: an infinite Hausdorff space X is either a disjoint union of countable[1] discrete subspaces or X is a one-point compactification of an infinite discrete space. To extend this statement onto non-Hausdorff spaces, we need the following definitions.

2000 *Mathematics Subject Classification.* 54D10, 37B20.
Key words and phrases. Almost discrete space, cofinite space, recurrent point, totally recurrent space.

[1]Throughout this paper "countable" means "countably infinite".

We say that a topological space X is *almost discrete* if, for every point $x \in X$, there exists a finite neighbourhood U of x. Every almost discrete T_1-space is discrete.

We say that an infinite topological space X is *cofinite* if each proper closed subspace of X is finite. Given an infinite set X, there exists only one cofinite T_1-topology τ on X: $U \in \tau$ if and only if either $U = \varnothing$ or $X \setminus U$ is finite.

Theorem 1. *Every topological space X can be partitioned $X = F \cup AD \cup CF$ where F is finite, AD is a disjoint union of countable almost discrete subspaces, CF is a disjoint union of cofinite subspaces.*

Theorem 2. *Let X be a topological space without cofinite subspaces. Then one of the following statements holds*

(i) *X is a disjoint union of countable almost discrete subspaces;*
(ii) *there exists a point $x \in X$ such that $X \setminus U$ is finite for every neighbourhood U of x.*

Theorem 3. *For every countable topological space X, one of the following statements hold*

(i) *X is a disjoint union of countable almost discrete subspaces;*
(ii) *there exists a point $x \in X$ such that, for every neighbourhood U of x, $X \setminus U$ has no infinite almost discrete subspaces.*

For decomposition of another type, we use the following definitions.

A subspace Y of a topological space X is called *Hausdorff* if, for any $x, y \in Y$, there exist the neighbourhoods U, V of x, y in X such that $U \cap V = \varnothing$

A subspace Y of a topological space X is called *linked* (or *anti-Hausdorff*) if, for any $x, y \in Y$ and any neighbourhoods U, V of x, y in X, one has $U \cap V \neq \varnothing$.

Theorem 4. *Every topological space X can be partitioned $X = F \cup H \cup L$ where F is finite, H is a disjoint union of infinite Hausdorff subspaces, L is a disjoint union of infinite linked subspaces.*

We apply the above decomposition theorems to total and bijective recurrence.

Theorem 5. *A topological space X is totally recurrent provided that there exists a point $x \in X$ such that, for every neighbourhood U of x, $X \setminus U$ has no infinite almost discrete subspaces.*

Theorem 6. *Let X be a topological space without cofinite subspaces. Then the following statements are equivalent*

(i) *X is totally recurrent;*
(ii) *X is bijectively recurrent;*
(iii) *there exists a point $x \in X$ such that $X \setminus U$ is finite for every neighbourhood U of x.*

Theorem 7. *For a countable topological space X, the following statements are equivalent*

(i) *X is totally recurrent;*
(ii) *X is bijectively recurrent;*
(iii) *there exists a point $x \in X$ such that, for every neighbourhood U of x, $X \setminus U$ has no infinite almost discrete subspaces.*

After some auxiliary lemmas, we prove all above theorems and conclude the paper with some remarks and open questions.

Lemma 1. *Every infinite topological space X contains either a cofinite subspace or a countable almost discrete subspace.*

Proof. We assume that X has no cofinite subspaces, take a point $x_0 \in X$ and its open neighbourhood U_0 such that $X \setminus U_0$ is infinite. Suppose that we have chosen the points $x_0, \dots, x_n \in X$ and its open neighbourhoods $U_0, \dots, U_n$ such that $X \setminus (U_0 \cup \dots \cup U_n)$ is infinite and $x_{i+1} \notin U_0 \cup \dots \cup U_i$ for each $i \in \{0, \dots, n-1\}$. Since $X \setminus (U_0 \cup \dots \cup U_n)$ is not cofinite, there exists $x_{n+1} \in X \setminus (U_0 \cup \dots \cup U_n)$ and its open neighbourhood U_{n+1} such that $X \setminus (U_0 \cup \dots \cup U_{n+1})$ is infinite. After ω steps we get the sequence $(x_n)_{n \in \omega}$ of elements of X and the sequence $(U_n)_{n \in \omega}$ of its open neighbourhoods such that $x_{n+1} \notin U_0 \cup \dots \cup U_n$ for each $n \in \omega$. Then the subspace $Y = \{x_n : n \in \omega\}$ of X is almost discrete. □

Lemma 2. *Let X be an infinite topological space, Y be an infinite subspace of X. Then Y contains either an infinite Hausdorff subspace of X or an infinite linked subspace of X.*

Proof. We define a coloring $\chi : [Y]_2 \to \{0, 1\}$ of 2-subsets of Y by the rule: $\chi\{x, y\} = 1$ if and only if there exist disjoint neighbourhoods of x and y in X. By the Ramsey theorem [1, p.16], there exists an infinite subset $Z \subseteq Y$ such that $\chi|_{[Z]_2}$ is constant. If $\chi|_{[Z]_2} \equiv 1$ then Z is Hausdorff, otherwise Z is linked. $\square$

Lemma 3. *Let $\{A_n : n \in \omega\}$ be a family of infinite subsets of a set X. Then there exists a disjoint family $\{B_n : n \in \omega\}$ of countable subsets of X such that $B_n \subseteq A_n$ for each $n \in \omega$.*

Proof. It suffices to choose a countable subset $B_0 \subseteq A_0$ such that $A_n \setminus B_0$ is infinite for each $n \in \omega$ (and then repeat the argument for $\{A_{n+1} \setminus B_0 : n \in \omega\}$). We choose inductively an injective sequence

$$a_{00}, a_{10}, a_{11}, \dots, a_{n0}, a_{n1}, \dots, a_{nn}, \dots$$

in X such that $a_{n0} \in A_0, a_{n1} \in A_1, a_{n2} \in A_2, \dots, a_{nn} \in A_n$ for each $n \in \omega$. Put $B_0 = \{a_{n0} : n \in \omega\}$. $\square$

Proof of Theorem 1. We denote by $\mathcal{F}$ the family of ordered pairs (Y, Z) of subsets of X where Y is a disjoint union of countable almost discrete subspaces, Z is a disjoint union of cofinite subspaces and $Y \cap Z = \varnothing$. The family $\mathcal{F}$ is partially ordered by the rule:

$$(Y, Z) \leqslant (Y', Z') \Leftrightarrow Y \subseteq Y', Z \subseteq Z'.$$

By Zorn lemma, $\mathcal{F}$ has a maximal element (AD, CF). Applying Lemma 1, we conclude that $X \setminus (AD \cup CF)$ is finite. $\square$

Proof of Theorem 2. By Theorem 1, $X = F \cup AD$ where F is finite, AD is a disjoint union of countable almost discrete subspaces. We may suppose that $F \cup AD$ is the partition of X with minimal F by cardinality. If $F = \varnothing$ then (i) holds. Assume that $F \neq \varnothing$, $x \in F$ and U is a neighbourhood of x. Let AD be a union of a disjoint family $\{Y_\alpha : \alpha \in I\}$ of countable almost discrete subspaces. If $Y_\alpha \cap U$ is finite for some $\alpha \in I$, then $Y_\alpha \cup \{x\}$ is almost discrete, so we can replace Y_α to $Y_\alpha \cup \{x\}$ and get a contradiction with the choice of F. Thus, $Y_\alpha \cap U$ is infinite for each $\alpha \in I$.

At last, suppose that $X \setminus U$ is infinite. By Lemma 1, there exists a countable almost discrete subspace Y of X such that $Y \subseteq X \setminus U$. Clearly, $Y \cup \{x\}$ is almost discrete. Since $Y_\alpha \cap U$ is infinite, $Y_\alpha \setminus U$ is also infinite. We consider the family $\mathcal{F}$ of countable almost discrete pairwise disjoint subsets

$$\{Y \cup \{x\}\} \cup \{Y_\alpha \setminus Y : \alpha \in I\}.$$

Then $X \setminus \bigcup \mathcal{F} \subseteq F \setminus \{x\}$ and again we get a contradiction with the choice of F. Hence $X \setminus U$ is finite for every neighbourhood U of x, and (ii) holds. □

Proof of Theorem 3. We assume that (ii) does not hold and prove (i). We enumerate $X = \{x_n : n \in \omega\}$ and, for every $n \in \omega$, choose a neighbourhood U_n of x_n and a countable almost discrete subspace $A_n \subseteq X \setminus U_n$. Then we apply Lemma 3 to choose a family $\{B_n : n \in \omega\}$ of pairwise disjoint countable almost discrete subspaces of X such that $B_n \subseteq A_n$ for each $n \in \omega$. We put $D_0 = \{x_0\} \cup B_0$, pick the minimal $m \in \omega$ such that $x_m \notin D_0$ and put $D_1 = (\{x_m\} \cup B_m) \setminus \{x_0\}$. We pick the minimal $k \in \omega$ such that $x_k \notin D_0 \cup D_1$, put $D_2 = (\{x_k\} \cup B_k) \setminus \{x_0, x_m\}$ and so on. After ω steps we get a decomposition of X in the almost discrete countable subspaces $\{D_n : n \in \omega\}$. □

Proof of Theorem 4. Repeat proof of Theorem 1 with Lemma 2 in place of Lemma 1. □

Proof of Theorem 5. Let $f : X \to X$ be an arbitrary mapping. If x is not a recurrent point, we choose a neighbourhood U of x and $n_0 \in \omega$ such that $f^n(x) \in X \setminus U$ for each $n \in \omega$. We put $x_0 = f^{n_0}(x)$. If x_0 is not a recurrent point, we choose a neighbourhood U_0 of x_0 and $n_1 \in \omega$ such that $f^n(x_0) \notin U_0$ for each $n \geqslant n_1$. We put $x_1 = f^{n_1}(x_0)$. If x_1 is not a recurrent point, we choose a neighbourhood U_1 of x_1 and $n_2 \in \omega$ such that $f^n(x_1) \notin U_1$ for each $n \geqslant n_2$. Repeating these arguments, we either find a recurrent point or construct the sequence $(x_n)_{n\in\omega}$ in $X \setminus U$ and the sequence $(U_n)_{n\in\omega}$ of its neighbourhoods such that $x_{n+1} \notin U_0 \cup \ldots \cup U_n$ for each $n \in \omega$. But the second variant is impossible because the subspace $\{x_n : n \in \omega\}$ of $X \setminus U$ is almost discrete. □

Proof of Theorem 6. The implication (iii)⇒(i) follows from Theorem 5, (i)⇒(ii) is trivial. We assume that X satisfies (ii) but (iii) does not hold. By Theorem 2, X is a disjoint union of family $\{Y_\alpha : \alpha \in I\}$ of countable almost discrete subspaces. Let $f : X \to X$ be a bijection of X without periodic points such that each subspace Y_α is f-invariant. Since each Y_α is almost discrete, f has no recurrent points contradicting (ii). □

Proof of Theorem 7. The implication (iii)⇒(i) follows from Theorem 5, (i)⇒(ii) is trivial. We assume that X satisfies (ii) but (iii) does not hold. By Theorem 3, X is a disjoint union of a family $\{Y_n : n \in I\}$ of countable almost discrete subspaces. Let $f : X \to X$ be a bijection of X without periodic points such that each subspace Y_n is f-invariant. Since each Y_n is almost discrete, f has no recurrent points contradicting (ii). □

Remark 1. We construct a bijectively recurrent space X which is not totally recurrent. This example also shows that the implication (ii)⇒(iii) in Theorem 7 does not hold for uncountable spaces. Let $X = C \cup D$ where C is an open uncountable cofinite subspace of X, D is an open countable discrete subspace of X. Let $f : X \to X$ be an arbitrary bijection. Since D is countable, X is uncountable and f is a bijection, there exists $x \in C$ such that $\{f^n(x) : n \in \omega\} \subseteq C$. Using the arguments proving Theorem 5, we conclude that at least one point $f^m(x)$ is recurrent, so X is bijectively recurrent. To see that X is not totally recurrent, we note that D is not totally recurrent and closed in X. By [2, Lemma 1], every closed subspace of totally recurrent space is totally recurrent.

Question 1. Let X be a totally recurrent space. Does there exists a point $x \in X$ such that, for every neighbourhood U of x, $X \setminus U$ has no infinite almost discrete subspaces?

Remark 2. Let us assume that a topological space X is partitioned $X = Y \cup Z$ so that Y is totally recurrent space without cofinite subspaces, Z has no infinite almost discrete subspaces. We show that X is totally recurrent. By Theorem 6, there exists a point $y \in Y$ such that $Y \setminus U$ is finite for every neighbourhood U of y. Let $f : X \to X$ be an arbitrary mapping. If y is not recurrent, there exist a neighbourhood U of y and $m \in \omega$ such that $f^n(y) \notin U$ for each $n \geqslant m$. If the sequence $(f^n(y))_{n\in\omega}$ meets the finite subset $Y \setminus U$ infinitely often, then at least one point of this sequence is recurrent. Otherwise, there exist $k \in \omega$ such that $f^n(y) \in Z$ for each $n \geqslant k$. Using the arguments proving Theorem 5, we see that at least one point of the sequence $(f^n(y))_{n\geqslant k}$ is recurrent.

Question 2. Can every totally recurrent space X be partitioned as $X = Y \cup Z$ so that Y is totally recurrent space without cofinite subspaces, Z has no infinite almost discrete subspaces?

Remark 3. Let (X, τ) be a topological space. We say that a topology τ_1 on X is a T_1-*refinement* of τ if τ_1 is the smallest T_1-topology such that $\tau \subseteq \tau_1$. The family

$$\{U \setminus F : U \in \tau, \ F \text{ is a finite subset of } X\}$$

forms a base for τ_1. Let $f : X \to X$ be an arbitrary mapping. If the orbit $\{f^n(x) : n \in \omega\}$ is infinite then x is recurrent in (X, τ) if and only if x is recurrent in (X, τ_1). It follows that (X, τ) is totally (bijectively) recurrent if (X, τ_1) is also totally (bijectively) recurrent. Thus, the problem of characterization of totally (bijectively) recurrent spaces is reduced to the case of T_1-spaces.

Remark 4. Let φ be a filter on a set X, $\dot{\varphi} = \varphi \cup \{\varnothing\}$. Then $\dot{\varphi}$ is a linked topology on X. If $X \setminus \Phi$ is finite for every $\Phi \in \varphi$ then $(X, \dot{\varphi})$ is cofinite and, by Theorem 5 $(X, \dot{\varphi})$ is totally recurrent. On the other hand, let $(X, \dot{\varphi})$ be totally recurrent. Clearly, the subspace $X \setminus \Phi$, $\Phi \in \phi$ is closed and discrete in $(X, \dot{\varphi})$. By [2, Lemma 1], $X \setminus \Phi$ is finite.

Let (X, τ) be an arbitrary linked space. Then τ is a base for some filter φ on X and $\tau \subseteq \dot{\varphi}$. If $(X, \dot{\varphi})$ is totally recurrent then so is (X, τ) but the converse statement is not true.

References

[1] R. Graham, B. Rothschild, and J. Spencer, *Ramsey Theory*, Wiley, New York, 1990.

[2] O. Petrenko, I. V. Protasov, *Discontinuous Birkhoff Theorem*, Ukr. Math. Bull., **4**(2007), 429-431.

[3] O. Petrenko, I. V. Protasov, *Remark on Birkhoff Theorem*, Travaux Math (to appear)

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