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by

Dennis K. Burke and Sheldon W. Davis

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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SPACES WITH A σ-WEAKLY HEREDITARILY CLOSURE PRESERVING BASE

DENNIS K. BURKE AND SHELDON W. DAVIS

Abstract. A collection $\mathcal{W}$ of subsets of a space X is said to be *weakly hereditarily closure-preserving* (wHCP) if every choice $x(W) \in W$, $W \in \mathcal{W}$, yields a closed discrete set $\{x(W) : W \in \mathcal{W}\}$. Spaces with a σ-wHCP base were introduced in [BEL] and were more recently studied by X. Ge in [G], by C. Liu and L. Ludwig in [LL] and by C. Liu, S. Lin and L. Ludwig in [LLL].

Proposition. If $\chi(x, X)$ denotes the character of x in X and $\Psi(x, X)$ denotes the pseudo-character then $\chi(x, X)$ has countable cofinality for all non-isolated x and $\Psi(x, X) < \chi(x, X)$ whenever $\chi(x, X)$ is uncountable.

This helps explain the nature of non-metrizable examples.

A natural question about products is answered by the following. This theorem does not extend to products with infinitely many factors.

Theorem. A product $X \times Y$, of spaces with a σ-wHCP base, has a σ-wHCP base if and only if all non-isolated points in X and Y have the same character.

Previous known examples of spaces with a σ-wHCP base were hereditarily paracompact. Answering a question from [LL] and [LLL] we have the following.

Example. There is an example of a space X with a σ-wHCP base which is not meta-Lindelöf. In fact, for any cardinal κ there is such a space where every "canonical" open cover has point-order $\geq \kappa$.

Answering another question from [LL], an example is given showing the non-preservation of spaces with a σ-wHCP base under a perfect map.

Example. There is a space X with a σ-wHCP base and a perfect mapping $g : X \to Y$, with exactly one non-trivial fiber, onto a space without a σ-wHCP base.

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1. Introduction

A collection $\mathcal{W}$ of subsets of a space X is said to be *weakly hereditarily closure-preserving* (wHCP) if every choice $x(W) \in W$, $W \in \mathcal{W}$, yields a closed discrete set $\{x(W) : W \in \mathcal{W}\}$. Spaces with a σ-wHCP base were introduced by Burke, Engelking and Lutzer in [BEL] as a weakening of the σ-HCP (σ-hereditarily closure-preserving) base condition. It was shown in [BEL] that regular spaces with a σ-HCP base are metrizable and that a regular space with a σ-wHCP base is metrizable if the space is also a k-space. An example was given of a non-metrizable space with a σ-wHCP base to illustrate that the existence of a σ-wHCP base is not enough to insure metrizability. This example, which will be described later in this introductory section, was a hereditarily paracompact space X with exactly one non-isolated point p of character $\chi(p, X) = \aleph_\omega$. This property of having all non-isolated points with character of countable cofinality turns out to be a characteristic of all spaces with a σ-wHCP base. (See Corollary 2.2.) We take advantage of this property in the following sections to prove results on convergence, products and perfect mappings. All topological spaces are assumed to be at least T_2.

Spaces with a σ-wHCP base were more recently studied by X. Ge in [G] and by C. Liu and L. Ludwig in [LL]. It was shown in [BEL] that a regular space X with a σ-wHCP base is metrizable if X is a first-countable space (or even a k-space). Liu and Ludwig generalized the result in [BEL] by showing that such a space X is metrizable if the character $\chi(X) < \aleph_\omega$ or if X has countable tightness. They also showed that a regular space X with σ-wHCP base is metrizable if it is separable or $\aleph_1$-compact. The reasons for these results being true become much more apparent with the convergence and point character properties given in Section 2.

We conclude the introduction with the following example. This example with specific $\kappa = \aleph_\omega$ is essentially the same as that given in [BEL] (Example 9) of a nonmetrizable, hereditarily paracompact space with a σ-wHCP base. It is worthwhile to repeat this construction here using an arbitrary uncountable cardinal κ, of countable cofinality. Notice that any such space, as in Example 1.1, has a σ-closure-preserving base (i.e. is an M_1-space [C]) and is hereditarily paracompact.

Example 1.1. For any uncountable cardinal κ, of countable cofinality, there is a space E_κ with a σ-wHCP base such that E_κ has exactly one non-isolated element p and $\chi(p, E_\kappa) = \kappa$.

Construction. Express $\kappa = \bigcup_{n\in\omega} \mu_n$ where $\langle \mu_n \rangle_{n\in\omega}$ is a strictly increasing sequence of regular cardinals. Let Z denote the product space 2^κ and let p denote the zero element of Z having $p(\alpha) = 0$ for all $\alpha \in \kappa$. Let

$$E_\kappa = \{p\} \cup \{z \in Z : |\kappa \smallsetminus \operatorname{supp} z| < \omega\},$$

where in this expression, $\operatorname{supp} z$ indicates $\{\alpha \in \kappa : z_\alpha = 1\}$ (the usual "support of z").

Induce the topology on E_κ by declaring every element of $E_\kappa \smallsetminus \{p\}$ to be isolated and by letting the neighborhoods of p be given by the relative topology neighborhoods inherited from the product topology on $Z = 2^\kappa$. To clarify the neighborhoods of p, we introduce the following notation. For the elements $a \in [\kappa]^{<\omega}$ (i.e., finite subsets of κ) the sets $U(a) = \{z \in Z : a \subseteq \kappa \smallsetminus \operatorname{supp} z\} \cap E_\kappa$ give the relative basic neighborhoods of p in E_κ.

For every $n \in \omega$, let

$$\mathcal{B}'(n) = \{\{z\} : z \in E_\kappa \text{ and } |\{\alpha \in \kappa : z_\alpha = 0\}| = n\}.$$

It is easy to verify that every $\mathcal{B}'(n)$, $n \in \omega$, is a discrete collection in E_κ, hence certainly wHCP.

For every $n \in \omega$, let $\mathcal{B}''(n) = \{U(a) : a \in [\kappa]^{<\omega}, a \subseteq \mu_n\}$. Notice that $|\mathcal{B}''(n)| = \mu_n$. To show that every $\mathcal{B}''(n)$ is wHCP pick $x(B) \in B$ for every $B \in \mathcal{B}''(n)$. Now, observe that

$$|\bigcup\{\kappa \smallsetminus \operatorname{supp} x(B) : B \in \mathcal{B}''(n)\}| \leq \mu_n$$

so there must exist some $\beta \in \kappa \smallsetminus \bigcup\{\kappa \smallsetminus \operatorname{supp} x(B) : B \in \mathcal{B}''(n)\}$. Because of this choice of β we see that

$$U(\{\beta\}) \cap \{x(B) : B \in \mathcal{B}''(n)\} = \emptyset.$$

This says that p is not a cluster point of $\bigcup\{x(B) : B \in \mathcal{B}''(n)\}$ and so $\{x(B) : B \in \mathcal{B}''(n)\}$ is a closed discrete set, as desired.

Now, it is clear that $\mathcal{B} = \left(\bigcup_{n\in\omega} \mathcal{B}'(n)\right) \cup \left(\bigcup_{n\in\omega} \mathcal{B}''(n)\right)$ is a σ-wHCP base for E_κ and $\chi(p, E_\kappa) = \kappa$. $\square$

These spaces E_κ, κ of countable cofinality, can generate other spaces of interest to this study. For example, we will see from results in Section 3 that $E_\kappa \times E_\lambda$ always has a σ-wHCP base when $\kappa = \lambda$ and **does not** have a σ-wHCP base when $\kappa \neq \lambda$. Clearly one can take a topological sum of spaces with a σ-wHCP base and obtain a space with a σ-wHCP base containing points of different local character.

2. Character and convergence properties

The results in this section seem to strictly control the nature of spaces with a σ-wHCP base. These results will be fundamental in arguments throughout the rest of the paper.

The following proposition is a refinement of the proof by Liu-Ludwig [LL] that a space X with a σ-wHCP base is metrizable if $\chi(X) < \aleph_\omega$.

Proposition 2.1. *Suppose $x \in X$ has character $\chi(x, X) = \kappa$. Then, for any wHCP collection $\mathcal{W}$ in X, the collection $\mathcal{W}$ must be "locally $< \kappa$ at x." In particular, any wHCP collection $\mathcal{W}$, of neighborhoods of x, has cardinality $|\mathcal{W}| < \kappa$.*

Proof. Suppose $\mathcal{B}(x) = \{B_\alpha(x) : \alpha \in \kappa\}$ is a local base at x and $\mathcal{W} = \{W_\alpha : \alpha \in \lambda\}$ is a wHCP collection (indexed one-to-one) of subsets of X. For contradiction assume that for any neighborhood V of x, $|\{\alpha : V \cap W_\alpha \neq \emptyset\}| \geq \kappa$. For all $\beta \in \kappa$ we can recursively pick $\gamma(\beta) \in \lambda$, where $\gamma(\beta) \neq \gamma(\alpha)$ if $\alpha < \beta$, and

$$z_\beta \in B_\beta(x) \cap W_{\gamma(\beta)} \setminus \{z_\alpha : \alpha < \beta\}.$$

The set $A = \{z_\beta : \beta < \kappa\}$ must be closed and discrete since $\mathcal{W}$ is wHCP, but A must also have x as a cluster point since $\mathcal{B}(x)$ is a local base at x and $A \cap B_\beta(x) \neq \emptyset$ for all $\beta \in \kappa$. This contradiction finishes the proof. □

Corollary 2.2. *If X is any space with a σ-wHCP base then the character $\chi(x, X)$ of every non-isolated $x \in X$ must have countable cofinality.*

The above corollary was stated here for emphasis but a slightly stronger result is stated and proved as 2.3(a) below.

Proposition 2.3. *Suppose non-isolated $x \in X$ has a local σ-wHCP base and local character $\chi(x, X) = \kappa$.*

(a) *The character $\chi(x, X)$ has countable cofinality.*
(b) *If κ is uncountable, then the pseudo-character $\Psi(x, X) < \kappa$.*
(c) *If $A \subseteq X$, with $|A| = \sigma < \kappa$, then x is not a limit point of A.*
(d) *If x has a neighborhood V with $|V| \leq \kappa$ then $\{x\}$ is a G_δ-set.*

Proof. It is clear that for (a) and (b) we may assume there is a local base $\mathcal{W}(x) = \bigcup_{n\in\omega} \mathcal{W}_n(x)$ at x where each $\mathcal{W}_n(x)$ is wHCP and $|\mathcal{W}_n(x)| \leq \kappa$.

Proof of **(a)**: Proposition 2.1 says that every $|\mathcal{W}_n(x)| < \kappa$. Since $\kappa = \sup_{n\in\omega} |\mathcal{W}_n(x)|$ it follows that κ has countable cofinality.

Proof of **(b)**: Express $\kappa = \bigcup_{n\in\omega} \mu_n$ where $\langle\mu_n\rangle_{n\in\omega}$ is a strictly increasing sequence of regular cardinals. By Proposition 2.1 we know that $|\mathcal{W}_n(x)| < \kappa$ for all $n \in \omega$. In fact, with a little rearrangement we may assume $|\mathcal{W}_n(x)| = \mu_n$ and $\mathcal{W}_n \subseteq \mathcal{W}_{n+1}$ all $n \in \omega$. Express each $\mathcal{W}_n(x) = \{W_{n\alpha}(x) : \alpha \in \mu_n\}$ with a one-to-one indexing.

If $\Psi(x, X) = \kappa$ then for all $n < \omega$ we may pick $z_n \in \bigcap_{\alpha\in\mu_n} W_{n\alpha}(x) \setminus \{x\}$. It is easily verified that $z_n \to x$. However, $\{z_n : n \in \omega\}$ could also represent a choice from $\{W_{0n} : n \in \omega\}$. This would imply that $\{z_n : n \in \omega\}$ is closed and discrete, a contradiction. Hence, $\Psi(x, X) < \kappa$.

Proof of **(c)**: Let $\mathcal{V} = \{V_\alpha : \alpha < \sigma^+\}$ be a wHCP collection of neighborhoods of x. If x is a limit point of A then for all $\beta \in \sigma^+$ we may choose

$$x_\beta \in V_\beta \cap A \setminus \{x_\alpha : \alpha \in \beta\}.$$

However, this gives a subset of A with $|\{x_\beta : \beta \in \sigma^+\}| = \sigma^+ > |A|$, a contradiction.

Proof of **(d)**: If $|V| \leq \kappa$ then $V \setminus \{x\}$ can be expressed as $V \setminus \{x\} = \bigcup_{n\in\omega} A_n$, where each $|A_n| < \kappa$. Then part (c) gives that x is not a limit point of A_n so $\{x\} = \bigcap_{n\in\omega} (V \setminus \overline{A_n})$, a G_δ-set. $\square$

Notice that 2.3(c) says that if κ is uncountable then there are no nontrivial sequences converging to x. Proposition 2.3(d) can be thought of as a partial answer to a question in [LL] about whether singleton sets are G_δ-sets in a space with a σ-wHCP base.

3. Products of spaces with a σ-wHCP base

Theorem 3.1. *Let spaces X, Y each have a σ-wHCP base and suppose $\chi(x, X) = \chi(y, Y)$ for all non-isolated $x \in X$, $y \in Y$. If $\mathcal{G}, \mathcal{H}$ are wHCP collections in X, Y respectively then the collection $\{G \times H : G \in \mathcal{G}, H \in \mathcal{H}\}$ is a wHCP collection in $X \times Y$. Hence, it follows that $X \times Y$ has a σ-wHCP base.*

Proof. Express $\mathcal{G} = \{G_\alpha : \alpha < \lambda\}$ and $\mathcal{H} = \{H_\alpha : \alpha < \mu\}$ with a one-to-one indexing. Suppose, for all $(\alpha, \beta) \in \lambda \times \mu$, $(x_{\alpha\beta}, y_{\alpha\beta}) \in G_\alpha \times H_\beta$ and $A = \{(x_{\alpha\beta}, y_{\alpha\beta}) : (\alpha, \beta) \in \lambda \times \mu\}$. Assume that A has a limit point (s,t) and let $A_1 = \{x_{\alpha\beta} : (\alpha, \beta) \in \lambda \times \mu\}$ and $A_2 = \{y_{\alpha\beta} : (\alpha, \beta) \in \lambda \times \mu\}$. If $\chi(s, X) = \chi(t, Y) = \kappa$ then Proposition 2.1 says that $\mathcal{G}$ must be locally $< \kappa$ at s and $\mathcal{H}$ must be locally $< \kappa$ at t, so without loss of generality we may assume $\lambda < \kappa$ and $\mu < \kappa$. Now, either s is a limit point of A_1 in X or t is a limit point of A_2 in Y. Assume s is a limit point of A_1. Since $|A_1| \leq \lambda \cdot \mu < \kappa$ Proposition 2.3(c) says that s cannot be a limit point of A_1, a contradiction. □

Theorem 3.2. *If X, Y each have a σ-wHCP base and there exist non-isolated $s \in X$, $t \in Y$ with $\chi(s, X) \neq \chi(t, Y)$ then $X \times Y$ does* **not** *have a σ-wHCP base. In particular, (s, t) cannot even have a local σ-wHCP base.*

Proof. Without loss of generality we may assume $\chi(s, X) < \chi(t, Y)$. If $p = (s, t)$ did have a local σ-wHCP base then since $\chi(p, X \times Y) = \chi(t, Y) > \chi(s, X)$ there would have to be a wHCP collection $\mathcal{W}$ of neighborhoods of p in $X \times Y$ where $|\mathcal{W}| = \chi(s, X)$. If $\mathcal{W} = \{W_\alpha : \alpha < \chi(s, X)\}$ and $\{U_\alpha : \alpha < \chi(s, X)\}$ is a local base at s in X then we may pick $(x_\alpha, t) \in W_\alpha \cap (U_\alpha \setminus \{s\}) \times \{t\})$, all $\alpha < \chi(s, X)$. The set $A = \{x_\alpha : \alpha < \chi(s, X)\}$ has s as a limit point in X so (s, t) must be a limit point of $A \times \{t\}$. However, since $\mathcal{W}$ is a wHCP collection the choice set A and so also $A \times \{t\}$ must be closed discrete sets. This contradication finishes the proof. □

Clearly, the results in Theorem 3.1 extend (by induction) to products with any finite number of factors. In [LLL] it was shown that X^n has a σ-wHCP base, for $X = E_\kappa$, $\kappa = \aleph_\omega$ and $n \in \mathbb{N}$. However, any expectation of some extension to a product with infinitely many factors (even countable) is doomed by the next result. This is a direct application of Theorem 3.2. We note that a related result, shown independently in [LLL], says that if a product X^ω has a σ-wHCP base then X must be metrizable.

Theorem 3.3. *If $\langle X_n \rangle_{n \in \omega}$ is any sequence of topological spaces, each having at least two elements, and X_0 is any non-metrizable space with a σ-wHCP base then $\prod_{n \in \omega} X_n$ cannot have a σ-wHCP base.*

Proof. By the two-element condition on each X_k it is clear that $\prod_{n\in\omega} X_n$ must contain a subspace homeomorpic to $X_0 \times 2^\omega$. Since X_0 must contain a point of uncountable character and 2^ω is first-countable, Theorem 3.2 implies that $X_0 \times 2^\omega$ does not have a σ-wHCP base. Because the σ-wHCP base condition is hereditary, we see that $\prod_{n\in\omega} X_n$ cannot have a σ-wHCP base. □

Remark 3.4. If Y is any non-metrizable space with a σ-wHCP base we have seen that $Y \times 2^\omega$ does not have a σ-wHCP base. The projection map $\pi_1 : Y \times 2^\omega \to Y$ shows that a perfect preimage of a space with a σ-wHCP base need not have a σ-wHCP base – even with the perfect map also being an open mapping and having metrizable compact fibers.

4. Other examples

Recall that a space X is said to be *meta-Lindelöf* [B] if every open cover of X has a point-countable open refinement. In [LL], the authors ask whether every space with a σ-wHCP base is meta-Lindelöf. At that time the only known examples of spaces with a σ-wHCP base were hereditarily paracompact. We give a negative answer to this question with an example using a subset of a product 2^λ as the base set.

Example 4.1. There is a Tychonoff space X with a σ-wHCP base and not meta-Lindelöf. Given any cardinal κ there is such a space with a canonical open cover such that every open refinement has point-order $\geq \kappa$.

Construction. For an uncountable regular cardinal κ, let $\delta_0 = \kappa$. Extend to a strictly increasing sequence $\langle \delta_n \rangle_{n\in\omega}$ of regular cardinals and let $\lambda = \sup_{n\in\omega} \delta_n$. Let $Z = 2^\lambda$. For $t \in \lambda$, let e_t denote the unique element of Z with $\operatorname{supp} e_t = \{t\}$ and $E = \{e_t : t \in \delta_0\}$. For $n \in \omega$, let

$$J_n = \{z \in Z : |\lambda \setminus \operatorname{supp} z| \leq \kappa \text{ and } \lambda \setminus \operatorname{supp} z \subseteq \delta_n\}.$$

Let $X = E \cup \bigcup_{n\in\omega} J_n$. A topology τ on X is generated by letting the elements of $\bigcup_{n\in\omega} J_n$ be isolated and the neighborhoods of each e_t in X, $t \in \delta_0$, are inherited from the product topology on Z.

For $t \in \delta_0$ and finite $F \in [\lambda \setminus \{t\}]^{<\omega}$ let $U(t, F)$ denote the basic neighborhood of e_t given by

$$U(t, F) = \{x \in X : x(t) = 1 \text{ and } x(\alpha) = 0, \text{all } \alpha \in F\}.$$

For $n \in \omega$, let $\mathcal{B}_n = \{U(t,F) : t \in \delta_0, F \in [\delta_n \smallsetminus \{t\}]^{<\omega}\}$ and let $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n \cup \bigcup_{n\in\omega} \{\{x\} : x \in J_n\}$.

The desired properties of X will now follow below with the verification of Claims 0–3. In particular, Claim 3 shows that the open cover $\{U(t,\emptyset) : t \in \delta_0\}$ does not have a point-countable open refinement.

Claim 0. $\mathcal{B}$ is a base for the topology on X.

Claim 1. For all $n \in \omega$, J_n is a closed, discrete set and E is a closed, discrete set.

It is enough to verify that no element $e_t \in E$ can be a limit point of J_n so pick any $\beta \in \lambda \smallsetminus \delta_n$. Then $U(t,\{\beta\}) \cap J_n = \emptyset$, showing that e_t is not a limit point of J_n. Also, $U(t,\emptyset) \cap E = \{e_t\}$ so this will show that E is closed and discrete.

Claim 2. For all $n \in \omega$, $\mathcal{B}_n$ is wHCP.

Observe that the cardinality of $\mathcal{B}_n$ is δ_n and we may express $\mathcal{B}_n = \{B_\alpha : \alpha \in \delta_n\}$ (with a one-to-one indexing). To check that $\mathcal{B}_n$ is wHCP, pick $p_\alpha \in B_\alpha$ for all $\alpha \in \delta_n$ and let $P = \{p_\alpha : \alpha \in \delta_n\}$. Since E is closed and discrete we may assume that $E \cap P = \emptyset$. Now, $|\bigcup_{\alpha\in\delta_n}(\lambda \smallsetminus \operatorname{supp} p_\alpha)| \leq \delta_n$ so we may pick

$$\gamma \in \lambda \smallsetminus \textstyle\bigcup_{\alpha\in\delta_n}(\lambda \smallsetminus \operatorname{supp} p_\alpha).$$

Also, we may pick such γ so $\gamma \notin \delta_0$. It follows that, for all $t \in \delta_0$, $U(t,\{\gamma\}) \cap P = \emptyset$. That is, P is closed and discrete.

Claim 3. For every basic open collection $\mathcal{W}$ of form $\mathcal{W} = \{U(t,F_t) : t \in \delta_0\}$ there is some $p \in X$ such that $\operatorname{ord}(p,\mathcal{W}) = \kappa$.

By the Delta System Lemma [K], the collection $\{\{t\} \cup F_t : t \in \delta_0\}$ has a subcollection $\{\{t\} \cup F_t : t \in \Lambda\}$ of cardinality κ and root R. (For distinct $t, t' \in \Lambda$, $(\{t\} \cup F_t) \cap (\{t'\} \cup F_{t'}) = R$.) We may also assume $\Lambda \cap R = \emptyset$ and we may assume there exists some $m \in \omega$ such that $\bigcup_{t\in\Lambda} F_t \subseteq \delta_m$. Certainly, $\left|\bigcup_{t\in\Lambda} F_t\right| \leq \kappa$ so we may let p be the element of Z with $\operatorname{supp} p = \lambda \smallsetminus \bigcup_{t\in\Lambda} F_t$. Then, $p \in J_m$ and $p \in U(t,F_t)$, for all $t \in \Lambda$. That is, $\operatorname{ord}(p,\mathcal{W}) = \kappa$. □

The next two examples answer another question from [LL]. It is natural to ask whether a σ-wHCP base is preserved under a perfect map as this seems to be the case for many base axioms.

However, we give examples to show that the property of having a σ-wHCP base is not always preserved under a perfect mapping.

Example 4.2. There is a space X with a σ-wHCP base and a perfect mapping $g : X \to Y$ onto a space Y which does not have a σ-wHCP base. Furthermore, the map g can be found having exactly one non-trivial fiber of cardinality larger than 1.

Construction. Let $\{Z_n : n \in \mathbb{N}\}$ be any pairwise disjoint collection of non-metrizable spaces with a σ-wHCP base and for each $k \in \mathbb{N}$ identify some $q_k \in Z_k$ having $\chi(q_k, Z_k) > \omega$. If p is any element not in $\bigcup_{n\in\mathbb{N}} Z_n$, let $X = \{p\} \cup \left(\bigcup_{n\in\mathbb{N}} Z_n\right)$. If $W_n = \{p\} \cup \left(\bigcup_{k\geq n} Z_k\right)$ the collection $\{W_n : n \in N\}$ will be a local base at p. If τ_n denotes the topology on Z_n then the topology σ on X be given by the base $\mathcal{B} = \{W_n : n \in N\} \cup \left(\bigcup_{n\in\mathbb{N}} \tau_n\right)$. It is straightforward to verify that $\mathcal{B}$ is a σ-wHCP base for X. For later use notice that $q_n \to p$ in X so the set $P = \{p\} \cup \{q_n : n \in \mathbb{N}\}$ is compact. (Also notice that for any choice $z_n \in Z_n$ we have $z_n \to p$.)

Let Y be the quotient space obtained from X by identifying all of the elements of P to a single element and let $g : Z \to Y$ be the corresponding quotient map. It is clear that g is a perfect mapping. It remains to show that Y cannot have a σ-wHCP base. For contradiction suppose that $\mathcal{G} = \bigcup\limits_{n\in\omega} \mathcal{G}_n$ is a base for Y where each $\mathcal{G}_n$ is wHCP. Observe that the element $t = g(p)$ has uncountable character in Y so there must be some $m \in \mathbb{N}$ such that we can find, for $i \in \mathbb{N}$, distinct $H_i \in \mathcal{G}_m$ having $t \in H_i$. Now, each $g^{-1}(H_i)$ is an open set about p so there is a strictly increasing sequence $\{n_i\}_{i=1}^{\infty}$ from $\mathbb{N}$ and corresponding choices $x_i \in (Z_{n_i} \setminus \{q_{n_i}\}) \cap g^{-1}(H_i)$. Since $x_i \to p$ we must have $g(x_i) \to g(p)$ and hence $g(p)$ is certainly a limit point of the set $A = \{g(x_i) : i \in \mathbb{N}\}$. However, this contradicts Proposition 2.3(c) (since $|A| = \omega < \chi(g(p), Y)$). That concludes the proof. □

Remark. In the above construction, if each Z_n is paracompact (certainly possible) then the domain space X is also paracompact. The next example could be considered the simplest possible example of a perfect mapping not preserving a σ-wHCP base. As in Example 4.2 this example takes advantage of Proposition 2.3(c) to argue effectively that Y does not have a σ-wHCP base.

Example 4.3. There is a space X with a σ-wHCP base and a perfect mapping $h : X \to Y$ onto a space Y which does not have a σ-wHCP base. Furthermore, the map h can be found having exactly one non-trivial fiber of cardinality larger or equal to 2.

Construction. Let $X = E'_\kappa \cup E'_\lambda$ be a topological sum of disjoint copies of E_κ, E_λ, with $\kappa < \lambda$. (See Example 1.1.) Let $s \in E'_\kappa, t \in E'_\lambda$ be the elements with $\chi(s, E'_\kappa) = \kappa$ and $\chi(t, E'_\lambda) = \lambda$. Let Y be the quotient space obtained from X by indentifying the pair $\{s,t\}$ to the point q and let $h : X \to Y$ be the corresponding quotient map. It is clear that h is a perfect mapping with the only nontrivial point-inverse of $f^{-1}(\{q\}) = \{s,t\}$. Now, observe that $\chi(q, Y) = \lambda$ and q is a limit point of the set $A = h(E'_\kappa)$. However, $|A| = \kappa < \lambda$ – so Proposition 2.3(c) says that Y cannot have a σ-wHCP base. That completes the argument for the construction. □

While the previous two examples were slightly different from one another the reason for the given perfect image not having a σ-wHCP base was essentially the same. In each case, the domain space X had two non-isolated points having different local character in X.

Remark 4.4. The approach in Example 4.3 shows that if X is any space with a σ-wHCP base and X has two non-isolated points of different local character then there is a perfect image of X not having a σ-wHCP base.

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Miami University
E-mail address: burkedk@muohio.edu

Miami University, University of Texas at Tyler
E-mail address: Sheldon_Davis@uttyler.edu