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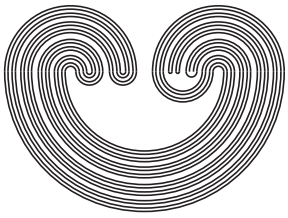

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by

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INCREASING WHITNEY PROPERTIES

FERNANDO OROZCO-ZITLI

ABSTRACT. Let X be a metric continuum and $C(X)$ the hyperspace of subcontinua of X. A topological property $\mathcal{P}$ is said to be an *increasing Whitney property* provided that if μ is a Whitney map for $C(X)$ and $t_0 \in [0,1)$ such that $\mu^{-1}(t_0)$ has property $\mathcal{P}$, then $\mu^{-1}(t)$ has the property for each $t \in [t_0, 1)$. In this paper we show that the following properties are increasing Whitney properties: being a particular solenoid, being the Buckethandle continuum, being proper circle-like, being planar and being non-planar for the class of circle-like continua, chainability, being the pseudo-arc, being a particular pseudo-solenoid, being a pseudo-circle, being hereditarily indecomposable and tree-like, being indecomposable and chainable, pathwise connectedness, uniform pathwise connectedness, continuum-chainability and uniform continuum-chainability. We also show that decomposability and aposyndesis are not increasing Whitney properties.

1. INTRODUCTION

A *continuum* is a nonempty, nondegenerate, compact, connected metric space. Throughout this paper X will denote a continuum. The hyperspace of subcontinua of a given continuum X is denoted by $C(X)$. We consider that $C(X)$ is metrized with the Hausdorff metric H. We denote by H^2 the corresponding Hausdorff metric for $C(C(X))$. The closure of a subset A of X is denoted by $\mathrm{cl}_X(A)$,

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Key words and phrases. Aposyndesis, Buckethandle, chainability, circle-like, continuum, continuum-chainability, decomposability, hereditarily indecomposable, hyperspace, increasing Whitney property, planar, pseudo-arc, pseudo-circle, pseudo-solenoid, solenoid, uniform pathwise connectedness, uniform continuum-chainability and Whitney property.

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the interior of A by $\text{int}_X(A)$ and the diameter of A by diameter$[A]$. The set of positive integers is denoted by $\mathbb{N}$.

Let X be a continuum, and let $\mathcal{H} \subset C(X)$. A *Whitney map for* $\mathcal{H}$ is a continuous function $\mu : \mathcal{H} \to [0,1]$ that satisfies the following two conditions:

i) for any A, $B \in \mathcal{H}$ such that $A \subset B$ and $A \neq B$, $\mu(A) < \mu(B)$;

ii) $\mu(A) = 0$ if and only if $A \in \mathcal{H} \cap \{\{x\} : x \in X\}$.

It is known that each space $C(X)$ admits Whitney maps (see [12, p. 25 and 26]).

A *Whitney level for* $C(X)$ is a set of the form $\mu^{-1}(t)$, where μ is a Whitney map for $C(X)$ and $t \in [0,1)$.

Let X be a continuum, a Whitney map μ for $C(X)$ and a number $t \in [0,1)$. The function $\sigma : C(\mu^{-1}(t)) \to \mu^{-1}([t,1])$ given by $\sigma(\mathcal{A}) = \bigcup \mathcal{A}$ is continuous (see [6, Lemma 1.1, p. 23]).

A topological property $\mathcal{P}$ is said to be:

– a *Whitney property* provided that if X has property $\mathcal{P}$, then each Whitney level for $C(X)$ has property $\mathcal{P}$,

– an *increasing Whitney property* provided that if μ is a Whitney map for $C(X)$ and $t_0 \in [0,1)$ is such that $\mu^{-1}(t_0)$ has property $\mathcal{P}$, then $\mu^{-1}(t)$ has property $\mathcal{P}$ for each $t \in [t_0, 1)$,

– a *sequential strong Whitney-reversible property* provided that whenever X is a continuum such that there is a Whitney map μ for $C(X)$ and a sequence $\{t_n\}_{n=1}^{\infty}$ in $(0,1)$ such that $\lim t_n = 0$ and $\mu^{-1}(t_n)$ has property $\mathcal{P}$ for each $n \in \mathbb{N}$, then X has property $\mathcal{P}$,

– a *decreasing Whitney property*, provided that whenever X is a continuum such that if μ is a Whitney map for $C(X)$, $t \in [0,1)$ and $\{t_n\}_{n=1}^{\infty}$ is a sequence in $(t,1)$ such that $\lim t_n = t$ and $\mu^{-1}(t_n)$ has property $\mathcal{P}$ for each $n \in \mathbb{N}$, then $\mu^{-1}(t)$ has property $\mathcal{P}$.

Whitney properties were defined in [8, p. 165], and sequential strong Whitney-reversible properties were defined in [13, p. 237]. Since then, many authors have studied them. In [5, Chapter VIII] a very complete discussion about this subject is presented. Decreasing Whitney properties were introduced in [14] and [15].

The following properties are known to be increasing Whitney properties:

a) arcwise connectedness ([16, Proposition 2, p. 151]),
b) being an arc ([10, Theorem 3, p. 174]),
c) being a simple closed curve ([10, Theorem 3, p.174]),
d) being unicoherent, for the class of Peano continua ([3, Theorem A, p. 252]),
e) hereditary indecomposability ([16, Proposition 8, p. 155]),
f) having span zero ([2, Theorem 3.2, p. 37]),
g) local connectedness ([16, Proposition 1, p. 150]).

In this paper we show that the following properties are increasing Whitney properties. The references on the right correspond to the place where they were proved to be Whitney properties:

(1) being the Buckethandle continuum ([5, Theorem 37.9, p. 258]),
(2) chainability ([7, Theorem 6.2 (a), p. 161]),
(3) being hereditarily indecomposable and tree-like ([5, Theorem 63.1, p. 292]),
(4) being indecomposable and chainable ([8, Theorem 4.3, p. 172]),
(5) being a particular pseudo-solenoid ([5, Theorem 57.2, p. 286]),
(6) being a particular solenoid ([12, Theorem 14.2, p. 422]),
(7) being planar and being non-planar for the class of circle-like continua ([5, Theorems 54.1 and 54.2, p. 283]),
(8) being proper circle-like ([7, Theorem 6.2 (b), p. 161]),
(9) being the pseudo-arc ([5, Theorem 56.1, p. 286]),
(10) being the pseudo-circle ([5, Theorem 57.4, p. 286]),
(11) continuum-chainability ([5, Corollary 33.6, p. 249]),
(12) pathwise connectedness ([5, Theorem 33.1, p. 247]),
(13) uniform continuum-chainability ([9, Theorem 2, p. 4]),
(14) uniform pathwise connectedness ([9, Theorem 2, p. 4]).

We also show that, even when decomposability and aposyndesis are Whitney properties (see [8, Theorem 3.4, p. 170] and [16, Proposition 9, p. 156], resp.), they are not increasing Whitney properties.

2. A GENERAL RESULT

If X is a continuum, μ is a Whitney map for $C(X)$, $A \in C(X)$ and $\mu(A) > t$, let $C(A,t) = (\mu \mid C(A))^{-1}(t)$.

Lemma 2.1. *Let X be a continuum, μ a Whitney map for $C(X)$, $A \in C(X)$ and $\mu(A) > t$. Then $C(A,t)$ is a subcontinuum of $\mu^{-1}(t)$.*

Proof. Since $\mu \mid C(A)$ is a Whitney map for $C(A)$, we see that $C(A,t)$ is a Whitney level for $C(A)$. Hence, $C(A,t)$ is a subcontinuum of $\mu^{-1}(t)$ (see [1, p. 1032] or [5, Theorem 19.9, p. 160]). □

Lemma 2.2. *Let X be a continuum, μ a Whitney map for $C(X)$ and $t \in [0,1)$. Let $\mathfrak{L} = \mu^{-1}([t,1))$. If any nondegenerate proper subcontinuum of $\mu^{-1}(t)$ is hereditarily irreducible, then $\sigma \mid \sigma^{-1}(\mathfrak{L}) : \sigma^{-1}(\mathfrak{L}) \to \mathfrak{L}$ is one-to-one.*

Proof. We prove that, for every $A \in \mathfrak{L}$, $\sigma^{-1}(A) = \{C(A,t)\}$. Let $A \in \mathfrak{L}$. Suppose that $A \in \mu^{-1}(t)$. Let $\mathcal{M} \in C\left(\mu^{-1}(t)\right)$ such that $\sigma(\mathcal{M}) = A$. Let $D \in \mathcal{M}$. Since $D \subset \sigma(\mathcal{M})$ and D, $A \in \mu^{-1}(t)$, we infer that $D = A$. Thus $\mathcal{M} = \{A\} = C(A,t)$.

Now, suppose that $t < \mu(A) < 1$. We need to show that:

(1) for every $\mathcal{K} \in C(C(A,t)) \setminus \{C(A,t)\}$, $\sigma(\mathcal{K}) \neq A$.

Since $C(A,t)$ is a nondegenerate proper subcontinuum of $\mu^{-1}(t)$ (see Lemma 2.1), by hypothesis $C(A,t)$ is hereditarily irreducible. Thus, by Theorem 14.73.2 of [12], we have that, for every proper subcontinuum $\mathcal{K}$ of $C(A,t)$, $\sigma(\mathcal{K}) \neq A$. This completes the proof of (1).

If there is $\mathcal{M} \in C\left(\mu^{-1}(t)\right)$ such that $\sigma(\mathcal{M}) = A$, then $\mathcal{M} \subset C(A,t)$. By (1), $\mathcal{M} = C(A,t)$.

Therefore $\sigma \mid \sigma^{-1}(\mathfrak{L})$ is one-to-one. □

Using Exercise 23.8, p. 206 of [5], it is easy to prove the following result (see [15]).

Lemma 2.3. *Let X be a continuum, μ a Whitney map for $C(X)$ and $t_0 \in [0,1)$. Let $\mathfrak{L} = \mu^{-1}([t_0,1))$. If $\sigma \mid \sigma^{-1}(\mathfrak{L}) : \sigma^{-1}(\mathfrak{L}) \to \mathfrak{L}$ is one-to-one, then for each $t \in (t_0,1)$, $\sigma^{-1}\left(\mu^{-1}(t)\right)$ is a Whitney level for $C\left(\mu^{-1}(t_0)\right)$.*

Theorem 2.4. *Let X be a continuum, μ a Whitney map for $C(X)$, $\mathcal{P}$ a Whitney property and $t_0 \in [0,1)$. Suppose that any nondegenerate proper subcontinuum of $\mu^{-1}(t_0)$ is hereditarily irreducible and $\mu^{-1}(t_0)$ has property $\mathcal{P}$. Then $\mu^{-1}(t)$ has property $\mathcal{P}$ for each $t \in [t_0,1)$.*

Proof. Let $\mathfrak{L} = \mu^{-1}([t_0,1))$. By Lemma 2.2, the function $\sigma \mid \sigma^{-1}(\mathfrak{L}) : \sigma^{-1}(\mathfrak{L}) \to \mathfrak{L}$ is one-to-one. By Lemma 2.3, for each $t \in (t_0,1)$, $\sigma^{-1}\left(\mu^{-1}(t)\right)$ is a Whitney level for $C\left(\mu^{-1}(t_0)\right)$.

So, since $\mathcal{P}$ is a Whitney property, $\sigma^{-1}\left(\mu^{-1}(t)\right)$ has property $\mathcal{P}$ for each $t \in (t_0, 1)$. Thus, since $\sigma^{-1}\left(\mu^{-1}(t)\right)$ is homeomorphic to $\mu^{-1}(t)$ for every $t \in (t_0, 1)$, we infer that $\mu^{-1}(t)$ has property $\mathcal{P}$ for each $t \in [t_0, 1)$. □

Corollary 2.5. *Each one of the properties (1)-(10) is an increasing Whitney property.*

Proof. Each one of the continua described in (1)-(10) has the property that its nondegenerate proper subcontinua are hereditarily irreducible (see [12, Exercises 1.209.3 and 1.209.4, p.201] and [11, Theorem 12.5, p. 233]). □

3. PATHWISE CONNECTEDNESS AND CONTINUUM-CHAINABILITY

A continuum X is said to be *uniformly pathwise connected* provided that there is a family $\mathcal{F}$ of paths satisfying:
(a) for each pair of points $x, y \in X$ there is a path $\alpha \in \mathcal{F}$ such that $\alpha(0) = x$ and $\alpha(1) = y$,
(b) for each $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for each $\alpha \in \mathcal{F}$ there are numbers $0 = t_0 < t_1 < \cdots < t_k = 1$ satisfying diameter$[\alpha([t_{i-1}, t_i])] < \varepsilon$ for each $i \in \{1, ..., k\}$.

A finite sequence of sets $\{A_1, ..., A_k\}$ is said to be a *weak chain* provided that $A_i \cap A_j \neq \emptyset$ if $|i - j| \leq 1$.

A continuum X is said to be *continuum-chainable* provided that for each $\varepsilon > 0$ and for each pair of points $p, q \in X$, there exists a weak chain of continua $\{A_1, ..., A_k\}$ such that diameter$[A_i] < \varepsilon$ for each $i \in \{1, ..., k\}$, $p \in A_1$ and $q \in A_k$.

A continuum X is said to be *uniformly continuum-chainable* provided that for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that for each pair of points $p, q \in X$, there exists a weak chain of continua $\{A_1, ..., A_k\}$ such that diameter$[A_i] < \varepsilon$ for each $i \in \{1, ..., k\}$, $p \in A_1$ and $q \in A_k$.

It is known that the union map from 2^{2^X} onto 2^X is continuous and $\bigcup \mathcal{A} \in C(X)$ for each $\mathcal{A}$ subcontinuum of 2^X such that $\mathcal{A} \cap C(X) \neq \emptyset$ (see [5, Exercise 11.5, p. 91] and [6, Lemma 1.2, p. 23]).

Theorem 3.1. *Let X be a continuum, μ a Whitney map for $C(X)$ and $t_0 \in [0, 1)$. If $\mu^{-1}(t_0)$ is uniformly continuum-chainable, then $\mu^{-1}(t)$ is uniformly pathwise connected for each $t \in (t_0, 1)$.*

Proof. Let $t \in (t_0, 1)$. Since the composite of μ and the union map is uniformly continuous, there exists $\delta > 0$ such that if $\mathcal{A} \in C\left(\mu^{-1}(t_0)\right)$ and diameter$[\mathcal{A}] < \delta$, then $\mu(\bigcup \mathcal{A}) < t$. Let k be as in the definition of the uniform continuum-chainability of $\mu^{-1}(t_0)$ for the positive number δ.

Let $A, B \in \mu^{-1}(t)$. Using order arcs (see [12, Theorem 1.8, p. 59]) it is possible to find elements $A_0, B_0 \in \mu^{-1}(t_0)$ such that $A_0 \subset A$ and $B_0 \subset B$. By the choice of k, there exists a weak chain $\{\mathcal{A}_1, ..., \mathcal{A}_k\}$ of subcontinua of $\mu^{-1}(t_0)$ such that $A_0 \in \mathcal{A}_1$, $B_0 \in \mathcal{A}_k$ and diameter$[\mathcal{A}_i] < \delta$ for each $i \in \{1, ..., k\}$. Given $i \in \{1, ..., k\}$, by the choice of δ, we have that $\mu(\bigcup \mathcal{A}_i) < t$. Using again order arcs it is possible to find an element $B_i \in \mu^{-1}(t)$ such that $\bigcup \mathcal{A}_i \subset B_i$. It is easy to see that $\{A, B_1, B_2, ..., B_k, B\}$ is a weak chain.

In [9, Lemma 2, p. 2], I. Krzemińska proved that: for each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ (depending only on μ, t and ε) such that if $C, D \in \mu^{-1}(t)$ and $C \cap D \neq \emptyset$, then there exists a path $\alpha : [0, 1] \to \mu^{-1}(t)$ such that $\alpha(0) = C$, $\alpha(1) = D$ and diameter$[\alpha([\frac{i-1}{n}, \frac{i}{n}])] < \varepsilon$ for each $i \in \{1, ..., n\}$. Applying this fact to each one of the pairs A and B_1, B_1 and B_2, B_2 and B_3,..., B_{k-1} and B_k, B_k and B, it is possible to find a path $\beta : [0, 1] \to \mu^{-1}(t)$ and a sequence of numbers $0 = t_0 < t_1 < \cdots < t_{(k+1)n} = 1$ such that $\beta(0) = A$, $\beta(1) = B$ and diameter$[\beta([t_{i-1}, t_i])] < \varepsilon$.

Therefore, $\mu^{-1}(t)$ is uniformly pathwise connected. □

In the proof of Theorem 2 of [9], I. Krzemińska proved that if a continuum X is uniformly pathwise connected, then X is uniformly continuum-chainable. Thus the following result is immediate from Theorem 3.1.

Theorem 3.2. *Uniform pathwise connectedness and uniform continuum-chainability are increasing Whitney properties.*

With similar ideas as those in Theorem 3.1 it is possible to prove the following result.

Theorem 3.3. *Let X be a continuum, μ a Whitney map for $C(X)$ and $t_0 \in [0, 1)$. If $\mu^{-1}(t_0)$ is continuum-chainable, then $\mu^{-1}(t)$ is pathwise connected for each $t \in (t_0, 1)$.*

Since pathwise connectedness implies continuum-chainability, the following corollary is a consequence of Theorem 3.3.

Corollary 3.4. *Pathwise connectedness and continuum-chainability are increasing Whitney properties.*

4. DECOMPOSABILITY

In [8, Theorem 3.4, p. 170], J. Krasinkiewicz and S.B. Nadler, Jr., showed that decomposability is a Whitney property. In [16, Proposition 6, p. 154], A. Petrus proved that if a Whitney level $\mu^{-1}(t_0)$ is decomposable, then there exists $t_1 \in (t_0, 1)$ such that $\mu^{-1}(t)$ is decomposable for each $t \in [t_0, t_1]$.

Now, we present an example of a continuum X to show that decomposability is not an increasing Whitney property.

The Example:

Denote by $\mathcal{C}$ the usual Cantor ternary set contained in $[0, 1]$. Let ρ be the usual metric in the Euclidean space $\mathbb{R}^3$. For each $w \in \mathcal{C}$, let $S_w = \{(0, y, z) \in \mathbb{R}^3 : \rho((0, y, z), (0, \frac{1}{2}, 1)) = |w - \frac{1}{2}| \text{ and } z \geq 1\}$, and let $S = \bigcup\{S_w : w \in \mathcal{C}\}$. For each $n \in \mathbb{N}$ and for each $v \in [\frac{2}{3^n}, \frac{1}{3^{n-1}}] \cap \mathcal{C}$, let $R_v^n =$

$$\{(0, y, z) \in \mathbb{R}^3 : \rho((0, y, z), (0, \frac{5}{2 \cdot 3^n}, \frac{1}{n})) = |v - \frac{5}{2 \cdot 3^n}|$$

and $z \leq \frac{1}{n}\}$, and let $R^n = \bigcup\{R_v^n : v \in [\frac{2}{3^n}, \frac{1}{3^{n-1}}] \cap \mathcal{C}\}$. For each $n \in \mathbb{N}$, let $T_n = \{0\} \times ([\frac{2}{3^n}, \frac{1}{3^{n-1}}] \cap \mathcal{C}) \times [\frac{1}{n}, 1]$. We consider $W = \bigcup\{R^n \cup T_n : n \in \mathbb{N}\}$ (see Figure 1).

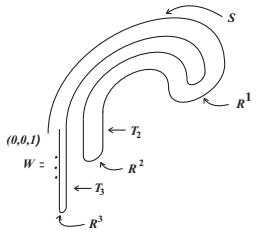

FIGURE 1. The set $W \cup S$

We consider the compactification Z of the ray $L = (0, 1]$ illustrated in Figure 2. The continuum Z is contained in the set $[-1, 1] \times \{0\} \times [0, 1]$ and the remainder of Z is the triod $T = ([-\frac{1}{2}, \frac{1}{2}] \times \{0\} \times \{0\}) \cup (\{0\} \times \{0\} \times [0, \frac{1}{2}])$. Let $\alpha : L \to Z$ be an embedding such that $\alpha(1) = (0, 0, 1)$. Thus α is of the form $\alpha = (\alpha_1, 0, \alpha_2)$.

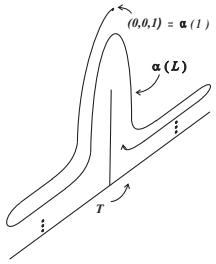

FIGURE 2. The continuum Z

Finally, let X be the continuum:

$$X = Z \cup S \cup \{(\alpha_1(z), y, \alpha_2(z)) : (0, y, z) \in W\}.$$

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^3$ be the map given by $\pi((x, y, z)) = (x, 0, z)$.

Note that the only subcontinua of T that are limits of subcontinua of X contained in $\alpha(L)$ are:
I. Subcontinua of one of the following arcs $([-\frac{1}{2}, 0] \times \{0\} \times \{0\}) \cup (\{0\} \times \{0\} \times [0, \frac{1}{2}])$ or $(\{0\} \times \{0\} \times [0, \frac{1}{2}]) \cup ([0, \frac{1}{2}] \times \{0\} \times \{0\})$; or subtriods of T of the form $(\{0\} \times \{0\} \times [0, \frac{1}{2}]) \cup ([a, b] \times \{0\} \times \{0\})$.

From this observation and the fact that $\pi(X \setminus (S \cup T)) \subset \alpha(L)$, the following claim is easy to prove.

Claim 1. If $\{A_n\}_{n=1}^{\infty}$ is a sequence in $C(X) \setminus C(T)$ converging to an element $A \in C(T)$, then A is of one of the forms described in I.

Let μ be a Whitney map for $C(X)$. Let $t_0 \in (0, 1)$ be such that $t_0 < \mu(T)$. We are going to prove that $\mu^{-1}(t_0)$ is decomposable. In order to prove that $\mu^{-1}(t_0)$ is decomposable, let E be a subtriod of T such that $\mu(E) = t_0$ and $(0, 0, \frac{1}{2}) \notin E$. By Claim 1, E is an interior point of the subcontinuum $C(T, t_0)$ (see Lemma 2.1). Since $C(T, t_0)$ is a proper subcontinuum of $\mu^{-1}(t_0)$, we conclude that $\mu^{-1}(t_0)$ is decomposable.
On the other hand, let $\mu(T) = t_1$. Clearly $\mu^{-1}(t_1)$ is homeomorphic to the Buckethandle continuum. Thus $\mu^{-1}(t_1)$ is indecomposable.

Therefore, decomposability is not an increasing Whitney property.

5. APOSYNDESIS

A continuum X is said to be *aposyndetic* provided that for every $p \neq q$ in X there exists a subcontinuum M such that $p \in \operatorname{int}_X(M)$ and $q \notin M$.

In [16, Proposition 9, p. 156], A. Petrus showed that aposyndesis is a Whitney property. In this section we construct a continuum X to show that aposyndesis is not an increasing Whitney property.

The Example:

Let $p_0 = (-\frac{1}{2}, -\frac{1}{2})$, $J_1 = \{(x, x) \in \mathbb{R}^2 : -\frac{1}{2} \leq x \leq 0\}$, $J_2 = \{(x, x) \in \mathbb{R}^2 : 1 \leq x \leq 2\}$ and $C = [0, 1] \times [0, 1]$.

For each $n \in \mathbb{N}$, let $L'_n = \bigcup\{[0, 1] \times \{\frac{k}{2n}\} : k \in \{0, 1, ..., 2n\}\}$ and $M'_n = \bigcup\{\{\frac{k}{2n}\} \times [0, 1] : k \in \{0, 1, ..., 2n\}\}$.

Let L_n (resp., M_n) be the unique arc contained in $J_1 \cup J_2 \cup (\{0, 1\} \times [0, 1]) \cup L'_n$ (resp., $J_1 \cup J_2 \cup ([0, 1] \times \{0, 1\}) \cup M'_n$) that joins p_0 and $(2, 2)$ and $L'_n \subset L_n$ (resp., $M'_n \subset M_n$).

Let $K_n = (L_n \times \{\frac{1}{2n-1}\}) \cup (M_n \times \{\frac{1}{2n}\}) \cup (\{(2, 2)\} \times [\frac{1}{2n}, \frac{1}{2n-1}]) \subset \mathbb{R}^3$. Let $C_1 = J_1 \cup J_2 \cup C$, and finally, let $X = (C_1 \times \{0\}) \cup (\bigcup\{K_n : n \in \mathbb{N}\}) \cup (\{p_0\} \times [0, 1]) \subset \mathbb{R}^3$ (see Figure 3).

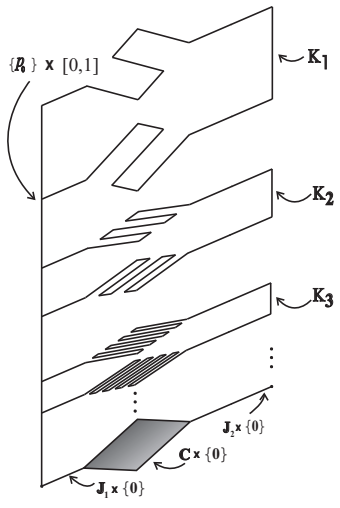

FIGURE 3. The continuum X

Let $R_0 = (J_1 \times \{0\}) \cup ([0, \frac{1}{2}] \times \{0\} \times \{0\})$ and $S_0 = (J_1 \times \{0\}) \cup (\{0\} \times [0, \frac{1}{2}] \times \{0\})$. For each $n \in \mathbb{N}$, let $R_n = (J_1 \times \{\frac{1}{2n-1}\}) \cup ([0, \frac{1}{2}] \times \{0\} \times \{\frac{1}{2n-1}\})$ and $S_n = (J_1 \times \{\frac{1}{2n}\}) \cup (\{0\} \times [0, \frac{1}{2}] \times \{\frac{1}{2n}\})$. Thus $R_n, S_n \subset K_n$ for each $n \in \mathbb{N}$ and the sequences $\{R_n\}_{n=1}^{\infty}$ and $\{S_n\}_{n=1}^{\infty}$, converge to R_0 and S_0, respectively. We consider:

$\mathcal{B}_1 = \big\{[0,1] \times \{y\} \times \{z\} \subset X : (y,z) \in \mathbb{R}^2\big\} \cup \big\{J_2 \times \{\frac{1}{n}\} : n \in \mathbb{N}\big\} \cup \{J_2 \times \{0\}\} \cup \big\{\{u\} \times [0,1] \times \{w\} \subset X : (u,w) \in \mathbb{R}^2\big\} \cup \{R_n\}_{n=0}^{\infty} \cup \{S_n\}_{n=0}^{\infty}$ and
$\mathcal{B}_2 = \big\{(C \times \{z\}) \cap X : z \in \{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}\big\}$.

Notice that $\mathcal{B}_1 \cup \mathcal{B}_2$ is a closed subset of $C(X)$. Let $0 < t_1 < t_2 < 1$. Define $\mu_1 : \mathcal{B}_1 \cup \mathcal{B}_2 \to \{t_1, t_2\}$ by $\mu_1(\mathcal{B}_i) = \{t_i\}$ for each $i \in \{1, 2\}$. It is easy to check that μ_1 is a Whitney map for $\mathcal{B}_1 \cup \mathcal{B}_2$. By Theorem 16.10 of [5], μ_1 can be extended to a Whitney map, μ, for $C(X)$.

We prove that $\mathcal{A} = \mu^{-1}(t_1)$ is aposyndetic. Let $A, B \in \mathcal{A}$ be such that $A \neq B$. We need to find a subcontinuum $\mathcal{B}$ of $\mathcal{A}$ such that $B \in \operatorname{int}_{\mathcal{A}}(\mathcal{B})$ and $A \notin \mathcal{B}$. Clearly X is locally connected at each point of $(X \setminus (C_1 \times \{0\})) \cup \{\{p_0\} \times \{0\}\}$. If the set B intersects $(X \setminus (C_1 \times \{0\})) \cup \{\{p_0\} \times \{0\}\}$, then B contains a point of connectedness im kleinen of X. Thus, by [5, Exercise 29.8, p. 239], $\mathcal{A}$ is connected im kleinen at B. Therefore, it is possible to find the continuum $\mathcal{B}$. Thus we may assume that $B \subset (C_1 \times \{0\}) \setminus \{\{p_0\} \times \{0\}\}$. First we analize the case that A is not contained in $C_1 \times \{0\}$. In this case there exists a point $a = (x, y, z) \in A$ such that $z > 0$. Let $n \in \mathbb{N}$ be such that $\frac{1}{n} < z$. Then the set $D = (C_1 \times [0, \frac{1}{n}]) \cap X$ is a subcontinuum of X such that $B \subset (C_1 \times [0, \frac{1}{n})) \cap X \subset \operatorname{int}_X(D)$ and A is not contained in D. Let $\mathcal{B} = C(D, t_1)$. Notice that $\mathcal{B}$ is a subcontinuum of $\mathcal{A}$ (see Lemma 2.1). Then $B \in \operatorname{int}_{\mathcal{A}}(\mathcal{B})$ and $A \notin \mathcal{B}$. Therefore, the proof in this case is finished.

Now, suppose that $A \subset C_1 \times \{0\}$. We need some auxiliary constructions and results. For each $n \in \mathbb{N}$, let $\mathcal{C}_n = C((L_n \times \{\frac{1}{2n-1}\}) \cup (\{(2,2)\} \times [\frac{1}{2n}, \frac{1}{2n-1}]) \cup (J_2 \times \{\frac{1}{2n}\}), t_1)$ and $\mathcal{D}_n = C(M_n \times \{\frac{1}{2n}\}, t_1)$. Then each of the sets $\mathcal{C}_n$ and $\mathcal{D}_n$ are Whitney levels of arcs, so they are also arcs (see [5, Theorem 31.1, p. 245]) and they intersect at $\{J_2 \times \{\frac{1}{2n}\}\}$. Thus the set $\mathcal{C}_n \cup \mathcal{D}_n$ is an arc that joins R_n and S_n.

Claim 1. The sequences $\{\mathcal{C}_n\}_{n=1}^{\infty}$ and $\{\mathcal{D}_n\}_{n=1}^{\infty}$ converge, respectively, to the sets:

$\mathcal{C}_0 = \{[0,1] \times \{y\} \times \{0\} : y \in [0,1]\} \cup C(([0,1] \times \{1\} \times \{0\}) \cup (J_2 \times \{0\}), t_1) \cup C(([0,1] \times \{0\} \times \{0\}) \cup (J_1 \times \{0\}), t_1)$ and

$\mathcal{D}_0 = \{\{x\} \times [0,1] \times \{0\} : x \in [0,1]\} \cup C(([1] \times [0,1] \times \{0\}) \cup (J_2 \times \{0\}), t_1) \cup C(([0] \times [0,1] \times \{0\}) \cup (J_1 \times \{0\}), t_1)$.

We only prove that the sequence $\{\mathcal{C}_n\}_{n=1}^{\infty}$ converges to $\mathcal{C}_0$; the proof for $\mathcal{D}_0$ is similar.

Clearly each one of the elements of $\mathcal{C}_0$ is the limit of a sequence $\{C_n\}_{n=1}^{\infty}$, where $C_n \in \mathcal{C}_n$ for each $n \in \mathbb{N}$.

Now, let $D \in \mathcal{A}$ be such that $D = \operatorname{Lim} D_k$, where, for each $k \in \mathbb{N}$, $D_k \in \mathcal{C}_{n_k}$ and $n_1 < n_2 < \cdots$. Then $D \subset C_1 \times \{0\}$. If $D \subset ([0,1] \times \{1\} \times \{0\}) \cup (J_2 \times \{0\})$ or $D \subset ([0,1] \times \{0\} \times \{0\}) \cup (J_1 \times \{0\})$, then, by definition, $D \in \mathcal{C}_0$.

Suppose that D is not contained in any of the sets $([0,1] \times \{1\} \times \{0\}) \cup (J_2 \times \{0\})$ and $([0,1] \times \{0\} \times \{0\}) \cup (J_1 \times \{0\})$. Then there exists a point $p = (u, v, 0) \in D$ such that $0 < v < 1$. If $(2,2,0) \in D$, then $D \subset J_2 \times \{0\}$ or $J_2 \times \{0\} \subset D$. Since both sets have the same value under μ, we have that $D = J_2 \times \{0\}$. This contradicts our assumption. Therefore $(2,2,0) \notin D$. Since $D \cap (\{(2,2)\} \times \mathbb{R}) = \emptyset$, we may assume that $D_k \cap (\{(2,2)\} \times \mathbb{R}) = \emptyset$ for each $k \in \mathbb{N}$. From the connectedness of D_k and since $D_k \subset (L_{n_k} \times \{\frac{1}{2n_k - 1}\}) \cup (J_2 \times \{\frac{1}{2n_k}\})$, we have that $D_k \subset L_{n_k} \times \{\frac{1}{2n_k - 1}\}$ or $D_k \subset J_2 \times \{\frac{1}{2n_k}\}$. Since D is not contained in $J_2 \times \{0\}$, we may assume that $D_k \subset L_{n_k} \times \{\frac{1}{2n_k - 1}\}$ for each $k \in \mathbb{N}$.

If there exists a point $(u_1, v_1, 0) \in D$ with $0 \leq v_1$ and $v_1 \neq v$, then $v < v_1$ or $v_1 < v$. We only analyse $v < v_1$ since the other case is similar. Choose numbers $v < v_2 < v_3 < v_1$. Since $D = \operatorname{Lim} D_k$, $(u_1, v_1, 0) \in \operatorname{int}_{\mathbb{R}^3}([-1,2] \times [v_3, 3] \times [-1,2]) \cap X$ and $(u, v, 0) \in \operatorname{int}_{\mathbb{R}^3}([-1,2] \times [-1, v_2] \times [-1,2]) \cap X$, we have that there exists $K \in \mathbb{N}$ such that, for each $k \geq K$, $D_k \cap ([-1,2] \times [-1, v_2] \times [-1,2]) \neq \emptyset$ and $D_k \cap ([-1,2] \times [v_3, 3] \times [-1,2]) \neq \emptyset$. We may assume that for each $k \geq K$, there is an element of the form $\frac{j_k}{2n_k}$ in the set (v_2, v_3). By the construction of L_{n_k} and the connectedness of D_k,

we have that $[0,1]\times\{\frac{j_k}{2n_k}\}\times\{\frac{1}{2n_k-1}\}\subset D_k$. We may assume that the sequence $\{\frac{j_k}{2n_k}\}_{k=1}^{\infty}$ has a subsequence converging to a number $v_0\in[v_2,v_1]$. Thus $[0,1]\times\{v_0\}\times\{0\}\subset D$. Since both sets have the same value under μ, we have that $D=[0,1]\times\{v_0\}\times\{0\}\in\mathcal{C}_0$. Hence $v=v_0=v_1$, which is a contradiction. This proves that there is not such a point $(u_1,v_1,0)\in D$. Hence $D\cap(C\times\{0\})\subset[0,1]\times\{v\}\times\{0\}$. From the connectedness of D, we have that $D\subset[0,1]\times\{v\}\times\{0\}$. Since both sets have the same value under μ, we have that $D=[0,1]\times\{v\}\times\{0\}\in\mathcal{C}_0$. This completes the proof that the sequence $\{\mathcal{C}_n\}_{n=1}^{\infty}$ converges to $\mathcal{C}_0$ and the proof of Claim 1.

Claim 2. $\mathcal{C}_0$ and $\mathcal{D}_0$ are two arcs which meet at $\{J_2\times\{0\}\}$. Thus $\mathcal{C}_0\cup\mathcal{D}_0$ is an arc joining R_0 and S_0.

It is easy to see that the set $\{[0,1]\times\{y\}\times\{0\}:y\in[0,1]\}$ is an arc joining $[0,1]\times\{0\}\times\{0\}$ and $[0,1]\times\{1\}\times\{0\}$. Since each of the sets $C(([0,1]\times\{1\}\times\{0\})\cup(J_2\times\{0\}),t_1)$ and $C(([0,1]\times\{0\}\times\{0\})\cup(J_1\times\{0\}),t_1)$ are Whitney levels of arcs, so they are also arcs (see [5, Theorem 31.1, p. 245]) joining $[0,1]\times\{1\}\times\{0\}$, $J_2\times\{0\}$ and R_0, $[0,1]\times\{0\}\times\{0\}$, respectively. Hence the set $\mathcal{C}_0$ is an arc that joins R_0 and $J_2\times\{0\}$. Similarly $\mathcal{D}_0$ is an arc that joins S_0 and $J_2\times\{0\}$. This completes the proof of Claim 2.

For each $n\in\mathbb{N}\cup\{0\}$, let $<_n$ be the natural order in the arc $\mathcal{C}_n\cup\mathcal{D}_n$ in which $R_n<_n S_n$. Given $D,E\in\mathcal{C}_n\cup\mathcal{D}_n$, with $D\leq_n E$, denote by $[D,E]_n$ the set $\{F\in\mathcal{C}_n\cup\mathcal{D}_n:D\leq_n F\leq_n E\}$.

The following claim is clear.

Claim 3. Let $\{D_n\}_{n=1}^{\infty}$ and $\{E_n\}_{n=1}^{\infty}$ be sequences in $\mathcal{C}_n\cup\mathcal{D}_n$ such that $\operatorname{Lim} D_n=D_0$ and $\operatorname{Lim} E_n=E_0$. If $D_n\leq_n E_n$ for each $n\in\mathbb{N}$, then $D_0\leq_0 E_0$.

From Claim 3, the following claim is immediate.

Claim 4. Let $D_0\in\mathcal{C}_0\cup\mathcal{D}_0$ and let $\{D_n\}_{n=1}^{\infty}$ be a sequence of elements of $C(X)$ such that $\operatorname{Lim} D_n=D_0$ and $D_n\in\mathcal{C}_n\cup\mathcal{D}_n$ for each $n\in\mathbb{N}$. Then the sequences $\{[R_n,D_n]_n\}_{n=1}^{\infty}$ and $\{[D_n,S_n]_n\}_{n=1}^{\infty}$ converge, respectively, to $[R_0,D_0]_0$ and $[D_0,S_0]_0$.

Now we return to the proof of the aposyndesis of $\mathcal{A}$. Recall that we are assuming that $A \subset C_1 \times \{0\}$ and $B \subset (C_1 \times \{0\}) \setminus \{\{p_0\} \times \{0\}\}$. Since C_1 is locally connected, $\mathcal{A} \cap C(C_1 \times \{0\})$ is locally connected (see [5, Exercise 29.8, p. 239]). Thus there exists a subcontinuum $\mathcal{B}_1$ of $\mathcal{A} \cap C(C_1 \times \{0\})$ such that $B \in \operatorname{int}_{\mathcal{A}\cap C(C_1\times\{0\})}(\mathcal{B}_1)$ and $A \notin \mathcal{B}_1$. Let $\mathcal{U}$ be an open subset of $C(X)$ such that $B \in \mathcal{U} \cap \mathcal{A} \cap C(C_1 \times \{0\}) \subset \mathcal{B}_1$. We consider two cases.

Case 1. Suppose that $B \notin \mathcal{C}_0 \cup \mathcal{D}_0$. In this case $B \notin \bigcup\{\mathcal{C}_n \cup \mathcal{D}_n : n \in \mathbb{N} \cup \{0\}\}$. In particular, $B \neq J_2 \times \{0\}$. If $(2, 2, 0) \in B$, then $B \subset J_2 \times \{0\}$ or $J_2 \times \{0\} \subset B$. Since both sets have the same value under μ, $B = J_2 \times \{0\} \in \mathcal{C}_0 \cup \mathcal{D}_0$, a contradiction. Thus $(2, 2, 0) \notin B$ and $B \cap (\{(2, 2)\} \times \mathbb{R}) = \emptyset$. Hence we may assume that no element of $\mathcal{U}$ intersects $(\{(2, 2)\} \times \mathbb{R}) \cup (\{p_0\} \times [0, 1])$. Since $\bigcup\{\mathcal{C}_n \cup \mathcal{D}_n : n \in \mathbb{N} \cup \{0\}\}$ is closed, we may also assume that $\mathcal{U} \cap (\bigcup\{\mathcal{C}_n \cup \mathcal{D}_n : n \in \mathbb{N} \cup \{0\}\}) = \emptyset$.

Under these conditions it is easy to see that $\mathcal{U} \subset C(C_1 \times \{0\})$. Thus $B \in \mathcal{U} \cap \mathcal{A} = \mathcal{U} \cap \mathcal{A} \cap C(C_1 \times \{0\}) \subset \mathcal{B}_1$. Therefore $B \in \operatorname{int}_{\mathcal{A}}(\mathcal{B}_1)$. Since $A \notin \mathcal{B}_1$, the proof in this case is finished.

Case 2. Suppose that $B \in \mathcal{C}_0 \cup \mathcal{D}_0$. Note that $A \notin [R_0, B]_0$ or $A \notin [B, S_0]_0$ (it can happen that A does not belong to any of these sets). We only consider the case that $A \notin [R_0, B]_0$, the other one is similar. Then there exists $D_0 \in \mathcal{C}_0 \cup \mathcal{D}_0$ such that $B <_0 D_0$ and $A \notin [R_0, D_0]_0$. Let $\{D_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{A}$ such that $\operatorname{Lim} D_n = D_0$ and $D_n \in \mathcal{C}_n \cup \mathcal{D}_n$ for each $n \in \mathbb{N}$. By Claim 4, $A \notin \bigcup\{[R_n, D_n]_n : n \in \mathbb{N} \cup \{0\}\}$.

Let $\pi : X \to \mathbb{R}^3$ be given by $\pi((x, y, z)) = (x, y, 0)$. For each $n \in \mathbb{N}$, let $T_n = R_n \cup R_0 \cup (\{p_0\} \times [0, 1])$. Observe that T_n is an arc or a simple triod. By Theorem 33.1 of [5], we have that $\mathcal{A} \cap C(T_n)$ is arcwise connected. Thus there exists an arc $\gamma_n \subset \mathcal{A} \cap C(T_n)$ joining R_n and R_0. Since $A \neq R_0$ and $\mu(A) = \mu(R_0)$, $A \nsubseteq R_0$. Notice that, if $E \in \bigcup\{\gamma_n : n \in \mathbb{N}\}$, then $\pi(E) \subset R_0$. Hence, since $\pi(A) = A$, $A \notin \operatorname{cl}_{C(X)}(\bigcup\{\gamma_n : n \in \mathbb{N}\})$. Let $\mathcal{B} = (\mathcal{B}_1 \cup \operatorname{cl}_{C(X)}(\bigcup\{\gamma_n : n \in \mathbb{N}\})) \cup (\bigcup\{[R_n, D_n]_n : n \in \mathbb{N} \cup \{0\}\})$. Since $B \in \mathcal{B}_1 \cap [R_0, D_0]_0$, $\mathcal{B}$ is a subcontinuum of $\mathcal{A}$. Clearly $A \notin \mathcal{B}$. Let $\mathcal{M} = \{F \in \mathcal{A} : F \cap (\{p_0\} \times [0, 1]) \neq \emptyset\} \cup (\bigcup\{[D_n, S_n]_n : n \in \mathbb{N} \cup \{0\}\})$. By Claim 4, $\mathcal{M}$ is a closed subset of $\mathcal{A}$ and $B \notin \mathcal{M}$. Thus we may assume that $\mathcal{U} \cap \mathcal{M} = \emptyset$. Finally, we prove that $\mathcal{U} \cap \mathcal{A} \subset \mathcal{B}$.

Let $E \in \mathcal{U} \cap \mathcal{A}$. Then $E \cap (\{p_0\} \times [0,1]) = \emptyset$. This implies that $E \subset C_1 \times \{0\}$ or $E \in [R_n, S_n]_n$ for some $n \in \mathbb{N} \cup \{0\}$. If $E \subset C_1 \times \{0\}$, then $E \in \mathcal{B}_1 \subset \mathcal{B}$. If $E \in [R_n, S_n]_n$, since $E \notin \mathcal{M}$, we have that $E \in \bigcup\{[R_n, D_n]_n : n \in \mathbb{N} \cup \{0\}\} \subset \mathcal{B}$. Hence, $\mathcal{U} \cap \mathcal{A} \subset \mathcal{B}$. So $B \in \text{int}_{\mathcal{A}}(\mathcal{B})$. This completes the analysis of this case.

This finishes the proof that $\mathcal{A}$ is aposyndetic.

Now, we see that $\mathcal{C} = \mu^{-1}(t_2)$ is not aposyndetic. Recall that, for each $z \in \{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}$, $\mu((C \times \{z\}) \cap X) = t_2$. For each $n \in \mathbb{N}$, let $Z_n = (K_n \setminus ((J_1 \times \{\frac{1}{2n-1}\}) \cup (J_1 \times \{\frac{1}{2n}\}))) \cup \{(0, 0, \frac{1}{2n-1}), (0, 0, \frac{1}{2n})\}$. Notice that Z_n is an arc and, by Theorem 31.1 of [5], $C(Z_n) \cap \mathcal{C}$ is an arc joining the elements $(C \times \{\frac{1}{2n-1}\}) \cap X$ and $(C \times \{\frac{1}{2n}\}) \cap X$. Fix an element $Q_n \in C(Z_n) \cap \mathcal{C}$ such that $(2, 2, \frac{1}{2n}) \in Q_n$. Let Q be a limit element of the sequence $\{Q_n\}_{n=1}^{\infty}$. Then $Q \subset C_1 \times \{0\}$ and $(2, 2, 0) \in Q$. Notice that, for each $n \in \mathbb{N}$, $\{(C \times \{\frac{1}{2n-1}\}) \cap X, (C \times \{\frac{1}{2n}\}) \cap X\}$ separates $\{Q_n\}$ and $\{Q\}$ in $\mathcal{C}$ (for the definition see [11, Exercise 5.30, p. 86]).

In order to show that $\mathcal{C}$ is not aposyndetic, suppose the contrary, then there exists a subcontinuum $\mathcal{D}$ of $\mathcal{C}$ such that $Q \in \text{int}_{\mathcal{C}}(\mathcal{D})$ and $C \times \{0\} \notin \mathcal{D}$. Then there exists $M \in \mathbb{N}$ such that $\{(C \times \{\frac{1}{2n-1}\}) \cap X, (C \times \{\frac{1}{2n}\}) \cap X\} \cap \mathcal{D} = \emptyset$ for each $n \geq M$. Let $n \geq M$ be such that $Q_n \in \mathcal{D}$. Since $\{(C \times \{\frac{1}{2n-1}\}) \cap X, (C \times \{\frac{1}{2n}\}) \cap X\}$ separates $\{Q_n\}$ and $\{Q\}$ in $\mathcal{C}$ and $\mathcal{D}$ is connected, we have that $\{(C \times \{\frac{1}{2n-1}\}) \cap X, (C \times \{\frac{1}{2n}\}) \cap X\} \cap \mathcal{D} \neq \emptyset$. This contradiction proves that $\mathcal{C}$ is not aposyndetic.

Therefore aposyndesis is not an increasing Whitney property.

In Section 5, we showed that aposyndesis is not an increasing Whitney property, in this direction, we have the following question:

Is the property of being finitely aposyndetic an increasing Whitney property? (A continuum X is said to be *finitely aposyndetic*, if for every $p \in X$ and any finite subset F of X such that $p \notin F$, there exists a subcontinuum M of X such that $p \in \text{int}_X(M)$ and $F \cap M = \emptyset$).

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