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CONTINUOUS SINGULAR COHOMOLOGY AND FIBRATIONS

L. D. MDZINARISHVILI

Abstract. Let Top be the category of topological spaces and continuous maps, Ab_T be the category of topological abelian groups and continuous homomorphisms, Ab be the category of abelian groups and homomorphisms.

In the work [5], the cohomology functor of two arguments $h_s^* : \mathrm{Top} \times \mathrm{Ab}_T \to \mathrm{Ab}$ contravariant in the first argument and covariant in the second is defined. This functor is called continuous singular cohomology.

In the present work, continuous singular cohomology with respect to the second argument is studied. It is proved that if a continuous homomorphism is a fibration, then for any topological space and $q \geq 2$ there is an exact cohomology sequence (theorem 1). In particular, if a continuous homomorphism is a covering projection, then for $q \geq 2$ cohomology with coefficients in the covering space and cohomology with coefficients in the base space are isomorphic (corollary 2).

If given an inverse sequence of fibrations in the category Ab_T, then there is a connection by short exact sequence between continuous singular cohomology with coefficients in the inverse limit of this sequence and the inverse limit of the sequence of continuous singular cohomology generated by this sequence (theorem 4).

In particular, if given an inverse sequence of covering projections, then for $q \neq 1$ there is an isomorphism (corollary 10).

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Let X be a topological space, $F(X) = \{F_q(X),\ \overline{e}_q^{\,i}\}$ be a topological simplicial set of X, where $F_q(X)$ is the space of all continuous maps from Δ_q to X given the compact-open topology, and $\overline{e}_q^{\,i} : F_q(X) \to F_{q-1}(X)$ is the continuous map induced by the standard face inclusion $e_q^i : \Delta_{q-1} \to \Delta_q$, $0 \le i \le q$. Let G be a topological abelian group. Denote by $M^q(X,G)$ the group of all continuous maps $\varphi : F_q(X) \to G$. The cochain complex $M^*(X,G) = \{M^q(X,G),\ \delta^q\}$, where $\delta^q(\psi) = \sum\limits_{i=0}^{q}(-1)^i \psi \overline{e}_q^{\,i}$, $\psi \in M^q(X,G)$, is generated by $F(X)$. The cohomology of the cochain complex $M^*(X,G)$ is called continuous singular cohomology and is denoted by $h_s^*(X,G)$.

Denote by $[F_q(X),G]$ the group of homotopy classes of continuous maps $F_q(X) \to G$ and by $\widetilde{M}^q(X,G)$ the subgroup of $M^q(X,G)$ consisting of all continuous maps $\varphi : F_q(X) \to G$ such that φ is homotopic to the constant map $\overline{g}_0 : F_q(X) \to G$, $\overline{g}_0(F_q(X)) = g_0$, where g_0 is the identity element of G. Therefore there is an exact sequence of cochain complexes

$$0 \longrightarrow \widetilde{M}^*(X,G) \longrightarrow M^*(X,G) \longrightarrow [F_*(X),G] \longrightarrow 0. \tag{1}$$

Denote by $\widetilde{h}^*(X,G)$ cohomology of the cochain complex $\widetilde{M}^*(X,G)$ and by $\pi^*(X,G)$ cohomology of the cochain complex $[F_*(X),G]$.

The sequence (1) induces an exact cohomology sequence

$$\begin{aligned} \cdots \longrightarrow \widetilde{h}^q(X,G) \longrightarrow h_s^q(X,G) \longrightarrow \pi^q(X,G) \\ \longrightarrow \widetilde{h}^{q+1}(X,G) \longrightarrow \cdots . \end{aligned} \tag{2}$$

Lemma 1. *If X is a Hausdorff space, then there are:*
a) $\pi^q(X,G) = 0$ *for* $q > 0$*;*
b) $\pi^0(X,G) = [X,G]$ *for* $q = 0$.

Proof. For $q \ge 0$ define the continuous map $j_q : X \to F_q(X)$ by $j_q(x) = T_x$, where $T_x : \Delta_q \to X$ is the constant map, $T_x(\Delta_q) = x$. Let $a = d^0 \in \Delta_q$, d^0 be the 0-th vertex of Δ_q. For $q \ge 0$ define the continuous map $p_q : F_q(X) \to X$ by $p_q(T) = T(a)$.

Clearly the composite $p_q j_q$ is the identity map.

For $0 \le i \le q$ there is the equality $j_{q-1} = \overline{e}\,^i_q j_q$, which induces the commutative diagram

$$
\text{(3)} \qquad
\begin{array}{ccc}
[F_{q-1}(X), G] & \xrightarrow{\;\widetilde{e}\,^i_q\;} & [F_q(X), G] \\
 & {\scriptstyle \overline{j}_{q-1}} \searrow \quad \swarrow {\scriptstyle \overline{j}_q} & \\
 & [X, G] &
\end{array}
$$

Let us show that $\overline{j}_q$ is an isomorphism, $q \ge 0$:

1) $\overline{j}_q$ is a monomorphism. Let $[\psi] \in [F_q(X), G]$ and $\overline{j}_q[\psi] = 0$. Then $\psi j_q \sim \overline{g}_0$ and $\psi j_q p_q \sim \overline{g}_0 p_q = \overline{g}_0$. By proposition III.9.10 from [3], the space $j_q(X)$ is a strong deformation retract of $F_q(X)$, $q \ge 0$. Hence $j_q p_q \sim 1_{F_q(X)}$ and $\psi = \psi 1_{F_q(X)} \sim \psi j_q p_q \sim \overline{g}_0$. Therefore $[\psi] = [\overline{g}_0]$.

2) $\overline{j}_q$ is an epimorphism. Let $[\varphi] \in [X, G]$. Then $[\varphi p_q] \in [F_q(X), G]$. Since $p_q j_q = 1_X$, then $\overline{j}_q[\varphi p_q] = [\varphi p_q j_q] = [\varphi]$.

Therefore $\overline{j}_q$ is an isomorphism, $q \ge 0$.

From the commutative diagram (3) it follows that for $0 \le i \le q$ the homomorphism $\widetilde{e}^i_q$ is an isomorphism and for $i \ne j$ there is the equality $\widetilde{e}^i_q = \widetilde{e}^j_q$. Hence for $q = 2n$ the coboundary operator $\widetilde{\delta}^{q+1} : [F_q(X), G] \to [F_{q+1}(X), G]$ is trivial and for $q = 2n+1$ it is an isomorphism. □

Corollary 1. 1) *If X is a Hausdorff space, then there are*

a) *for $q \ge 2$ an isomorphism*

$$\widetilde{h}^q(X, G) \approx h^q_s(X, G);$$

b) *for $q = 0, 1$ an exact sequence*

$$0 \to \widetilde{h}^0(X, G) \to h^0_s(X, G) \to [X, G] \to \widetilde{h}^1(X, G) \to h^1_s(X, G) \to 0\,.$$

2) *If X is a path connected space and G is a path connected abelian group, then there is an exact sequence*

$$0 \longrightarrow [X, G] \longrightarrow \widetilde{h}^1(X, G) \longrightarrow h^1_s(X, G) \longrightarrow 0$$

and

$$\widetilde{h}^0(X, G) = h^0_s(X, G) = G.$$

(In fact the very last equality does not require path connectedness of G.)

Proof. The statement 1) follows from lemma 1 and the exact sequence (2).

2) We have $h_s^0(X,G)=Z^0(X,G)$ (here and everywhere below Z^* denote the groups of singular cocycles). For any $T\in F_1(X)$ and $\varphi\in Z^0(X,G)$ there is $\varphi\overline{e}\,_1^1(T)=\varphi\overline{e}\,_1^0(T)$. Since X is path connected, for arbitrary two points x_1 and x_2 there exists $\tau: I\to X$ such that $\tau(0)=x_1$, $\tau(1)=x_2$. Because I and Δ_1 are homeomorphic spaces, there is a continuous map $T:\Delta_1\to X$ such that $\overline{e}\,_1^0(T)=x_1$, $\overline{e}\,_1^1(T)=x_2$. Hence, if $\varphi\in Z^0(X,G)$, then φ is a constant map. Because G is path connected, $\varphi\sim\overline{e}_0$, $\varphi\in Z^0(X,G)$. A homotopy $H: X\times I\to G$ is defined by $H=hp_I$, where $p_I: X\times I\to I$ is a projection and $h: I\to G$ such that $h(0)=g$, $h(1)=g_0$, $g=\varphi(X)$. Therefore we have the equality $h_s^0(X,G)=Z^0(X,G)=\widetilde{Z}^0(X,G)=\widetilde{h}^0(X,G)$.

The statement 2) follows from exact sequence (2) and the statement 1). □

Let $p:E\to B$ be a continuous homomorphism, where E and B are topological abelian groups and e_0, b_0 are identity elements of E, B respectively.

The topological group $F=p^{-1}(b_0)$ will be called the fiber if p is a fibration.

A homomorphism p induces cochain homomorphisms

$$p^*: M^*(X,E)\to M^*(X,B),$$
$$\widetilde{p}^*: \widetilde{M}^*(X,E)\to \widetilde{M}^*(X,B),$$

which induce homomorphisms

$$p_s^*: h_s^*(X,E)\to h_s^*(X,B),$$
$$\widetilde{p}^*: \widetilde{h}^*(X,E)\to \widetilde{h}^*(X,B),$$

respectively.

Theorem 1. *For any topological space X and a fibration $p:E\to B$ for $q\geq 2$ there is an exact cohomology sequence*

$$\cdots\to h_s^q(X,F)\to h_s^q(X,E)\to h_s^q(X,B)\to h_s^{q+1}(X,F)\to\cdots.$$

Proof. A fibration p induces an epimorphism $\widetilde{p}^*: \widetilde{M}(X,E)\to \widetilde{M}^*(X,B)$. Hence the exact sequence of cochain complexes

$$0\longrightarrow \operatorname{Ker}\widetilde{p}^*\longrightarrow \widetilde{M}^*(X,E)\longrightarrow \widetilde{M}^*(X,B)\longrightarrow 0$$

induces an exact cohomology sequence

$$(4)\qquad \cdots \longrightarrow K^q \longrightarrow \widetilde{h}^q(X,E) \longrightarrow \widetilde{h}^q(X,B) \longrightarrow \cdots ,$$

where $K^q = H^q(\operatorname{Ker}\widetilde{p}^*)$.

Clearly for $q \geq 0$ there is $\operatorname{Ker}\widetilde{p}^q = M^q(X,F) \cap \widetilde{M}^q(X,E)$. Hence in the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Ker}\widetilde{p}^q & \longrightarrow & M^q(X,F) & \longrightarrow & \Phi^q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi^q} & & \\ 0 & \longrightarrow & \widetilde{M}^q(X,E) & \longrightarrow & M^q(X,E) & \longrightarrow & [F_q(X),E] & \longrightarrow & 0 \end{array}$$

the homomorphism φ^q is a monomorphism, where $\Phi^q = M^q(X,F) / \operatorname{Ker}\widetilde{p}^q$.

There is a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & \widetilde{M}^q(X,F) & \longrightarrow & M^q(X,F) & \longrightarrow & [F_q(X),F] & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow{\scriptstyle t^q} & & \\ 0 & \longrightarrow & \operatorname{Ker}\widetilde{p}^q & \longrightarrow & M^q(X,F) & \longrightarrow & \Phi^q & \longrightarrow & 0, \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

where t^q is an epimorphism.

Consider a commutative diagram

$$
\begin{array}{ccc}
[F_{q-1}(X),F] & \xrightarrow{\widetilde{e}^i_q} & [F_q(X),F] \\
\Big\downarrow{\scriptstyle t^{q-1}} & & \Big\downarrow{\scriptstyle t^q} \\
\Phi^{q-1} & \xrightarrow{\overline{\overline{e}}{}^i_q} & \Phi^q \\
\Big\downarrow{\scriptstyle \varphi^{q-1}} & & \Big\downarrow{\scriptstyle \varphi^q} \\
[F_{q-1}(X),E] & \xrightarrow[\widetilde{e}^i_q]{} & [F_q(X),E].
\end{array}
\tag{5}
$$

By lemma 1, $\widetilde{e}^i_q$ is an isomorphism, $0 \le i \le q$. Since φ^{q-1} is a monomorphism, from the diagram (5) it follows that $\overline{\overline{e}}{}^q$ is a monomorphism. Since t^q is an epimorphism, $q \ge 0$, from the diagram (5) it follows that $\overline{\overline{e}}{}^q$ is an epimorphism. Hence $\overline{\overline{e}}{}^q_i$ is an isomorphism, $0 \le i \le q$. Since $\widetilde{e}^i_q = \widetilde{e}^j_q$, $i \ne j$ (lemma 1), $\overline{\overline{e}}{}^i_q = \overline{\overline{e}}{}^j_q$. Therefore for $q = 2n$ the coboundary operator $\delta^{q+1} : \Phi^q \to \Phi^{q+1}$ is trivial, and for $q = 2n+1$ it is an isomorphism. Hence $H^q(\Phi^*) = 0$, $q > 0$, and for $q = 0$ there is

$$H^0(\Phi^*) = \Phi^0. \tag{6}$$

An exact sequence of cochain complexes

$$0 \longrightarrow \operatorname{Ker} \widetilde{p}^* \longrightarrow M^*(X,F) \longrightarrow \Phi^* \longrightarrow 0$$

generates an exact cohomology sequence

$$\cdots \longrightarrow K^q \longrightarrow h^q_s(X,F) \longrightarrow H^q(\Phi^*) \longrightarrow \cdots .$$

Using the equality (6) we have

1) for $q \ge 2$ an isomorphism

$$K^q \approx h^q_s(X,F); \tag{7}$$

2) for $q = 0, 1$ an exact sequence

$$0 \longrightarrow K^0 \longrightarrow h^0_s(X,F) \longrightarrow \Phi^0 \longrightarrow K^1 \longrightarrow h^1_s(X,F) \longrightarrow 0\,. \tag{8}$$

Now the result follows from (4), (7) and corollary 1. □

Lemma 2. *For any topological space X and a discrete abelian group G there are:*

a) $h^q_s(X,G) = 0$ *for* $q \ge 1$*;*

b) $h^0_s(X,G) = [X,G]$ *for* $q = 0$.

Proof. Consider a cochain complex $\widetilde{M}^*(X,G)$. We show that for $q \geq 0$ one has

$$\widetilde{M}^q(X,G) = 0\,. \tag{9}$$

Let $\varphi : F_q(X) \to G$ and $\varphi \sim \overline{g}_0$, where $\overline{g}_0$ is the constant map, $\overline{g}_0(F_q(X)) = g_0$, g_0 being the identity element of G.

Since $\varphi \sim \overline{g}_0$, there exists a continuous map $H : F_q(X) \times I \to G$ such that $H(T,0) = g_0$, $H(T,1) = \varphi(T)$, $T \in F_q(X)$. For each $T \in F_q(X)$ the subset $T \times I \subset F_q(X) \times I$ is connected, therefore the restriction $H_T = H|T \times I$ is a constant map, because G is a discrete group and $H_T(T \times I)$ is connected. Hence $H_T(T,0) = H_T(T,1) = g_0 = \varphi(T)$ and $\varphi = \overline{g}_0$. Therefore $\widetilde{h}^q(X,G) = 0$ for $q \geq 0$.

From the exact sequence (1) and equality (9) it follows that for $q \geq 0$ there is an isomorphism

$$h^q_s(X,G) \approx \pi^q(X,G)\,.$$

The result now follows from lemma 1. □

Corollary 2. *If the fiber F of a fibration $p : E \to B$ is a discrete subgroup, then there are:*

1_{a}) *for any topological space X and $q \geq 2$ an isomorphism*

$$p^q_s : h^q_s(X,E) \approx h^q_s(X,B);$$

1_{b}) *for $q = 1$ an epimorphism*

$$p^1_s : h^1_s(X,E) \to h^1_s(X,B) \to 0.$$

If moreover E is contractible, then for any path connected space X,

2_{a}) *for $q = 1$ there is an exact sequence*

$$0 \longrightarrow [X,B] \longrightarrow h^1_s(X,E) \longrightarrow h^1_s(X,B) \longrightarrow 0;$$

2_{b}) *for $q = 0$ an exact sequence*

$$0 \longrightarrow [X,F] \longrightarrow h^0_s(X,E) \longrightarrow h^0_s(X,B) \longrightarrow 0.$$

Proof. 1_{a}) Since $\operatorname{Ker} p = F$ is a discrete abelian group, by lemma 2 and isomorphism (7) for $q \geq 2$ we have

$$K^q = h^q_s(X,F) = 0. \tag{10}$$

Hence the statement 1_{a}) now follows from theorem 1.

$1_{\rm b}$) From exact sequence (4) and the equality (10) it follows that $\widetilde{p}^1 : \widetilde{h}^1(X, E) \to \widetilde{h}^1(X, B)$ is an epimorphism. Using corollary 1 we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widetilde{h}^0(X,E) & \longrightarrow & h^0_s(X,E) & \longrightarrow & [X,E] \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \widetilde{h}^0(X,B) & \longrightarrow & h^0_s(X,B) & \longrightarrow & [X,B] \\ \longrightarrow & \widetilde{h}^1(X,E) & \longrightarrow & h^1_s(X,E) & \longrightarrow & 0 & \\ & \downarrow{\scriptstyle \widetilde{p}^1} & & \downarrow{\scriptstyle p^1_s} & & & \\ \longrightarrow & \widetilde{h}^1(X,B) & \longrightarrow & h^1_s(X,B) & \longrightarrow & 0. & \end{array} \tag{11}$$

Because $\widetilde{p}^1$ is an epimorphism, p^1_s is an epimorphism.

$2_{\rm a}$) Since E is a contractible space, E and B are path connected spaces and, by corollary $1_{2)}$, there are an isomorphism

$$\widetilde{h}^0(X, B) \approx h^0_s(X, B)$$

and an exact sequence

$$0 \longrightarrow [X, B] \longrightarrow \widetilde{h}^1(X, B) \longrightarrow h^1_s(X, B) \longrightarrow 0. \tag{12}$$

Because E is a contractible space, $[X, E] = 0$ and from the diagram (11) it follows that there is an isomorphism

$$\widetilde{h}^1(X, E) \approx h^1_s(X, E). \tag{13}$$

Since $\varphi^0 : \Phi^0 \to [X, E]$ is a monomorphism (notation as in the proof of theorem 1), $\Phi^0 = 0$. Hence from exact sequence (8) and lemma 2 it follows that

$$K^0 \approx h^0_s(X, F) = [X, F] \tag{14}$$

and

$$K^1 \approx h^1_s(X, F) = 0. \tag{15}$$

Using the exact sequence (4) and equalities (10) and (15), we have an isomorphism

$$\widetilde{p}^1 : \widetilde{h}^1(X, E) \xrightarrow{\approx} \widetilde{h}^1(X, B). \tag{16}$$

The statement $2_{\rm a}$) follows from the exact sequence (12) and isomorphisms (13) and (16). The statement $2_{\rm b}$) follows from corollary $1_{2)}$, the exact sequence (4) and isomorphisms (14) and (15). □

Lemma 3. *If X is a topological space and G_1 and G_2 are topological abelian groups, then there is an isomorphism of cochain complexes*

$$M^*(X, G_1 \times G_2) \approx M^*(X, G_1) \oplus M^*(X, G_2).$$

Proof. Since for $q \geq 0$ there are natural homomorphisms

$$p^q_\lambda : M^q(X, G_1 \times G_2) \to M^q(X, G_\lambda)$$

and

$$i^q_\lambda : M^q(X, G_\lambda) \to M^q(X, G_1 \times G_2), \quad \lambda = 1, 2,$$

such that 1) $p^q_\lambda i^q_\lambda = 1^q_\lambda$; 2) $p^q_\mu i^q_\lambda = 0$, $\mu \neq \lambda$; 3) $i^q_1 p^q_1 + i^q_2 p^q_2 = 1^q$, then by proposition 1.2.16 [1] there is an isomorphism

$$p^q : M^q(X, G_1 \times G_2) \approx M^q(X, G_1) \oplus M^*(X, G_2)$$

defined by $p^q(\varphi) = (p^q_1(\varphi), p^1_2(\varphi))$. Because $\delta^{q+1}_\lambda p^q_\lambda = p^{q+1}_\lambda \delta^{q+1}$, one has $\delta^{q+1} p^q = p^{q+1} \delta^{q+1}$. □

Corollary 3. *There is an isomorphism*

$$h^*_s(X, G_1 \times G_2) \approx h^*_s(X, G_1) \oplus h^*_s(X, G_2).$$

Example 1. Let $E = B \times F$ be a direct product of topological abelian groups B and F. Then $p : E \to B$ is a fibration [7], where $p = p_B$ is a projection. By corollary 3 there is an isomorphism

$$h^*_s(X, E) \approx h^*_s(X, B) \oplus h^*_s(X, F). \tag{17}$$

If F is a discrete abelian group, then p is a covering projection [6]. Using lemma 2 and an isomorphism (17), we have for $q \geq 1$ an isomorphism

$$h^q_s(X, E) \approx h^q_s(X, B)$$

and for $q = 0$ an isomorphism

$$h^0_s(X, E) \approx h^0_s(X, B) \oplus [X, F].$$

Example 2. Let $p : R \to S^1$ be the exponential map. Then we have:

1) for $q \geq 2$ an isomorphism

$$p^q_s : h^q_s(X, R) \approx h^q_s(X, S^1);$$

2) for $q = 1$ an epimorphism

$$p^1_s : h^1_s(X, R) \to h^1_s(X, S^1) \to 0;$$

3) if X is a path connected space, then for $q=1$ there is an exact sequence

$$0 \longrightarrow [X,S^1] \longrightarrow h_s^1(X,R) \longrightarrow h_s^1(X,S^1) \longrightarrow 0;$$

for $q=0$ there is an exact sequence

$$0 \longrightarrow [X,Z] \longrightarrow h_s^0(X,R) \longrightarrow h_s^0(X,S^1) \longrightarrow 0.$$

In particular (see Corollary 14 of [5]), if a) $X=S^1$, then

$$h_s^q(S^1,S^1)=\begin{cases} S^1, & \text{if } q=0,1,\\ 0, & \text{if } q\neq 0,1;\end{cases}$$

b) $X=S^m$, $m\geq 2$, then

$$h_s^q(S^m,S^1)=\begin{cases} R, & \text{if } q=m,\\ 0, & \text{if } q\neq 0,m\\ S^1, & \text{if } q=0.\end{cases}$$

Let G be a topological abelian group. Consider the fibration $p:PG\to G$, where PG is the space of continuous maps $\tau: I\to G$, $\tau(0)=g_0$, topologized by the compact-open topology and the fiber $F=\Omega G$ is the loops space.

Corollary 4. 1) *For any topological space X and the fibration $p:PG\to G$ there is an exact cohomology sequence*

$$\begin{aligned} &0 \longrightarrow h_s^0(X,\Omega G) \longrightarrow h_s^0(X,PG) \longrightarrow \widetilde{h}^0(X,G) \\ &\longrightarrow h_s^1(X,\Omega G) \longrightarrow h_s^1(X,PG) \longrightarrow \widetilde{h}^1(X,G) \longrightarrow h_s^2(X,\Omega G) \\ &\longrightarrow \cdots \longrightarrow h_s^q(X,\Omega G) \longrightarrow h_s^q(X,PG) \longrightarrow h_s^q(X,G) \longrightarrow \cdots . \end{aligned} \tag{18}$$

2) *If X and G are path connected spaces, then*
a) $K^0\approx \Omega G$*;*
b) $\widetilde{h}^0(X,PG)\approx PG$*;*
c) $\widetilde{h}^0(X,G)\approx G$*;*
d) *there is a monomorphism*

$$0 \longrightarrow h_s^1(X,\Omega G) \longrightarrow h_s^1(X,PG);$$

e) *there is an exact sequence*

$$0 \longrightarrow [X,G] \longrightarrow \widetilde{h}^1(X,G) \longrightarrow h_s^1(X,G) \longrightarrow 0.$$

Proof. 1) Since PG is a contractible space, $[X, PG] = 0$ and $\Phi^0 = 0$. Hence by corollary 1 there is an isomorphism $\widetilde{h}^q(X, PG) \approx h^q_s(X, PG)$, $q = 0, 1$ and, using the exact sequence (8), we have isomorphisms

$$K^0 \approx h^0_s(X, \Omega G), \quad K^1 \approx h^1_s(X, \Omega G).$$

The result now follows from the exact cohomology sequence (4) and from Theorem 1.

2) Since X and G are path connected spaces, by corollary 1 we have statements b) and c). Since G is a path connected space, $p : PG \to G$ is an epimorphism, whose kernel is $K^0 \approx \Omega G$ (the statement a)), and from the exact sequence of the statement 1) follows the statement d). The statement e) follows from the statement 2) of corollary 1. □

Example 3. For any topological space X and the fibration $p : PR \to R$ there is an exact cohomology sequence

$$\begin{aligned} 0 \longrightarrow h^0_s(X, \Omega R) \longrightarrow h^0_s(X, PR) \longrightarrow h^0_s(X, R) \\ \longrightarrow h^1_s(X, \Omega R) \longrightarrow h^1_s(X, PR) \longrightarrow h^1_s(X, R) \longrightarrow h^2_s(X, \Omega R) \\ \longrightarrow h^2_s(X, PR) \longrightarrow h^2_s(X, R) \longrightarrow \cdots . \end{aligned} \tag{19}$$

Since R is a contractible space, by corollary 1 there are isomorphisms

$$\widetilde{h}^0(X, R) \approx h^0_s(X, R), \qquad \widetilde{h}^1(X, R) \approx h^1_s(X, R) \tag{20}$$

and we have an exact sequence (19).

Theorem 2. 1) *For any topological space X and a covering projection $p : E \to B$ there are:*

a) *for $q \geq 3$ an isomorphism*

$$\Omega(p)^q : h^q_s(X, \Omega E) \xrightarrow{\approx} h^q_s(X, \Omega B);$$

b) *for $q = 2$ an epimorphism*

$$\Omega(p)^2 : h^2_s(X, \Omega E) \to h^2_s(X, \Omega B);$$

c) *for $q = 0$ a monomorphism*

$$h^0_s(X, \Omega E) \to h^0_s(X, \Omega B).$$

2) *If E is a contractible space, then there are*

a) *for* $q = 2$ *an isomorphism*

$$\Omega(p)^2 : h_s^2(X, \Omega E) \approx h_s^2(X, \Omega B);$$

b) *for* $q = 1$ *an epimorphism*

$$\Omega(p)^1 : h_s^1(X, \Omega E) \to h_s^1(X, \Omega B).$$

3) *If* X *and* E *are path connected spaces, then there are*
a) *for* $q \geq 1$ *an isomorphism*

$$\Omega(p)^q : h_s^q(X, \Omega E) \approx h_s^q(X, \Omega B);$$

b) *for* $q = 0$ *an exact sequence*

$$0 \longrightarrow h_s^0(X, \Omega E) \longrightarrow h_s^0(X, \Omega B) \longrightarrow F \longrightarrow 0.$$

Proof. 1) Since $p : E \to B$ is a covering projection, it is a fibration (theorem 2.2.3 [6]) and induces a fibration $P(p) : PE \to PB$ (proposition 4.5 [7]). We show that $P(p)$ is a topological isomorphism. Let $\omega \in PE$ and $P(p)\omega = \overline{b}_0$. Since $P(p)\omega = p\omega = \overline{b}_0$, $\omega(I) \subset F$. Since p is a covering projection, the fiber $F = p^{-1}(b_0)$ is discrete (2.D [6]). Since I is a connected space, $\omega(I) = e_0$. Hence $P(p)$ is a monomorphism.

Since $p : E \to B$ is a fibration with unique path lifting (theorem 2.2.2 [6]) and $p(e_0) = b_0$, for each $\tau \in PB$ there exists $\omega \in PE$ such that $P(p)\omega = \tau$. Therefore $P(p)$ is an epimorphism.

Since p is an open map (corollary 2.10.10 [6]), $P(p)$ is an open map. Let $\langle F, U\rangle$ be a prebase of the space PE. Since p is an open map, $P(p)\langle F, U\rangle = \langle F, p(U)\rangle$ is a prebase of PB. Since $P(p)$ is a bijection map, $P(p)(A \cap B) = P(p)(A) \cap P(p)(B)$ and $P(p)(\cup A_i) = \cup P(p)(A_i)$. Hence $P(p)(\cap\langle F_i, U_i\rangle) = \cap P(p)\langle F_i, U_i\rangle$ is an open subset of PB. If A_i are elements of a base for the topology of PE, then $P(p)(\cup A_i) = \cup P(p)(A_i)$ is an open subset of PB. Therefore $P(p)$ is a topological isomorphism.

Using theorem 1 and a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega E & \longrightarrow & PE & \longrightarrow & E \\ & & \big\downarrow{\scriptstyle \Omega(p)} & & \big\downarrow{\scriptstyle P(p)} & & \big\downarrow{\scriptstyle p} \\ 0 & \longrightarrow & \Omega B & \longrightarrow & PB & \longrightarrow & B \end{array} \tag{21}$$

we have for $q \geq 2$ a commutative diagram with exact rows

$$\begin{array}{ccccc} h^q_s(X,PE) & \longrightarrow & h^q_s(X,E) & \longrightarrow & h^{q+1}_s(X,\Omega E) \\ \downarrow{\scriptstyle P(p)^q_s} & & \downarrow{\scriptstyle p^q_s} & & \downarrow{\scriptstyle \Omega(p)^{q+1}_s} \\ h^q_s(X,PB) & \longrightarrow & h^q_s(X,B) & \longrightarrow & h^{q+1}_s(X,\Omega B) \end{array}$$

$$\begin{array}{ccccc} \longrightarrow & h^{q+1}_s(X,PE) & \longrightarrow & h^{q+1}_s(X,E) \\ & \downarrow{\scriptstyle P(p)^{q+1}_s} & & \downarrow{\scriptstyle p^{q+1}_s} \\ \longrightarrow & h^{q+1}_s(X,PB) & \longrightarrow & h^{q+1}_s(X,B). \end{array}$$

Since $P(p)$ is an isomorphism, for $q \geq 0$ $P(p)^q$ is an isomorphism. Since for $q \geq 2$ the homomorphism p^q_s is an isomorphism (corollary 2), by the five lemma for $q \geq 3$ there is an isomorphism $\Omega(p)^q : h^q_s(X,\Omega E) \approx h^q_s(X,\Omega B)$. The diagram (21) generates (corollary 4) a commutative diagram with exact rows

$$(22)\qquad \begin{array}{ccccc} h^1_s(X,PE) & \longrightarrow & \widetilde{h}^1(X,E) & \longrightarrow & h^2_s(X,\Omega E) \\ \downarrow{\scriptstyle P(p)^1} & & \downarrow{\scriptstyle \widetilde{p}^1} & & \downarrow{\scriptstyle \Omega(p)^2} \\ h^1_s(X,PB) & \longrightarrow & \widetilde{h}^1(X,B) & \longrightarrow & h^2_s(X,\Omega B) \\ \longrightarrow & h^2_s(X,PE) & \longrightarrow & h^2_s(X,E) & \\ & \downarrow{\scriptstyle P(p)^2} & & \downarrow{\scriptstyle p^2_s} & \\ \longrightarrow & h^2_s(X,PB) & \longrightarrow & h^2_s(X,B). & \end{array}$$

Since F is the discrete fibre of the covering projection p, $\widetilde{p}^1 : \widetilde{h}^1(X,E) \to \widetilde{h}^1(X,B)$ is an epimorphism (see the beginning of proof of 1_b of corollary 2). Using the five lemma for the diagram (22) we have an epimorphism $\Omega(p)^2 : h^2_s(X,\Omega E) \to h^2_s(X,\Omega B)$. Since for a fibration $PE \to E$ there is an exact sequence (8)

$$0 \longrightarrow K^0_E \longrightarrow h^0_s(X,\Omega E) \longrightarrow \Phi^0_E \longrightarrow K^1_E \longrightarrow h^1_s(X,\Omega E) \longrightarrow 0$$

and by contractibility of PE one has $\Phi^0_E = 0$, it follows that

$$(23)\qquad K^0_E \approx h^0_s(X,\Omega E), \quad K^1_E \approx h^1_s(X,\Omega E).$$

Since for a fibration $PB \to B$ there is an exact sequence (8)

$$0 \longrightarrow K^0_B \longrightarrow h^0_s(X,\Omega B) \longrightarrow \Phi^0_B \longrightarrow K^1_B \longrightarrow h^1_s(X,\Omega B) \longrightarrow 0$$

and similarly by contractibility of PB one has $\Phi^0_B = 0$, it follows that

$$(24)\qquad K^0_B \approx h^0_s(X,\Omega B), \quad K^1_B \approx h^1_s(X,\Omega B).$$

The diagram (21) generates from (4) a commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & K_E^0 & \longrightarrow & \widetilde{h}^0(X,PE) & \longrightarrow & \widetilde{h}^0(X,E) \\
 & & \downarrow & & \downarrow {\scriptstyle \widetilde{P}(p)^0}\,\approx & & \downarrow {\scriptstyle \widetilde{p}^1} \\
0 & \longrightarrow & K_B^0 & \longrightarrow & \widetilde{h}^0(X,PB) & \longrightarrow & \widetilde{h}^0(X,B) \\
 & \longrightarrow & K_E^1 & \longrightarrow & \widetilde{h}^1(X,PE) & & \\
 & & \downarrow & & \downarrow {\scriptstyle \widetilde{P}(p)^1}\,\approx & & \\
 & \longrightarrow & K_B^1 & \longrightarrow & \widetilde{h}^1(X,PB). & &
\end{array} \tag{25}
$$

Using isomorphisms (23) and (24) and the diagram (25) we have a commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & h_s^0(X,\Omega E) & \longrightarrow & \widetilde{h}^0(X,PE) & \longrightarrow & \widetilde{h}^0(X,E) \\
 & & \downarrow {\scriptstyle \Omega(p)^0} & & \downarrow \approx & & \downarrow \\
0 & \longrightarrow & h_s^0(X,\Omega B) & \longrightarrow & \widetilde{h}^0(X,PB) & \longrightarrow & \widetilde{h}^0(X,B) \\
 & \longrightarrow & h_s^1(X,\Omega E) & \longrightarrow & \widetilde{h}^1(X,PE) & & \\
 & & \downarrow & & \downarrow \approx & & \\
 & \longrightarrow & h_s^1(X,\Omega B) & \longrightarrow & \widetilde{h}^1(X,PB), & &
\end{array} \tag{26}
$$

which implies that $\Omega(p)^0 : h_s^0(X,\Omega E) \to h_s^0(X,\Omega B)$ is a monomorphism.

2) If E is a contractible space, then for a fibration $p : E \to B$ with fiber $F = p^{-1}(b_0)$ there is an exact sequence

$$0 \longrightarrow K^0 \longrightarrow h_s^0(X,F) \longrightarrow \Phi^0 \longrightarrow K^1 \longrightarrow h_s^1(X,F) \longrightarrow 0$$

in which $\Phi^0 = 0$. Hence there are isomorphisms

$$K^0 \approx h_s^0(X,F) \quad \text{and} \quad K^1 \approx h_s^1(X,F). \tag{27}$$

Since F is a discrete group, by lemma 2 and isomorphisms (27) we have $K^0 \approx [X,F]$ and $K^1 = 0$. Since $K^2 = h_s^2(X,F) = 0$ (equality (10)) and $K^1 = 0$, from the exact sequence (4) it follows that there is an isomorphism

$$\widetilde{p}^1 : \widetilde{h}^1(X,E) \approx \widetilde{h}^1(X,B). \tag{28}$$

Therefore applying the five lemma to the diagram (22) we have an isomorphism

$$\Omega(p)^2 : h_s^2(X,\Omega E) \approx h_s^2(X,\Omega B).$$

Since $K^1 = 0$, there is an exact sequence

$$0 \longrightarrow K^0 \longrightarrow \widetilde{h}^0(X, E) \xrightarrow{\widetilde{p}^0} \widetilde{h}^0(X, B) \longrightarrow 0. \tag{29}$$

Applying the five lemma to the commutative diagram

$$\begin{array}{ccccc}
\widetilde{h}^0(X, PE) & \longrightarrow & \widetilde{h}^0(X, E) & \longrightarrow & h^1_s(X, \Omega E) \\
{\scriptstyle \widetilde{P}(p)^0}\big\downarrow \approx & & \big\downarrow {\scriptstyle \widetilde{p}^0} & & \big\downarrow {\scriptstyle \Omega(p)^1} \\
\widetilde{h}^0(X, PB) & \longrightarrow & \widetilde{h}^0(X, B) & \longrightarrow & h^1_s(X, \Omega B)
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & \widetilde{h}^1(X, PE) & \longrightarrow & \widetilde{h}^1(X, E) \\
& \approx \big\downarrow {\scriptstyle \widetilde{P}(p)^1} & & \approx \big\downarrow {\scriptstyle \widetilde{p}^1} \\
\longrightarrow & \widetilde{h}^1(X, PB) & \longrightarrow & \widetilde{h}^1(X, B)
\end{array}$$

and using the isomorphism (28) and the exact sequence (29), we have that $\Omega(p)^1 : h^1_s(X, \Omega E) \to h^1_s(X, \Omega B)$ is an epimorphism.

3) Let X and E be path connected spaces. We show that $\Phi^0 = 0$ in the exact sequence (8). Since the fiber $F = p^{-1}(b_0)$ is a discrete subgroup of E and X, E are path connected spaces, there is an equality $\operatorname{Ker} \widetilde{p}^0 = M^0(X, F)$. Hence $\Phi^0 = M^0(X, F)/\operatorname{Ker} \widetilde{p}^0 = 0$ and from the exact sequence (8) we have isomorphisms (27). By lemma 2 $h^q_s(X, F) = 0$ for $q \geq 1$, therefore $K^q = 0$ for $q \geq 1$ and using the exact sequence (4) we have an isomorphism

$$\widetilde{p}^1 : \widetilde{h}^1(X, E) \approx \widetilde{h}^1(X, B).$$

Since p is a surjection, B is path connected. Since PE and PB are contractible spaces, PE and PB are path connected. By corollary 1 we have isomorphisms

$$\widetilde{h}^0(X, E) \approx h^0_s(X, E), \quad \widetilde{h}^0(X, B) \approx h^0_s(X, B),$$
$$\widetilde{h}^0(X, PE) \approx h^0_s(X, PE), \quad \widetilde{h}^0(X, PB) \approx h^0_s(X, PB).$$

From these isomorphisms, (29), the first isomorphism in (27) and the snake lemma we have a commutative diagram

$$(30)\qquad \begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & h^0_s(X,F) & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & h^0_s(X,\Omega E) & \longrightarrow & h^0_s(X,PE) & \longrightarrow & h^0_s(X,E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow\approx & & \downarrow & & \\
0 & \longrightarrow & h^0_s(X,\Omega B) & \longrightarrow & h^0_s(X,PB) & \longrightarrow & h^0_s(X,B) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & h^0_s(X,F) & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Since $h^0_s(X,F)\approx F$, from the diagram (30) it follows that there is an exact sequence

$$0\longrightarrow h^0_s(X,\Omega E)\longrightarrow h^0_s(X,\Omega B)\longrightarrow F\longrightarrow 0.$$

Consider a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & h^1_s(X,\Omega E) & \longrightarrow & \widetilde{h}^1(X,PE) & \longrightarrow & \widetilde{h}^1(X,E) & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle \Omega(p)^1} & & \approx\Big\downarrow{\scriptstyle \widetilde{P}(p)^1} & & \approx\Big\downarrow{\scriptstyle \widetilde{p}^1} & & \\
0 & \longrightarrow & h^1_s(X,\Omega B) & \longrightarrow & \widetilde{h}^1(X,PB) & \longrightarrow & \widetilde{h}^1(X,B) & \longrightarrow & \cdots,
\end{array}$$

where $\widetilde{p}^1$ and $\widetilde{P}(p)^1$ are isomorphisms. Using the five lemma, we have an isomorphism

$$\Omega(p)^1 : h^1_s(X,\Omega E)\approx h^1_s(X,\Omega B). \qquad \square$$

Corollary 5. 1) *For any topological space X and the covering projection $p: R\to S^1$ there are*

a) *for $q\geq 2$ an isomorphism*

$$\Omega(p)^q : h^q_s(X,\Omega R)\approx h^q_s(X,\Omega S^1);$$

b) *for $q=1$ an epimorphism*

$$\Omega(p)^1 : h^1_s(X,\Omega R)\to h^1_s(X,\Omega S^1).$$

2) *If X is a path connected space, then there are*

a) *for* $q = 1$ *an isomorphism*

$$\Omega(p)^1 : h_s^1(X, \Omega R) \approx h_s^1(X, \Omega S^1);$$

b) *for* $q = 0$ *an exact sequence*

$$0 \longrightarrow h_s^0(X, \Omega R) \longrightarrow h_s^0(X, \Omega S^1) \longrightarrow Z \longrightarrow 0.$$

Corollary 6. 1) *For any topological space* X *and the covering projection* $p : S^1 \to S^1$, $p(z) = z^n$, *there are*

a) *for* $q \geq 3$ *an isomorphism*

$$\Omega(p)^2 : h_s^q(X, \Omega S^1) \approx h_s^q(X, \Omega S^1);$$

b) *for* $q = 2$ *an epimorphism*

$$\Omega(p)^2 : h_s^2(X, \Omega S^1) \longrightarrow h_s^2(X, \Omega S^1);$$

c) *for* $q = 0$ *a monomorphism*

$$\Omega(p)^0 : h_s^0(X, \Omega S^1) \to h_s^0(X, \Omega S^1).$$

2) *If* X *is a connected space, then there are*

a) *for* $q = 2$ *an isomorphism*

$$\Omega(p)^2 : h_s^2(X, \Omega S^1) \approx h_s^2(X, \Omega S^1);$$

b) *for* $q = 1$ *an epimorphism*

$$\Omega(p)^1 : h_s^1(X, \Omega S^1) \to h_s^1(X, \Omega S^1).$$

3) *If* X *is a path connected space, then there are*

a) *for* $q = 1$ *an isomorphism*

$$\Omega(p)^1 : h_s^1(X, \Omega S^1) \approx h_s^1(X, \Omega S^1);$$

b) *for* $q = 0$ *an exact sequence*

$$0 \longrightarrow h_s^0(X, \Omega S^1) \longrightarrow h_s^0(X, \Omega S^1) \longrightarrow Z_n \longrightarrow 0,$$

where $0 \longrightarrow Z_n \longrightarrow S^1 \xrightarrow{p} S^1 \longrightarrow 0.$

Proof. Let us prove 2_a and 2_b — the rest is obvious. If X is a connected space, then $M^0(X, Z_n)$ consists of constant maps only and $\operatorname{Ker} \widetilde{p}^0 = M^0(X, Z_n)$. Therefore $K^q = 0$, $q \geq 1$ and using the exact sequence (4) we have an isomorphism

$$\widetilde{p}^1 : \widetilde{h}^1(X, S^1) \xrightarrow{\approx} \widetilde{h}^1(X, S^1).$$

Using the five lemma we obtain 2_a and 2_b. □

Example 4. If $X = S^m$, $m \geq 1$, then by Corollary 9 of [5]

$$h_s^q(S^m, \Omega S^1) = \begin{cases} \Omega R, & \text{if } q = m, \\ 0, & \text{if } q \neq 0, m, \\ \Omega S^1, & \text{if } q = 0. \end{cases}$$

Lemma 4. *If G is a topological abelian group, then for $n \geq 0$ there is an isomorphism*

$$P\Omega^{n+1}G \approx \Omega P\Omega^n G.$$

Proof. There is an exact sequence

$$0 \longrightarrow \Omega G \xrightarrow{i} PG \xrightarrow{p} G,$$

which induces an exact sequence

$$0 \longrightarrow P\Omega G \xrightarrow{P(i)} P^2 G \xrightarrow{P(p)} PG. \tag{31}$$

Since $P(p)$ is fibration with fiber ΩPG, from the exact sequence (31) it follows that there is an isomorphism

$$P\Omega G \approx \Omega PG. \tag{32}$$

Assuming $\Omega^n G = H$ and using the isomorphism (32) we have $P\Omega H \approx \Omega PH$. Since $\Omega^{n+1}G = \Omega(\Omega^n G)$, the result follows from this. □

Corollary 7. *Let $p : E \to B$ be a covering projection, then for $n \geq 0$ there is an isomorphism*

$$P\Omega^{n+1}E \approx P\Omega^{n+1}B.$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega PE & \longrightarrow & P^2E & \longrightarrow & PE & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \Omega(P(p))} & & \Big\downarrow{\scriptstyle P^2(p)} & & \Big\downarrow{\scriptstyle P(p)} & & \\ 0 & \longrightarrow & \Omega PB & \longrightarrow & P^2B & \longrightarrow & PB & \longrightarrow & 0. \end{array}$$

By theorem 2, $P(p)$ is an isomorphism, hence $P^2(p)$ and $\Omega(P(p))$ are isomorphisms too. By lemma 4 there are isomorphisms

$$\Omega PE \approx P\Omega E \quad \text{and} \quad \Omega PB \approx P\Omega B.$$

Then for $n = 0$ there is an isomorphism

$$P\Omega E \approx \Omega PE \overset{\Omega(P(p))}{\underset{\sim}{\longrightarrow}} \Omega PB \approx P\Omega B.$$

For any topological abelian group G, using lemma 4 and the isomorphism (32) we have an isomorphism

$$\begin{aligned} P\Omega^{n+1}G = P\Omega(\Omega^n G) \approx \Omega P \Omega^n G \approx \Omega^2 P \Omega^{n-1} G \\ \approx \cdots \approx \Omega^n P \Omega G \approx \Omega^{n+1} PG. \end{aligned} \tag{33}$$

Since $PE \approx PB$ (theorem 2), there is an isomorphism

$$\Omega^{n+1}PE \approx \Omega^{n+1}PB. \tag{34}$$

Using isomorphisms (33) and (34) we have an isomorphism

$$P\Omega^{n+1}E \approx \Omega^{n+1}PE \approx \Omega^{n+1}PB \approx P\Omega^{n+1}B. \qquad \square$$

Theorem 3. 1) *For any topological space X and a covering projection $p : E \to B$ there are:*

a) *for $q \geq n+2$, $n \geq 0$, an isomorphism*

$$\Omega^n(p)^q : h_s^q(X, \Omega^n E) \approx h_s^q(X, \Omega^n B);$$

b) *for $q = n+1$, $n \geq 0$, an epimorphism*

$$\Omega^n(p)^{n+1} : h_s^{n+1}(X, \Omega^n E) \to h_s^{n+1}(X, \Omega^n B).$$

2) *For any topological space X and a covering projection $p : E \to B$ such that $\pi_2(E, e_0) = 0$ there are*

a) *for $q \geq 0$, $n \geq 3$ an isomorphism*

$$\Omega^n(p)^q : h_s^q(X, \Omega^n E) \approx h_s^q(X, \Omega^n B);$$

b) *for $q \geq 2$ an isomorphism*

$$\Omega^2(p)^q : h_s^q(X, \Omega^2 E) \approx h_s^q(X, \Omega^2 B);$$

c) *for $q = 1$ an epimorphism*

$$\Omega^2(p)^1 : h_s^1(X, \Omega^2 E) \to h_s^1(X, \Omega^2 B).$$

3) *For any topological space X and a covering projection $p : E \to B$ such that $\pi_2(E, e_0) = 0$ and $\pi_2(B, b_0) = 0$, for $q = 0, 1$ there is an isomorphism*

$$\Omega^2(p)^q : h_s^q(X, \Omega^2 E) \approx h_s^q(X, \Omega^2 B).$$

Proof. For $n \geq 1$ there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^n E & \longrightarrow & P\Omega^{n-1}E & \longrightarrow & \Omega^{n-1}E \\ & & \downarrow{\scriptstyle \Omega^n(p)} & & \downarrow{\scriptstyle P(\Omega^{n-1}(p))} & & \downarrow{\scriptstyle \Omega^{n-1}(p)} \\ 0 & \longrightarrow & \Omega^n B & \longrightarrow & P\Omega^{n-1}B & \longrightarrow & \Omega^{n-1}B, \end{array} \tag{35}$$

where by corollary 7, $P(\Omega^{n-1}(p))$ is an isomorphism.

The diagram (35) induces a commutative diagram with exact rows

$$
(36)\qquad
\begin{array}{ccccccc}
\cdots \longrightarrow & h_s^q(X, P\Omega^{n-1}E) & \longrightarrow & h_s^q(X, \Omega^{n-1}E) & \longrightarrow \\
& \Big\downarrow {\scriptstyle P(\Omega^{n-1}(p))^q} & & \Big\downarrow {\scriptstyle \Omega^{n-1}(p)^q} & \\
\cdots \longrightarrow & h_s^q(X, P\Omega^{n-1}B) & \longrightarrow & h_s^q(X, \Omega^{n-1}B) & \longrightarrow
\end{array}
$$

$$
\begin{array}{ccc}
\longrightarrow & h_s^{q+1}(X, \Omega^n E) & \longrightarrow \\
& \Big\downarrow {\scriptstyle \Omega^n(p)^{q+1}} & \\
\longrightarrow & h_s^{q+1}(X, \Omega^n B) & \longrightarrow
\end{array}
$$

$$
\begin{array}{ccccc}
\longrightarrow & h_s^{q+1}(X, P\Omega^{n-1}E) & \longrightarrow & h_s^{q+1}(X, \Omega^{n-1}E) & \longrightarrow \cdots \\
& \Big\downarrow {\scriptstyle P(\Omega^{n-1}(p))^{q+1}} & & \Big\downarrow {\scriptstyle \Omega^{n-1}(p)^{q+1}} & \\
\longrightarrow & h_s^{q+1}(X, P\Omega^{n-1}B) & \longrightarrow & h_s^{q+1}(X, \Omega^{n-1}B) & \longrightarrow \cdots
\end{array}
$$

For $n = 0$ by corollary $2.1_{\mathrm{a})}$ one has the statement $1_{\mathrm{a})}$.

For $n = 1$ by theorem $2.1_{\mathrm{a})}$ one has the statement $1_{\mathrm{a})}$.

For $n = 2$ and $q \geq 3$ by theorem 2, using the diagram (36) in which $P(\Omega^{n-1}(p))^q$, $\Omega^{n-1}(p)^q$ are isomorphisms, and the five lemma, one has the statement $1_{\mathrm{a})}$, i.e. for $q \geq 4$ there is an isomorphism

$$\Omega^2(p)^q : h_s^q(X, \Omega^2 E) \approx h_s^q(X, \Omega^2 B).$$

By induction, using the diagram (36) and the isomorphism $P(\Omega^{n-1}(p))$ one has the statement $1_{\mathrm{a})}$.

$1_{\mathrm{b})}$ For $n = 0$ by corollary $2.1_{\mathrm{b})}$ one has the statement $1_{\mathrm{b})}$.

For $n = 1$ by theorem $2.1_{\mathrm{b})}$ one has the statement $1_{\mathrm{b})}$.

For $n = 2$ and $q = 2$, using the diagram (36), in which $P(\Omega(p))^q$ is an isomorphism for $q \geq 2$ and by theorem $2.1_{\mathrm{b})}$, $\Omega(p)^2$ is an epimorphism, and the five lemma, one has the statement $1_{\mathrm{b})}$, i.e. for $q = 3$ there is an epimorphism

$$\Omega^2(p)^3 : h_s^3(X, \Omega^2 E) \to h_s^3(X, \Omega^2 B).$$

By induction, using the diagram (36) and the isomorphism $P(\Omega^{n-1}(p))$, one has the statement $1_{\mathrm{b})}$.

$2_{\mathrm{a})}$ As known [3], there are isomorphisms

$$\pi_0(\Omega^2 E, \omega_0) \approx \pi_1(\Omega E, \overline{e}_0) \approx \pi_2(E, e_0).$$

Since $\pi_2(E, e_0) = 0$, $\pi_0(\Omega^2 E, \omega_0) = 0$ and $\Omega^2 E$ is a path connected space. Since by corollary 7 there is an isomorphism $P(\Omega^{n-1}(p))$, from the diagram (35) it follows that for $n \geq 1$ the homomorphism $\Omega^n(p) : \Omega^n E \to \Omega^n B$ is a monomorphism. Since $\Omega^2 E$ is a path connected space, $P\Omega^2 E \to \Omega^2 E$ is an epimorphism. Therefore from the diagram (35) it follows that for $n \geq 3$ there is an isomorphism

$$\Omega^n(p) : \Omega^n E \to \Omega^n B. \tag{37}$$

Hence for $q \geq 0$ and $n \geq 3$ there is an isomorphism

$$\Omega^n(p)^q : h_s^q(X, \Omega^n E) \approx h_s^q(X, \Omega^n B). \tag{38}$$

$2_{\text{b})}$ For $n = 3$ and $q \geq 3$, using the diagram (36), isomorphisms $P(\Omega^{n-1}(p))$ and (38) and the five lemma, we have for $q \geq 2$ an isomorphism

$$\Omega^2(p)^q : h_s^q(X, \Omega^2 E) \approx h_s^q(X, \Omega^2 B). \tag{39}$$

$2_{\text{c})}$ By corollary 4 for $n = 3$ the diagram (35) induces a commutative diagram with exact rows

$$\begin{array}{ccccc}
h_s^1(X, \Omega^3 E) & \longrightarrow & \widetilde{h}^1(X, P\Omega^2 E) & \longrightarrow & \widetilde{h}^1(X, \Omega^2 E) \\
\Big\downarrow {\scriptstyle (\Omega^3(p))^1} & & \Big\downarrow {\scriptstyle \widetilde{P}(\Omega^2(p))^1} & & \Big\downarrow {\scriptstyle \widetilde{\Omega}^2(p))^1} \\
h_s^1(X, \Omega^3 B) & \longrightarrow & \widetilde{h}^1(X, P\Omega^2 B) & \longrightarrow & \widetilde{h}^1(X, \Omega^2 B)
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & h_s^2(X, \Omega^3 E) & \longrightarrow & h_s^2(X, P\Omega^2 E) \\
 & \Big\downarrow {\scriptstyle \Omega^3(p)^2} & & {\scriptstyle P(\Omega^2(p))^2} \Big\downarrow \\
\longrightarrow & h_s^2(X, \Omega^3 B) & \longrightarrow & h_s^2(X, P\Omega^2 B).
\end{array}$$

Using the five lemma, isomorphisms $P(\Omega^{n-1}(p))$ and (38) we have an isomorphism

$$\widetilde{\Omega}^2(p)^1 : \widetilde{h}^1(X, \Omega^2 E) \approx \widetilde{h}^1(X, \Omega^2 B). \tag{40}$$

By corollary 1, there is a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \widetilde{h}^0(X, \Omega^2 E) & \longrightarrow & h_s^0(X, \Omega^2 E) & \longrightarrow & [X, \Omega^2 E] \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
0 & \longrightarrow & \widetilde{h}^0(X, \Omega^2 B) & \longrightarrow & h_s^0(X, \Omega^2 B) & \longrightarrow & [X, \Omega^2 B]
\end{array}$$

$$\begin{array}{ccccccc}
\longrightarrow & \widetilde{h}^1(X, \Omega^2 E) & \longrightarrow & h_s^1(X, \Omega^2 E) & \longrightarrow & 0 \\
 & \Big\downarrow & & \Big\downarrow & & \\
\longrightarrow & \widetilde{h}^1(X, \Omega^2 B) & \longrightarrow & h_s^1(X, \Omega^2 B) & \longrightarrow & 0
\end{array}$$

from which by the isomorphism (40) it follows that there is an epimorphism

$$\Omega^2(p)^1 : h_s^1(X, \Omega^2 E) \to h_s^1(X, \Omega^2 B).$$

3) If $\pi_2(E, e_0) = 0$ and $\pi_2(B, b_0) = 0$, then $\Omega^2 E$ and $\Omega^2 B$ are path connected spaces.

For $n = 3$ there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^3 E & \longrightarrow & P\Omega^2 E & \longrightarrow & \Omega^2 E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^3 B & \longrightarrow & P\Omega^2 B & \longrightarrow & \Omega^2 B & \longrightarrow & 0, \end{array}$$

in which by corollary 7 the homomorphism $P(\Omega^2(p)) : P\Omega^2 E \to P\Omega^2 B$ is an isomorphism and $\Omega^3(p)$ is an epimorphism (37). Hence $\Omega^2(p) : \Omega^2 E \to \Omega^2 B$ is an isomorphism. Therefore for $q = 0, 1$ there is an isomorphism

$$\Omega^2(p)^q : h_s^q(X, \Omega^2 E) \approx h_s^q(X, \Omega^2 B). \qquad \square$$

Corollary 8. *For any topological space X and the covering projection $p : R \to S^1$ for $n \geq 2$ there is an isomorphism*

$$\Omega^n(p)^* : h_s^*(X, \Omega^n R) \approx h_s^*(X, \Omega^n S^1).$$

Corollary 9. *For any topological space X and the covering projection $p : S^1 \to S^1$, $p(z) = z^n$, for $q \geq 2$ there is an isomorphism*

$$\Omega^q(p)^* : h_s^*(X, \Omega^q S^1) \approx h_s^*(X, \Omega^q S^1).$$

Example 5. If $X = S^m$, $m \geq 1$, then by Corollary 9 of [5] for $n \geq 2$

$$h_s^q(S^m, \Omega^n S^1) \approx \begin{cases} \Omega^n R, & \text{if } q = m, 0, \\ 0, & \text{if } q \neq m, 0. \end{cases}$$

Now suppose given an inverse sequence of fibrations

$$E_0 \xleftarrow{p_1} E_1 \longleftarrow \cdots \longleftarrow E_{k-1} \xleftarrow{p_k} E_k \longleftarrow \cdots \tag{41}$$

and $E_\infty = \varprojlim E_k$, where each E_k is a topological abelian group and each p_k is a continuous homomorphism.

For each $k \in \mathbb{Z}^+$ ($\mathbb{Z}^+$ is the set of nonnegative integers) and any topological space X there is an exact sequence

$$0 \longrightarrow \widetilde{M}^*(X, E_k) \longrightarrow M^*(X, E_k) \longrightarrow [F_*(X), E_k] \longrightarrow 0.$$

Projections $p_k : E_k \to E_{k-1}$ generate an exact sequence of inverse systems of cochain complexes

$$0 \longrightarrow \{\widetilde{M}^*(X, E_k)\} \longrightarrow \{M^*(X, E_k)\} \longrightarrow \{[F_*(X), E_k]\} \longrightarrow 0.$$

Since for each $q \geq 0$ a fibration p_k induces an epimorphism $\widetilde{p}^q_k : \widetilde{M}^q(X, E_k) \to \widetilde{M}^q(X, E_{k-1})$, one has $\varprojlim^{(1)} \widetilde{M}^*(X, E_k) = 0$ (lemma A.15 [4]) and there is an exact sequence

$$0 \longrightarrow \varprojlim \widetilde{M}^*(X, E_k) \longrightarrow \varprojlim M^*(X, E_k) \longrightarrow \varprojlim\, [F_*(X), E_k] \longrightarrow 0. \tag{42}$$

By lemma 1, for $q = 2n$ the coboundary operator

$$\widetilde{\delta}^{q+1}_k : [F_q(X), E_k] \to [F_{q+1}(X), E_k]$$

is trivial, hence

$$\widetilde{\delta}^{q+1} : \varprojlim\, [F_q(X), E_k] \to \varprojlim\, [F_{q+1}(X), E_k]$$

is trivial. If $q = 2n+1$, then $\widetilde{\delta}^{q+1}_k$ is an isomorphism, hence $\widetilde{\delta}^{q+1}$ is an isomorphism. Therefore for $q \geq 1$ one has $H^q(\varprojlim\, [F_*(X), E_k]) = 0$ and for $q = 0$ one has

$$H^0\big(\varprojlim\, [F_*(X), E_k]\big) = \varprojlim\, [X, E_k]. \tag{43}$$

From an exact cohomology sequence induced by the exact sequence of cochain complexes (42) and the equality (43) we have

1) for $q \geq 2$, denoting $H^q(X) := H^q(\varprojlim M^*(X, E_k))$ (and similarly for $\widetilde{M}^*$), an isomorphism

$$\widetilde{H}^q(X) = H^q\big(\varprojlim \widetilde{M}^*(X, E_k)\big) \approx H^q\big(\varprojlim M^*(X, E_k)\big) = H^q(X); \tag{44}$$

2) for $q = 0, 1$ an exact sequence

$$0 \longrightarrow \widetilde{H}^0(X) \longrightarrow H^0(X) \longrightarrow \varprojlim\, [X, E_k] \longrightarrow \widetilde{H}^1(X) \longrightarrow H^1(X) \longrightarrow 0. \tag{45}$$

Since in the inverse system of cochain complexes $\{\widetilde{M}^*(X, E_k)\}$ the homomorphisms $\widetilde{p}^q_k$ induced by fibrations p_k are epimorphisms, by theorem A.19 [4] there is an exact sequence

$$0 \to \varprojlim^{(1)} \widetilde{h}^{q-1}(X, E_k) \to \widetilde{H}^q(X) \to \varprojlim \widetilde{h}^q(X, E_k) \to 0. \tag{46}$$

As known [6], there is a bijection $\mathrm{Hom}(Z, \varprojlim E_k) \approx \varprojlim \mathrm{Hom}(Z, E_k)$, where $\mathrm{Hom}(Z, Y)$ is the set of all continuous maps $Z \to Y$. Hence for any topological space X and $q \geq 0$ there is an isomorphism $M^q(X, E_\infty) \approx \varprojlim M^q(X, E_k)$, where $M^q(X, E_\infty) = \mathrm{Hom}(F_q(X), E_\infty)$, $M^q(X, E_k) = \mathrm{Hom}(F_q(X), E_k)$.

Since for each continuous map $f : Z' \to Z$ there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(Z, E_\infty) & \approx \varprojlim & \mathrm{Hom}(Z, E_k) \\ \big\downarrow \overline{f} & & \big\downarrow \varprojlim \overline{f} \\ \mathrm{Hom}(Z', E_\infty) & \approx \varprojlim & \mathrm{Hom}(Z', E_k), \end{array}$$

where $\overline{f}(\varphi) = \varphi f$, $\varprojlim \overline{f}\{\varphi_k\} = \{\varphi_k f\}$, there is an isomorphism of cochain complexes

$$M^*(X, E_\infty) \approx \varprojlim M^*(X, E_k).$$

Hence for $q \geq 0$ there is an isomorphism

$$h_s^q(X, E_\infty) \approx H^q(X). \tag{47}$$

Using isomorphisms (44) and (47) and the exact sequence (46), we have for $q \geq 2$ an exact sequence

$$\begin{aligned} 0 \longrightarrow \varprojlim^{(1)} \widetilde{h}^{q-1}(X, E_k) \longrightarrow h_s^q(X, E_\infty) \\ \longrightarrow \varprojlim \widetilde{h}^q(X, E_k) \longrightarrow 0. \end{aligned} \tag{48}$$

By corollary 1, for $q \geq 2$ there is an isomorphism

$$\widetilde{h}^q(X, E_k) \approx h_s^q(X, E_k). \tag{49}$$

Therefore using the exact sequence (48) and isomorphism (49), we have

Theorem 4. *For any topological space X and an inverse sequence of fibrations* (41) *there are:*

1) *for* $q \geq 3$ *an exact sequence*

$$0 \longrightarrow \varprojlim{}^{(1)} h_s^{q-1}(X, E_k) \longrightarrow h_s^q(X, E_\infty) \longrightarrow \varprojlim h_s^q(X, E_k) \longrightarrow 0; \tag{50}$$

2) *for* $q = 2$ *an exact sequence*

$$0 \longrightarrow \varprojlim{}^{(1)} \widetilde{h}^1(X, E_k) \longrightarrow h_s^2(X, E_\infty) \longrightarrow \varprojlim h_s^2(X, E_k) \longrightarrow 0. \tag{51}$$

Theorem 5. *If* X *is a path connected space and in an inverse sequence* (41) *each* E_k *is a path connected space, then there is an exact sequence*

$$0 \longrightarrow \varprojlim{}^{(1)} h_s^0(X, E_k) \longrightarrow h_s^1(X, E_\infty) \longrightarrow \varprojlim h_s^1(X, E_k)$$
$$\longrightarrow \varprojlim{}^{(1)} [X, E_k] \longrightarrow \varprojlim{}^{(1)} \widetilde{h}^1(X, E_k) \longrightarrow \varprojlim{}^{(1)} h_s^1(X, E_k) \longrightarrow 0.$$

Proof. By corollary 1.2) for each $k \in \mathbb{Z}^+$ we have

a) an isomorphism

$$\widetilde{h}^0(X, E_k) \approx h_s^0(X, E_k); \tag{52}$$

b) an exact sequence

$$0 \longrightarrow [X, E_k] \longrightarrow \widetilde{h}^1(X, E_k) \longrightarrow h_s^1(X, E_k) \longrightarrow 0. \tag{53}$$

Since $\widetilde{h}^0(X, E_k) = h_s^0(X, E_k) \approx E_k$ and

$$\widetilde{H}^0(X) = \widetilde{Z}^0(X) \approx \varprojlim \widetilde{Z}^0(X, E_k) \approx \varprojlim Z^0(X, E_k) \approx Z^0(X) = H^0(X) \approx \varprojlim E_k = E_\infty, \tag{54}$$

from the exact sequence (45) it follows that

$$0 \longrightarrow \varprojlim [X, E_k] \longrightarrow \widetilde{H}^1(X) \longrightarrow H^1(X) \longrightarrow 0. \tag{55}$$

Using exact sequences (46) and (55) and also the exact sequence generated by an exact sequence of inverse systems

$$0 \longrightarrow \{[X, E_k]\} \longrightarrow \{\widetilde{h}^1(X, E_k)\} \longrightarrow \{h_s^1(X, E_k)\} \longrightarrow 0$$

we get a commutative diagram with exact rows and columns

$$
(56)\qquad
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \varprojlim^{(1)}\widetilde{h}^0(X,E_k) & \rightarrow & \operatorname{Ker} t & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \varprojlim[X,E_k] & \rightarrow & \widetilde{H}^1(X) & \rightarrow & H^1(X) & \rightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow t & & \\
0 & \rightarrow & \varprojlim[X,E_k] & \rightarrow & \varprojlim\widetilde{h}^1(X,E_k) & \rightarrow & \varprojlim h^1_s(X,E_k) & \rightarrow & \varprojlim^{(1)}[X,E_k] \rightarrow \cdots. \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

From the diagram (56) using the isomorphism (47) one obtains that there is an isomorphism

$$
(57)\qquad \varprojlim^{(1)}\widetilde{h}^0(X,E_k) \approx \operatorname{Ker} t
$$

and an exact sequence

$$
(58)\qquad 0 \longrightarrow \varprojlim^{(1)}\widetilde{h}^0(X,E_k) \longrightarrow h^1_s(X,E_\infty) \longrightarrow \varprojlim h^1_s(X,E_k)
$$
$$
\longrightarrow \varprojlim^{(1)}[X,E_k] \longrightarrow \varprojlim^{(1)}\widetilde{h}^1(X,E_k) \longrightarrow \varprojlim^{(1)}h^1_s(X,E_k) \longrightarrow 0.
$$

Since there is an isomorphism (52), the inverse systems $\{\widetilde{h}^0(X,E_k)\}$ and $\{h^0_s(X,E_k)\}$ induce an isomorphism

$$
(59)\qquad \varprojlim^{(1)}\widetilde{h}^0(X,E_k) \approx \varprojlim^{(1)}h^0_s(X,E_k).
$$

The result follows from the exact sequence (58) using the isomorphism (59). □

Corollary 10. 1) *For any topological space X and an inverse sequence of fibrations* (41)*, where each $p_k : E_k \to E_{k-1}$ is a covering projection, for $q \neq 1$ there is an isomorphism*

$$
h^q_s(X,E_\infty) \approx \varprojlim h^q_s(X,E_k).
$$

2) *If X and each E_k are path connected spaces, then for $q = 1$ there is an exact sequence*

$$
0 \longrightarrow h^1_s(X,E_\infty) \longrightarrow \varprojlim h^1_s(X,E_k) \longrightarrow \varprojlim^{(1)}[X,E_k] \longrightarrow 0.
$$

Proof. By corollary 2 for $q \geq 2$ there is an isomorphism $p_k^q : h_s^q(X, E_k) \approx h_s^q(X, E_{k-1})$ and for $q = 1$ there is an epimorphism $\widetilde{p}_k^1 : \widetilde{h}^1(X, E_k) \to \widetilde{h}^1(X, E_{k-1})$ (see beginning of proof of 1_b of corollary 2). Hence by lemma A.15 [4] we have:

a) for $q \geq 2$ the equality

$$\varprojlim{}^{(1)} h_s^q(X, E_k) = 0; \tag{60}$$

b) for $q = 1$ the equality

$$\varprojlim{}^{(1)} \widetilde{h}^1(X, E_k) = 0. \tag{61}$$

The statement 1 for $q \geq 2$ follows from exact sequences (50) and (51) using equalities (60) and (61).

For $q = 0$, since for any space X one has $H^0(X) = Z^0(X)$ and by universality and left exactness of $\varprojlim$ there is an isomorphism

$$h_s^0(X, E_\infty) \approx H^0(X) = Z^0(X) \approx \varprojlim Z^0(X, E_k) = \varprojlim h_s^0(X, E_k).$$

2) Since p_k is a covering projection, by corollary $2.2_{\text{b})}$, $p_k^0 : h_s^0(X, E_k) \to h_s^0(X, E_{k-1})$ is an epimorphism. Hence by lemma A.15 [4] there is the equality

$$\varprojlim{}^{(1)} h_s^0(X, E_k) = 0. \tag{62}$$

The statement 2) follows from theorem 5 using equalities (61) and (62). □

Example 6. 1) Let X be an arbitrary space and

$$S^1 \xleftarrow{p_1} S^1 \xleftarrow{p_2} S^1 \longleftarrow \cdots \longleftarrow S^1 \xleftarrow{p_k} S^1 \longleftarrow$$

be the sequence, where p_k is the covering projection, $p_k(z) = z^p$ [3] for all $k \geq 1$, $k \in \mathbb{Z}^+$. The inverse limit $E_\infty = \varprojlim S^1 = \Sigma_p$ is the p-adic solenoid [2].

By corollary 10 for $q \neq 1$ there is an isomorphism

$$h_s^q(X, \Sigma_p) \approx \varprojlim h_s^q(X, S^1).$$

2) If X is a path connected space, then for $q = 1$ there is an exact sequence

$$0 \longrightarrow h_s^1(X, \Sigma_p) \longrightarrow \varprojlim h_s^1(X, S^1) \longrightarrow \varprojlim{}^{(1)} [X, S^1] \longrightarrow 0.$$

In particular, if $X = S^1$, then by example 2 we have:

a) for $q \geq 2$ the equality

$$h_s^q(S^1, \Sigma_p) \approx \varprojlim h_s^q(S^1, S^1) = 0;$$

b) for $q = 1$ the exact sequence

$$0 \longrightarrow h_s^1(S^1, \Sigma_p) \longrightarrow \Sigma_p \longrightarrow \varprojlim{}^{(1)}[S^1, S^1] \longrightarrow 0;$$

c) for $q = 0$ the isomorphism

$$h_s^0(S^1, \Sigma_p) \approx \varprojlim S^1 = \Sigma_p.$$

If $X = S^m$, $m \geq 2$, then by example 2 we have:
a) for $q \neq 0, 1, m$ the equality

$$h_s^q(S^m, \Sigma_p) \approx \varprojlim h_s^q(S^m, S^1) = 0;$$

b) for $q = 0$ the isomorphism

$$h_s^0(S^m, \Sigma_p) \approx \varprojlim h_s^0(S^m, S^1) \approx \varprojlim S^1 = \Sigma_p;$$

c) for $q = 1$ the exact sequence

$$0 \longrightarrow h_s^1(S^m, \Sigma_p) \longrightarrow \varprojlim h_s^1(S^m, S^1) \longrightarrow \varprojlim{}^{(1)}[S^m, S^1] \longrightarrow 0.$$

Since $h_s^1(S^m, S^1) = 0$, then $h_s^1(S^m, \Sigma_p) = 0$;
d) for $q = m$ the isomorphism

$$h_s^m(S^m, \Sigma_p) \approx \varprojlim h_s^m(S^m, S^1) \approx \varprojlim R \approx R.$$

Example 7. Let G be a topological abelian group, e_0 is the identity element of G.

For each $k \geq 1$ consider a fibration $p_k : P^k G \to P^{k-1} G$, where $P^k G = P(P^{k-1} G)$.

A topological abelian group G generates an inverse sequence of fibrations

$$P^* G = G \xleftarrow{p_1} PG \xleftarrow{p_2} P^2 G \longleftarrow \cdots \longleftarrow P^{k-1} G \xleftarrow{p_k} P^k G \longleftarrow \cdots .$$

By corollary 1 for any space X and $q \geq 2$ there is an isomorphism

$$\widetilde{h}^q(X, P^k G) \approx h_s^q(X, P^k G). \tag{63}$$

By proposition 4.4 [7], $P^k G$ is a contractible space, hence from the exact sequence

$$0 \longrightarrow \widetilde{h}^0(X, P^k G) \longrightarrow h_s^0(X, P^k G) \longrightarrow [X, P^k G]$$
$$\longrightarrow \widetilde{h}^1(X, P^k G) \longrightarrow h_s^1(X, P^k G) \longrightarrow 0$$

it follows that there are isomorphisms

$$\widetilde{h}^0(X, P^kG) \approx h_s^0(X, P^kG), \tag{64}$$

$$\widetilde{h}^1(X, P^kG) \approx h_s^1(X, P^kG), \tag{65}$$

and from the exact sequence (45) it follows that there are isomorphisms

$$\widetilde{H}^0(X) \approx H^0(X), \tag{66}$$

$$\widetilde{H}^1(X) \approx H^1(X). \tag{67}$$

Using isomorphisms (44), (47), (66) and (67), we have for $q \geq 0$ an isomorphism

$$h_s^q(X, P^\infty G) \approx \widetilde{H}^q(X), \tag{68}$$

where $P^\infty G = \varprojlim P^kG$.

Using isomorphisms (63), (64), (65) and (68) and also the exact sequence (46), we have for $q \geq 0$ an exact sequence

$$\begin{aligned} 0 \to \varprojlim{}^{(1)} h_s^{q-1}(X, P^kG) \to h_s^q(X, P^\infty G) \\ \to \varprojlim h_s^q(X, P^kG) \to 0. \end{aligned} \tag{69}$$

Let $\Omega = \{G_k\}$ be a countable set of topological abelian groups G_k, $k \in \mathbb{Z}^+$. Denote by $E_k = G_0 \times \cdots \times G_k$ the direct product. For each $k \in \mathbb{Z}^+$ is defined the projection $p_k : E_k \to E_{k-1}$, which is a fibration [6]. Hence there is an inverse sequence of fibrations

$$E_0 \xleftarrow{p_1} E_1 \longleftarrow \cdots \longleftarrow E_{k-1} \xleftarrow{p_k} E_k \longleftarrow \cdots$$

the inverse limit of which we will denote by $E_\infty = \varprojlim E_k$.

By lemma 3 for $k \geq 1$ there is an isomorphism of cochain complexes

$$M^*(X, E_k) \approx M^*(X, G_0) \oplus \cdots \oplus M^*(X, G_k).$$

Hence the projection p_k induces an epimorphism $p_k^* : M^*(X, E_k) \to M^*(X, E_{k-1})$ and the inverse system of cochain complexes $\{M^*(X, E_k)\}$ satisfies conditions of theorem A.19 [4]. Therefore for $q \geq 0$ there is an exact sequence

$$0 \longrightarrow \varprojlim{}^{(1)} h_s^{q-1}(X, E_k) \longrightarrow H^q(X) \longrightarrow \varprojlim h_s^q(X, E_k) \longrightarrow 0. \tag{70}$$

Using the isomorphisms (47) and the exact sequence (70), we have for $q \geq 0$ an exact sequence

$$(71) \qquad 0 \longrightarrow \varprojlim{}^{(1)} h_s^{q-1}(X, E_k) \longrightarrow h_s^q(X, E_\infty) \\ \longrightarrow \varprojlim h_s^q(X, E_k) \longrightarrow 0.$$

By corollary 3 for $k \geq 1$ there is an isomorphism

$$(72) \qquad h_s^*(X, E_k) \approx \prod_{0 \leq \lambda \leq k} h_s^*(X, G_\lambda).$$

Since the projection $p_k : E_k \to E_{k-1}$ induces an epimorphism $p_k^* : h_s^*(X, E_k) \to h_s^*(X, E_{k-1})$, for $q \geq 0$ there is the equality

$$(73) \qquad \varprojlim{}^{(1)} h_s^{q-1}(X, E_k) = 0.$$

Using the equality (73) and an exact sequence (71), we have an isomorphism

$$(74) \qquad h_s^*(X, E_\infty) \approx \varprojlim h_s^*(X, E_k).$$

Since there is an isomorphism

$$E_\infty \approx \prod_{k \in \mathbb{Z}^+} G_k\,,$$

using isomorphisms (72) and (74), we have

Theorem 6. *For any topological space X there is an isomorphism*

$$h_s^*\Big(X, \prod_{k \in \mathbb{Z}^+} G_k\Big) \approx \prod_{k \in \mathbb{Z}^+} h_s^*(X, G_k).$$

As known [6], if $\theta : E \to B$ is a basepoint preserving map, there is a fibration $p_\theta : E_\theta \to E$ induced from the path fibration $PB \to B$. This induced fibration is called the principal fibration induced by θ and has fiber $p_\theta^{-1}(e_0) = e_0 \times \Omega B$. We show that, if θ is a continuous homomorphism, then E_θ is a topological abelian group. Since E_θ is a subset of the topological abelian group $E_\theta \times PB$, elements of which are pairs (e, τ) such that $\theta(e) = p\tau = \tau(1)$, one has $(e_1, \tau_1) + (e_2, \tau_2) = (e_1 + e_2, \tau_1 + \tau_2)$ and $\theta(e_1 + e_2) = \theta(e_1) + \theta(e_2) = p\tau_1 + p\tau_2 = p(\tau_1 + \tau_2)$, i. e. $(e_1, \tau_1) + (e_2, \tau_2) \in E_\theta$. Since $\theta(e_0) = b_0 = p\tau_0$, where $\tau_0 : I \to B$, $\tau_0(I) = b_0$, one has $(e_0, \tau_0) \in E_\theta$. Since $\theta(-e) = -\theta(e) = -p\tau = p(-\tau)$, one has $(-e, -\tau) \in E_\theta$.

There is a commutative diagram

(75)
$$\begin{array}{ccccccc} & & & & 0 & & 0 \\ & & & & \downarrow & & \downarrow \\ & & & & F\times\tau_0 & \xrightarrow{\approx} & F \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & e_0\times\Omega B & \longrightarrow & E_\theta & \xrightarrow{p_\theta} & E \\ & & {\scriptstyle\Omega(\theta)}\downarrow{\scriptstyle\approx} & & {\scriptstyle\theta'}\downarrow & & \downarrow{\scriptstyle\theta} \\ 0 & \longrightarrow & \Omega B & \longrightarrow & PB & \xrightarrow[p]{} & B\,. \end{array}$$

If θ is a fibration, then θ' is a fibration [6], where $\theta'(e,\tau)=\tau$.

Corollary 11. *For any space X and a fibration $\theta: E\to B$ with discrete fiber F there is for $q\geq 2$ an isomorphism*

$$\theta_s'^q : h_s^q(X,E_\theta)\approx h_s^q(X,PB)$$

and for $q=1$ an epimorphism

$$\theta_s'^1 : h_s^1(X,E_\theta)\to h_s^1(X,PB).$$

Proof. Since fiber of a fibration $\theta' : E_\theta\to PB$ is a discrete space $F\times\tau_0$, then by corollary 2 statements of the corollary follow. □

Consider an inverse sequence of principal fibrations

$$E_{\theta_*}=E_{\theta_0}\xleftarrow{p_{\theta_1}} E_{\theta_1}\xleftarrow{p_{\theta_2}} E_{\theta_2}\longleftarrow\cdots\longleftarrow E_{\theta_{k-1}}\xleftarrow{p_{\theta_k}} E_{\theta_k}\longleftarrow\cdots,$$

where $E_{\theta_0}=E$, $E_{\theta_1}=E_\theta$, $p_{\theta_1}: E_{\theta_1}\to E_{\theta_0}$ is the principal fibration induced by $\theta_0=\theta : E\to B$, $p_{\theta_2}: E_{\theta_2}\to E_{\theta_1}$ is the principal fibration induced by $\theta_1 : E_{\theta_1}\to PB$, and $p_{\theta_k}: E_{\theta_k}\to E_{\theta_{k-1}}$ is the principal fibration induced by $\theta_{k-1}: E_{\theta_{k-1}}\to P^{k-1}B$.

Let E_∞ denote the inverse limit $\varprojlim E_{\theta_k}$ of the inverse sequence E_{θ_*}.

Since for $k\geq 1$ there is a commutative diagram

$$\begin{array}{ccc} E_{\theta_k} & \xrightarrow{\theta_k} & P^kB \\ {\scriptstyle p_{\theta_k}}\downarrow & & \downarrow{\scriptstyle p_k} \\ E_{\theta_{k-1}} & \xrightarrow{\theta_{k-1}} & P^{k-1}B, \end{array}$$

we have a natural map $\theta_* = \{\theta_k\} : E_{\theta_*} \to P^* B$.

Lemma 5. *If $\theta : E \to B$ is a fibration, then $\theta_\infty : E_\infty \to P^\infty B$ is a fibration.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & X \times I \\ {\scriptstyle f_\infty}\downarrow & & \downarrow{\scriptstyle H_\infty} \\ E_\infty & \xrightarrow{\ \theta_\infty\ } & P^\infty B. \end{array} \tag{76}$$

We show that there exists a continuous map $\widetilde{H} : X \times I \to E_\infty$ such that $f_\infty = \widetilde{H}_i$ and $\theta_\infty \widetilde{H} = H_\infty$.

For each $k \geq 0$ there is a commutative diagram

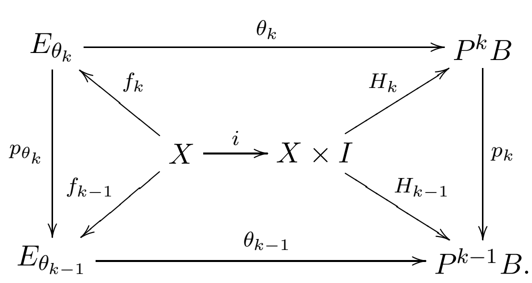
$$\begin{array}{ccc} E_\infty & \xrightarrow{\ \theta_\infty\ } & P^\infty B \\ {\scriptstyle q_k}\downarrow & & \downarrow{\scriptstyle \pi_k} \\ E_{\theta_k} & \xrightarrow{\ \theta_k\ } & P^k B. \end{array} \tag{77}$$

Diagrams (76) and (77) induce for each $k \geq 0$ a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & X \times I \\ {\scriptstyle f_k}\downarrow & & \downarrow{\scriptstyle H_k} \\ E_{\theta_k} & \xrightarrow{\ \theta_k\ } & P^k B, \end{array}$$

where $f_k = q_k f_\infty$, $H_k = \pi_k H_\infty$.

Consider for $k \geq 1$ the commutative diagram



Let $\widetilde{H}_{k-1} : X \times I \to E_{\theta_{k-1}}$ be such that k_1) $\widetilde{H}_{k-1} i = f_{k-1}$; k_2) $H_{k-1} = \theta_{k-1} \widetilde{H}_{k-1}$. Define the map $\widetilde{H}_k : X \times I \to E_{\theta_{k-1}} \times P^k B$ by

$\widetilde{H}_k = (\widetilde{H}_{k-1}, H_k)$. Since $\theta_{k-1}\widetilde{H}_{k-1} = H_{k-1} = p_k H_k$, by definition of E_{θ_k} it follows that $\widetilde{H}_k(X \times I) \subset E_{\theta_k}$. Hence we have $\widetilde{H}_k : X \times I \to E_{\theta_k}$ and $p_{\theta_k}\widetilde{H}_k = \widetilde{H}_{k-1}$, $\theta_k \widetilde{H}_k = H_k$. One has $\widetilde{H}_k i = (\widetilde{H}_{k-1} i, H_k i) = (f_{k-1}, \theta_k f_k) = (p_{\theta_k} f_k, \theta_k f_k) = f_k$. Since $\theta_0 : E_{\theta_0} \to B$ is a fibration, where $E_{\theta_0} = E$, there exists $\widetilde{H}_0 : X \times I \to E_{\theta_0}$ such that conditions 1_1) and 1_2) are satisfied. Therefore there is a map $\widetilde{H}_1 : X \times I \to E_{\theta_1}$ such that $p_{\theta_1}\widetilde{H}_1 = \widetilde{H}_0$; $\theta_1\widetilde{H}_1 = H_1$; $\widetilde{H}_1 i = f_1$. By induction there is a map $\{\widetilde{H}_k\} : X \times I \to E_\theta$, which generates the map $\widetilde{H} : X \times I \to \varprojlim E_{\theta_*} = E_\infty$.

Since one has a map $\{f_k\} : X \to E_{\theta_*}$ such that for $k \geq 0$ there is a commutative diagram

$$\begin{array}{ccc} X \times I & \xrightarrow{\{\widetilde{H}_k\}} & E_{\theta_*} \\ {\scriptstyle i}\uparrow & \nearrow{\scriptstyle \{f_k\}} & \\ X, & & \end{array}$$

one has $\widetilde{H} i = f_\infty$.

Since one has a map $\{H_k\} : X \times I \to P^* B$ such that for $k \geq 0$ there is a commutative diagram

$$\begin{array}{ccc} X \times I & \xrightarrow{\{\widetilde{H}_k\}} & E_{\theta_*} \\ & {\scriptstyle \{H_k\}}\searrow & \downarrow{\scriptstyle \{\theta_k\}} \\ & & P^* B, \end{array}$$

one has $\theta_\infty \widetilde{H} = H_\infty$. □

Corollary 12. *For any space X and a fibration $\theta : E \to B$ with discrete fiber F there is for $q \geq 2$ an isomorphism*

$$\theta^q_\infty : h^q_s(X, E_\infty) \approx h^q_s(X, P^\infty B)$$

and for $q = 1$ an epimorphism

$$\theta^1_\infty : h^1_s(X, E_\infty) \to h^1_s(X, P^\infty B).$$

Proof. Since $\operatorname{Ker}\theta_\infty = \varprojlim \operatorname{Ker}\theta_k$ and for all $k \geq 1$ there is an isomorphism $\operatorname{Ker}\theta_k \approx \operatorname{Ker}\theta_{k-1}$, by theorem VIII.3.4 [2] there is an isomorphism $\operatorname{Ker}\theta_\infty \approx \theta_0 = F$. Then by corollary 2 the statement follows. □

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