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ON DENSITY OF PERIODIC POINTS FOR INDUCED HYPERSPACE MAPS

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Abstract. Let X be a continuum and 2^X be the hyperspace of all nonempty closed subsets of X endowed with the Hausdorff metric. It is known that for each continuous map $f : X \to X$ the density of periodic points of the induced map $2^f : 2^X \to 2^X$ implies the density of periodic points of the base map f provided that X is a graph. In this note we describe a continuum X and a continuous map $f : X \to X$ where the density of periodic points of the induced map 2^f does not imply the density of periodic points of the base map f. Also we study a condition of f equivalent to the density of periodic points of 2^f.

1. Introduction and some definitions

Let (X, d) be a compact metric space without isolated points. Let $f : X \to X$ be a continuous mapping. As usual, $\mathbb{N}$ denotes the set of all positive integers. Let f^0 be identity map in X, $f^1 = f$, and for each $n \in \mathbb{N}$, $f^{n+1} = f \circ f^n$.

Given a point x in X, the *orbit of x under f* is the set

$$o(x, f) = \{f^n(x) : n \geq 0\},$$

and the *omega limit set of x under f* is the set

$$\omega(x, f) = \{y \in X : \exists \{n_i\} \subset \mathbb{N}, \lim_{n_i \to \infty} f^{n_i}(x) = y\}.$$

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It is not difficult to prove that for each $x \in X$, the $\omega(x, f)$ is a nonempty closed set in X, and $f(\omega(x, f)) = \omega(x, f)$.

Let $x \in X$. We say that x is a *periodic point of* f provided that $f^n(x) = x$ for some $n \in \mathbb{N}$. The set of all periodic points of f is denoted by $Per(f)$. If $x \in Per(f)$ then the smallest $n \in \mathbb{N}$ such that $f^n(x) = x$ is called the *period of* x. The point x is said to be *recurrent* if for every open set U containing x there exists $n \in \mathbb{N}$ such that $f^n(x) \in U$. And it is said to be *regularly recurrent* if for every open set U containing x there exists $N \in \mathbb{N}$ such that $f^{Nk}(x) \in U$ for every $k \in \mathbb{N}$. The set of all recurrent points of f and the set of all regularly recurrent points of f are denoted by $R(f)$ and $RR(f)$ respectively.

Notice that $x \in \omega(x, f)$ if and only if $x \in R(f)$, and that

$$Per(f) \subset RR(f) \subset R(f).$$

We say that $f : X \to X$ is:

- *transitive* if for each pair of nonempty open sets, U and W, in X there exists $n \in \mathbb{N}$ such that $f^n(U) \cap W \neq \emptyset$;

- *weakly mixing* provided that for each four nonempty open sets, A, B, C and D, in X, there exists $n \in \mathbb{N}$ such that $f^n(A) \cap C \neq \emptyset$ and $f^n(B) \cap D \neq \emptyset$; and

- *exact* if for every nonempty open set $U \subset X$ there exists $n \in \mathbb{N}$ such that $f^n(U) = X$.

Notice that if $f : X \to X$ is weakly mixing, then it is transitive.

Let 2^X denote the set of all nonempty closed subsets of X endowed with the Hausdorff metric d_H induced by d. Let $2^f : 2^X \to 2^X$ be the induced mapping by f in the hyperspace 2^X. Let us recall that for each $n \in \mathbb{N}$ and for each $A \in 2^X$, $\left(2^f\right)^n (A) = f^n(A)$.

We study in this note the connection between these two conditions: Density of $Per(f)$ in X, and density of $Per\left(2^f\right)$ in 2^X.

The reader can easily verify that the first condition implies the second one (see also lemma 1 in [1]). The reverse implication is more interesting.

A nonempty compact metric space X is a *continuum* if it is connected as well.

Let X be a continuum. We say that X is

- an *arc* if it is homeomorphic to the closed interval $[0, 1]$;

- a *graph* if X can be written as the finite union of arcs such that every two of them meet at a subset of their end points.

In [5], theorem 3.3, the authors proved the following claim: If X is a graph, then the density of $Per\left(2^f\right)$ implies the density of $Per(f)$.

In spite of the previous statement, the density of $Per\left(2^f\right)$ implies density of $Per(f)$ is not always true. In Theorem 14 of [4] and in [1] there are examples of transitive maps $f : X \to X$ where $Per\left(2^f\right)$ is dense in 2^X but $Per(f)$ is not dense in X. In the first example 2^f is transitive as well. In both cases the base space X is a Cantor set. So, it seems natural to wonder if it is possible to produce an example where X is a continuum and density of $Per\left(2^f\right)$ does not imply density of $Per(f)$. The main goal of this note is to describe such an example.

At the end of section 2 we present a condition of f that is equivalent to the density of $Per\left(2^f\right)$.

In the last section we present some conjectures.

2. The example

The continuum we are about to describe was previously presented, with different purposes, in [6].

First let us recall some results and definitions.

Let I be the closed interval $[0,1]$ in the real line $\mathbb{R}$. For each $n \in \mathbb{N}$, let $r \in \{0,1,\ldots,n-1\}$, and $g_n : I \to I$ be the piecewise linear function defined by the formula

$$g_n(x) = \begin{cases} nx - r, & \text{if } r \text{ is even and } x \in \left[\frac{r}{n}, \frac{r+1}{n}\right] \subset [0,1], \\ -nx + r + 1, & \text{if } r \text{ is odd and } x \in \left[\frac{r}{n}, \frac{r+1}{n}\right] \subset [0,1]. \end{cases}$$

Notice that g_1 is the identity function, and g_2 is the well known *tent map*. It is known that for any two positive integers, n and m, $g_n \circ g_m = g_m \circ g_n = g_{nm}$ (see [2]).

It is not difficult to prove that for each $n \geq 2$ and for each open interval $(a,b) \subset [0,1]$, $a < b$, there exists $m \in \mathbb{N}$ so that $g_n^m((a,b)) = [0,1]$. That is, each g_n is an exact mapping provided that $n \geq 2$. Furthermore, for each $n \geq 2$, $Per\left(g_n\right)$ is a dense set of $[0,1]$.

Let M be a sequence of positive integers ≥ 2,

$$M = \{n_1, n_2, n_3, \ldots\}.$$

Consider the *inverse limit space* taking g_{n_1}, g_{n_2}, g_{n_3},... as bonding maps,

$$K_{\{M\}} = \{\widehat{x} = (x_1, x_2, x_3, \ldots) : x_i = g_{n_i}(x_{i+1}) \text{ for each } i \in \mathbb{N}\}.$$

The metric in $K_{\{M\}}$ is given by the formula:

$$d(\widehat{x}, \widehat{y}) = d((x_1, x_2, x_3, \ldots), (y_1, y_2, y_3, \ldots)) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}.$$

It is known that $K_{\{M\}}$ is a continuum (see [7]). Some authors refer to this space as a Knaster Continuum.

Since for each $n \geq 2$, g_n and g_2 commute, g_2 induces a mapping of $K_{\{M\}}$ into itself, $G : K_{\{M\}} \to K_{\{M\}}$, given by

$$G(x_1, x_2, x_3, \ldots) = (g_2(x_1), g_2(x_2), g_2(x_3), \ldots).$$

Given $i \in \mathbb{N}$, $\pi_i : K_{\{M\}} \to I$ denotes the corresponding projection.

Notice that for each $i \in \mathbb{N}$ the following diagram commutes

$$\begin{array}{ccc} K_{\{M\}} & \xrightarrow{G} & K_{\{M\}} \\ \pi_i \downarrow & & \downarrow \pi_i \\ I & \xrightarrow{g_2} & I. \end{array}$$

Furthermore, for each $l \in \mathbb{N}$, we have that $\pi_i \circ G^l = g_2^l \circ \pi_i$. Also, for each $i \in \mathbb{N}$, π_i is an open function since $g_{n_k} : I \to I$ is an open function for each $n_k \in M$.

Given a point $\widehat{x} \in K_{\{M\}}$ and $\varepsilon > 0$, $B(\widehat{x}, \varepsilon)$ denotes the set

$$\left\{\widehat{y} \in K_{\{M\}} : d(\widehat{x}, \widehat{y}) < \varepsilon\right\}.$$

Example. From now on in the definition of $K_{\{M\}}$ we consider the sequence:

$$M = \{2, 4, 6, 8, \ldots\} = \{n_i = 2i\}_{i=1}^{\infty}.$$

In [6] it is proved that $G : K_{\{M\}} \to K_{\{M\}}$ is a homeomorphism.

Our first goal is to show that $Per(G)$ is not dense in $K_{\{M\}}$. This result is a consequence of the next proposition. The reader can find its proof in [6] as well.

Proposition 2.1. *If $\widehat{x} \in K_{\{M\}}$ is a periodic point under G, then $\widehat{x} = (0, 0, 0, \ldots)$.*

Proposition 2.2. *The mapping $G : K_{\{M\}} \to K_{\{M\}}$ is weakly mixing.*

Proof. Let A, B, C and E be four nonempty open subsets of $K_{\{M\}}$. Let $\widehat{a} \in A$, $\widehat{b} \in B$, $\widehat{c} \in C$ and $\widehat{e} \in E$. Let $\varepsilon > 0$ be such that $B(\widehat{a}, \varepsilon) \subset A$, $B\left(\widehat{b}, \varepsilon\right) \subset B$, $B(\widehat{c}, \varepsilon) \subset C$ and $B(\widehat{e}, \varepsilon) \subset E$.

Let $k \in \mathbb{N}$ be such that

$$\sum_{n=k+1}^{\infty} \frac{1}{2^n} < \varepsilon.$$

Then for each $t \in [0,1]$, $diam\left(\pi_k^{-1}(t)\right) < \varepsilon$.

Now, $\pi_k(B(\widehat{a}, \varepsilon))$ and $\pi_k\left(B\left(\widehat{b}, \varepsilon\right)\right)$ are open subsets of I. Hence there exists $l \in \mathbb{N}$ such that

$$g_2^l(\pi_k(B(\widehat{a}, \varepsilon))) = [0,1],$$

and

$$g_2^l\left(\pi_k\left(B\left(\widehat{b}, \varepsilon\right)\right)\right) = [0,1].$$

There exist $\widehat{t} \in B(\widehat{a}, \varepsilon)$ and $\widehat{s} \in B\left(\widehat{b}, \varepsilon\right)$ such that

$$g_2^l\left(\pi_k\left(\widehat{t}\right)\right) = c_k = \pi_k(\widehat{c}),$$

and

$$g_2^l(\pi_k(\widehat{s})) = e_k = \pi_k(\widehat{e}).$$

It follows that $\pi_k \circ G^l\left(\widehat{t}\right) = \pi_k(\widehat{c})$, and $\pi_k \circ G^l(\widehat{s}) = \pi_k(\widehat{e})$. Hence,

$$d\left(G^l\left(\widehat{t}\right), \widehat{c}\right) < \varepsilon, \text{ and } d\left(G^l(\widehat{s}), \widehat{e}\right) < \varepsilon.$$

Therefore,

$$G^l(A) \cap C \neq \emptyset, \text{ and } G^l(B) \cap E \neq \emptyset.$$

Thus, $G : K_{\{M\}} \to K_{\{M\}}$ is weakly mixing. □

Corollary 2.3. *The mapping $G : K_{\{M\}} \to K_{\{M\}}$ is transitive.*

In Theorem 2 of [1] J. Banks proved that for a compact metric space X and for each continuous map $f : X \to X$, f is weakly mixing if and only if the map $2^f : 2^X \to 2^X$ is transitive. Thus, using proposition 2.2 we have the following result.

Corollary 2.4. *The induced mapping $2^G : 2^{K_{\{M\}}} \to 2^{K_{\{M\}}}$ is transitive.*

We have already seen that $Per(G)$ is not dense in $K_{\{M\}}$ (proposition 2.1). In order to prove that $2^G : 2^{K_{\{M\}}} \to 2^{K_{\{M\}}}$ has dense set of periodic points we introduce another definition.

Given $f : X \to X$ and $\varepsilon > 0$, consider the following set:

$$RR_\varepsilon(f) = \left\{ x \in X : \exists N \in \mathbb{N}, \text{ so that } d\left(x, f^{Nk}(x)\right) < \varepsilon, \forall k \in \mathbb{N} \right\}.$$

The proof of the next proposition follows an idea given in the proof of Lemma 1 of [4].

Proposition 2.5. *Assume that for each $\varepsilon > 0$, the set $RR_\varepsilon(f)$ is dense in X. Then $Per\left(2^f\right)$ is dense in 2^X.*

Proof. Step one. Let $x_0 \in X$ and $\delta > 0$.

First we show that there exist $A \in 2^X$ and $N \in \mathbb{N}$ so that $A \subset B(x_0, \delta)$ and $f^N(A) = A$.

By hypothesis there exists $y \in RR_{\frac{\delta}{2}} \cap B\left(x_0, \frac{\delta}{2}\right)$. Then there exists $N \in \mathbb{N}$ such that $d\left(y, f^{Nk}(y)\right) < \frac{\delta}{2}$ for every $k \in \mathbb{N}$. It follows that

$$\omega\left(y, f^N\right) \subset Cl\left(B\left(y, \frac{\delta}{2}\right)\right) \subset B\left(x_0, \delta\right).$$

Take $A = \omega\left(y, f^N\right)$. Then $A \subset B(x_0, \delta)$ and $f^N(A) = A$.

Step two. Let $B \in 2^X$ and $\delta > 0$.

Since B is compact, there exist a finite collection of points

$$\{b_1, \ldots, b_m\} \subset B$$

so that $B \subset \cup_{i=1}^m B\left(b_i, \frac{\delta}{2}\right)$. For each $1 \leq i \leq m$, take A_i in 2^X and N_i in $\mathbb{N}$ so that $d_H\left(\{b_i\}, A_i\right) < \frac{\delta}{2}$ and $f^{N_i}\left(A_i\right) = A_i$. Let $A = \cup_{i=1}^m A_i$ and $N = lcm\left(N_1, \ldots, N_m\right)$. It follows that $d_H(B, A) < \delta$ and $f^N(A) = A$. $\square$

In the next proposition we return to our example.

Proposition 2.6. *For each $\varepsilon > 0$ the corresponding set $RR_\varepsilon\left(G\right)$ is dense in $K_{\{M\}}$.*

Proof. Fix $\varepsilon > 0$. Take $\widehat{t} = (t_1, t_2, \ldots)$ in $K_{\{M\}}$, and $\delta > 0$.

Consider N so that $\Sigma_{n=N+1}^\infty \frac{1}{2^n} < \min\left\{\frac{\delta}{2}, \varepsilon\right\}$.

Since g_2, g_4, ..., $g_{2(N-1)}$ are continuous functions, there exists γ, $0<\gamma<\frac{\delta}{2}$, such that if $|s-t_N|<\gamma$, then

$$\begin{gathered}\left|g_{2(N-1)}(s)-t_{N-1}\right|<\tfrac{\delta}{2},\\ \left|\left(g_{2(N-2)}\circ g_{2(N-1)}\right)(s)-t_{N-2}\right|<\tfrac{\delta}{2},\\ \vdots\\ \left|\left(g_2\circ g_4\circ\cdots\circ g_{2(N-2)}\circ g_{2(N-1)}\right)(s)-t_1\right|<\tfrac{\delta}{2}.\end{gathered}$$

Now, since $Per\,(g_2)$ is dense in $[0,1]$, there is a periodic point of g_2, say s_N, such that $|s_N-t_N|<\gamma$. Let $\widehat{s}\in K_{\{M\}}$ be a point such that $\pi_N\left(\widehat{s}\right)=s_N$.

Note that $d\left(\widehat{s},\widehat{t}\right)<\frac{\delta}{2}\left(\Sigma_{n=1}^{N}\frac{1}{2^n}\right)+\frac{\delta}{2}<\delta$.

Assume that $n_0\in\mathbb{N}$ is the period of s_N under g_2. Notice that for each $1\le i\le N-1$, we have that

$$\begin{aligned}g_2^{n_0}\left(\pi_i\left(\widehat{s}\right)\right) &= g_2^{n_0}\left(\left(g_{2i}\circ\cdots\circ g_{2(N-1)}\right)(s_N)\right)\\ &= \left(g_{2i}\circ\cdots\circ g_{2(N-1)}\right)\left(g_2^{n_0}(s_N)\right)\\ &= \left(g_{2i}\circ\cdots\circ g_{2(N-1)}\right)(s_N)\\ &= \pi_i\left(\widehat{s}\right).\end{aligned}$$

Therefore, for each $1\le i\le N$, $\pi_i\left(\widehat{s}\right)$ is a periodic point of g_2 of a period that is a factor of n_0. It follows that for each $1\le i\le N$ and for each $k\in\mathbb{N}$,

$$\pi_i\left(G^{n_0k}\left(\widehat{s}\right)\right)=\pi_i\left(\widehat{s}\right).$$

Hence

$$d\left(G^{n_0k}\left(\widehat{s}\right),\widehat{s}\right)\le\sum_{n=N+1}^{\infty}\frac{1}{2^n}<\varepsilon.$$

Thus, $\widehat{s}$ is in the set $RR_\varepsilon(G)$. $\square$

Corollary 2.7. *The mapping $2^G:2^{K_{\{M\}}}\to 2^{K_{\{M\}}}$ has dense set of periodic points.*

Remark 2.8. Corollaries 2.4 and 2.7 and proposition 2.1 say that the induced map $2^G:2^{K_{\{M\}}}\to 2^{K_{\{M\}}}$ is chaotic (according to the R. Devaney's definition, see [3]) while the map $G:K_{\{M\}}\to K_{\{M\}}$ is not.

The proposition 2.5 lead us to the following result.

Theorem 2.9. *The set $Per\left(2^f\right)$ is dense in 2^X if and only if for each $\varepsilon>0$, the set $RR_\varepsilon(f)$ is dense in X.*

Proof. Assume $Per\left(2^f\right)$ is dense in 2^X. Fix $\varepsilon > 0$.

Let $x_0 \in X$, and $\delta > 0$. There exist $A \in 2^X$ and $N \in \mathbb{N}$ such that $\left(2^f\right)^N(A) = A$ and $d_H(\{x_0\}, A) < \min\left\{\delta, \frac{\varepsilon}{2}\right\}$. Take $y \in A$. It follows that for each $k \in \mathbb{N}$,

$$f^{Nk}(y) \in A \subset B\left(x_0, \frac{\varepsilon}{2}\right).$$

Therefore, for each $k \in \mathbb{N}$, $d\left(y, f^{Nk}(y)\right) < \varepsilon$.

Since $d(x_0, y) < \delta$, $RR_\varepsilon(f)$ is a dense set of X.

The other part of this theorem is contained in proposition 2.5. □

3. Some conjectures

Given $f : X \to X$ we have two conditions at hand:

a) The set of regularly recurrent points is dense in X.

b) For each $\varepsilon > 0$, the set $RR_\varepsilon(f)$ is dense in X.

In [4] the authors proved that condition *a)* implies density of $Per\left(2^f\right)$ in 2^X.

Question 3.1. *Are conditions a) and b) equivalent?*

Remark 3.2. Since for each $\varepsilon > 0$ we have that $RR(f) \subset RR_\varepsilon(f)$, then condition *a)* implies condition *b)*.

It seems interesting to study in which continua the density of $Per\left(2^f\right)$ implies the density of $Per(f)$.

Let X be a continuum. We say that X is

- a *dendrite* if X is locally connected and it contains no simple closed curves;

- *decomposable* provided that X contains two proper subcontinua, A and B, such that $X = A \cup B$;

- *hereditarily decomposable* provided that each subcontinuum of X with more than one point is decomposable;

- *indecomposable* if X is not decomposable;

- *arc-like* provided that for each $\varepsilon > 0$ there exists an onto and continuous function $f : X \to [0, 1]$ such that for each $t \in [0, 1]$, $diam\left(f^{-1}(t)\right) < \varepsilon$.

The following result is already known. The proof we present here follows an idea from [5].

Proposition 3.3. *Let $f : [0,1] \to [0,1]$. If $Per(2^f)$ is dense in $2^{[0,1]}$ then $Per(f)$ is dense in $[0,1]$.*

Proof. Let $0 \leq a < b \leq 1$. Consider $x_0 = \frac{a+b}{2}$ and $\varepsilon = \frac{b-a}{2}$. Since $Per(2^f)$ is dense in $2^{[0,1]}$, there exist $A \in 2^{[0,1]}$ and $N \in \mathbb{N}$ such that $d_H(\{x_0\}, A) < \varepsilon$ and $\left(2^f\right)^N(A) = A$. It implies that $A \subset (a,b)$ and $f^N(A) = A$.

Let $\alpha = \min A$ and $\beta = \max A$. Then $f^N(\alpha) \geq \alpha$ and $f^N(\beta) \leq \beta$. Since f^N is continuous in $[0,1]$, there exists $x \in [\alpha, \beta] \subset (a,b)$ so that $f^N(x) = x$. □

In [5], theorem 3.3, the authors proved the following claim: If X is a graph, then the density of $Per\left(2^f\right)$ implies the density of $Per(f)$. This result is a generalization of the previous proposition. In this setting the following conjecture could be interesting.

Conjecture 3.4. *Let X be a dendrite. Then for each continuous map $f : X \to X$ the density of $Per\left(2^f\right)$ implies the density of $Per(f)$.*

The space $K_{\{M\}}$ described in the previous section is an arc-like continuum. It is indecomposable as well (see [7]). As we have seen before, in this example the density of $Per\left(2^f\right)$ does not imply the density of $Per(f)$. Hereditarily decomposable arc-like continua and the arc are, somehow, more alike. With this in mind, it seems possible that the following result were true.

Conjecture 3.5. *Let X be a hereditarily decomposable arc-like continuum. Then for each continuous map $f : X \to X$, the density of $Per\left(2^f\right)$ implies the density of $Per(f)$.*

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