TOPOLOGY PROCEEDINGS
Volume 35, 2010
Pages 291–330
http://topology.auburn.edu/tp/

A theory of convergence and cluster points based on κ-nets

by

R.E. Hodel

Electronically published on October 18, 2009

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
TOPOLOGY PROCEEDINGS
Volume 35 (2010)
Pages 291-330
E-Published on October 18, 2009

A THEORY OF CONVERGENCE AND CLUSTER POINTS BASED ON κ-NETS

R.E. HODEL

ABSTRACT. This paper is a systematic study of convergence and cluster points using a special kind of net called a *κ-net*. The paper is organized as follows.

1. Nets and filters; κ-nets
2. Φ-convergence and Φ-cluster points
3. κ-nets compared to nets with directed sets of cardinality at most κ; κ-Fréchet and κ-net spaces
4. Convergence structures defined in terms of κ-nets; cryptomorphisms with κ-Fréchet and κ-net spaces
5. Φ-compactness and product theorems; (κ,ω)-nets and Lindelöf spaces
6. Properties related to Φ-compactness
7. Applications of κ-nets to cardinal functions
8. κ-nets in analysis

Dedicated to Mary Ellen Rudin on the occasion of her eighty-fifth birthday.

2000 *Mathematics Subject Classification.* 54A20, 54A25, 54D55, 54A10, 54D30.

Key words and phrases. Nets and filters; convergence and cluster points; compactness-like properties; convergence structures; higher cardinality versions of Fréchet and sequential spaces; cardinal functions.

1. Nets and filters; κ-nets

Nets and filters are the two main approaches to a general theory of convergence and cluster points in topology; nets are due to Moore and Smith, filters to Bourbaki (especially Cartan). Each theory has its own advocates. Nets are more intuitive due to their close connection to sequences and are generally preferred by analysts. Filters, on the other hand, are perhaps more elegant and flexible and are usually the preferred choice of topologists.

In [4], Bartle gives a nice discussion of the two competing theories and describes constructions for going back and forth between nets and filters. He points out advantages of each thusly: "the use of nets parallels very closely standard constructions involving sequences, and has the pronounced advantage that sequential arguments may quite readily be adapted to them." On the other hand, filters have a certain "algebraic elegance," are "admirably suited for certain transfinite arguments," and "enjoy a uniqueness not possessed by nets."

In this paper we introduce a special class of nets called *κ-nets* (κ an infinite cardinal). A κ-net is a special type of net and therefore has the intuitive appeal of nets, but its special nature takes advantage of many of the nice properties of filters (more precisely, of ideals). Here is the official definition.

Definition. Let X be a set and let κ be an infinite cardinal. A *κ-net* in X is a function $\xi : \kappa^{<\omega} \to X$, where $\kappa^{<\omega} = \{F : F$ is a finite subset of $\kappa\}$ and is directed by $\subseteq$. The κ-net ξ is usually denoted by

$$\langle x_F : F \in \kappa^{<\omega}\rangle, \text{ or just } \langle x_F\rangle,$$

where $x_F = \xi(F)$ for all $F \in \kappa^{<\omega}$.

A κ-net is itself a special case of a phalanx as defined by Tukey (see p. 17 in [32]). By definition, a *phalanx* is a function $f : A^{<\omega} \to X$; thus, a κ-net is a phalanx with $A = \kappa$.

It is straightforward to go back and forth between κ-nets and filters.

- Let $\langle x_F\rangle$ be a κ-net, and for each $F \in \kappa^{<\omega}$ let $A_F = \{x_G : G \in \kappa^{<\omega}$ and $F \subseteq G\}$. The filter associated with $\langle x_F\rangle$ has $\{A_F : F \in \kappa^{<\omega}\}$ as a filter base.

• Let $\mathcal{L}$ be a filter of cardinality κ, say $\mathcal{L} = \{L_\alpha : \alpha < \kappa\}$. A κ-net associated with $\mathcal{L}$ is $\langle x_F : F \in \kappa^{<\omega}\rangle$, where $x_F \in \bigcap\{L_\alpha : \alpha \in F\}$.

The usual properties are preserved by these constructions; for example, $\langle x_F\rangle$ converges to q if and only if the associated filter converges to q, and so on.

This paper is organized as follows. In section 2 we introduce two basic ideas: convergence of κ-nets and cluster points of κ-nets. This is done in the general setting of an operator $\Phi : P(X) \to P(X)$ so that results apply not only to compact spaces but also to H-closed spaces, δ-compact spaces, and possibly other compactness-like properties. In section 3 we show that nets can be replaced by κ-nets in the study of convergence and cluster points; we then use κ-nets to extend Fréchet and sequential spaces to higher cardinality to obtain the κ-Fréchet and the κ-net spaces. This has already been done by Meyer [20] with nets whose directed sets have cardinality at most κ; we will show that the two approaches give the same classes of spaces. In section 4 we give axioms for two types of convergence structures and then show that one is cryptomorphic to κ-net spaces, the other to κ-Fréchet spaces. In section 5 we characterize various generalizations of compactness in terms of cluster points of κ-nets. As an application of these ideas, we are able to give a unified approach to a number of results on the preservation of compactness-like properties in product spaces. In section 6 we obtain results related to various characterizations of initial κ-compactness due to Alexandroff and Urysohn. In section 7 we discuss applications of κ-nets to cardinal functions, specifically to two theorems due to Gryzlov. Finally, in section 8 we give situations in analysis where κ-net ideas naturally apply.

We use κ, λ, and θ to denote infinite cardinals and α, β, γ and δ to denote ordinals. A space X is *Fréchet* if whenever $q \in A^-$, there is a sequence $\langle x_n\rangle$ in A that converges to q; also, X is *sequential* if every sequentially closed subset of X is a closed set (A is *sequentially closed* if it has the property that whenever $\langle x_n\rangle$ is a sequence in A that converges to q, then $q \in A$). For background on cardinal functions, see [16].

2. Φ-convergence and Φ-cluster points

Throughout this paper we will work with an operator Φ that applies to the open sets of topological spaces. More precisely: if V is an open set in X, then $\Phi(V) \subseteq X$ (but need not be open). We require that the following properties hold for every space X and all open sets U and V in X:

- $V \subseteq \Phi(V)$;
- if $U \subseteq V$, then $\Phi(U) \subseteq \Phi(V)$.

Given an operator Φ, the Φ-*closure* of $A \subseteq X$, denoted A^Φ, is defined by

$A^\Phi = \{x : x \in X$ and $\Phi(V) \cap A \neq \varnothing$ for every open neighborhood V of $x\}$.

If $A^\Phi = A$, we say that A is Φ-*closed*.

Throughout this paper we will emphasize three choices for Φ: $\Phi(V) = V$, $\Phi(V) = V^-$, and $\Phi(V) = V^{-o}$ (see the article *Modified open and closed sets* in [23]). For these three choices we have the following notation and terminology:

$\Phi(V) = V$ $\quad A^\Phi$ is denoted by A^- and is called the *closure* of A;
$\Phi(V) = V^-$ $\quad A^\Phi$ is denoted by A^θ and is called the θ-*closure* of A;
$\Phi(V) = V^{-o}$ $\quad A^\Phi$ is denoted by A^δ and is called the δ-*closure* of A.

Moreover, the following hold:

- for all $A \subseteq X$, $A^- \subseteq A^\delta \subseteq A^\theta$, and each is a closed set;
- for X regular and $A \subseteq X$, $A^- = A^\delta = A^\theta$.

Definition. Let Φ be an operator, let X be a space, let $q \in X$, and let $\langle x_F \rangle$ be a κ-net in X.

- The point q is a Φ-*cluster point* of $\langle x_F \rangle$ if given any open neighborhood V of q and any $F \in \kappa^{<\omega}$, there exists $G \in \kappa^{<\omega}$ such that $F \subseteq G$ and $x_G \in \Phi(V)$.
- The κ-net $\langle x_F \rangle$ Φ-*converges* to q, written

$$x_F \to_\Phi q,$$

if given any open neighborhood V of q, there exists $F \in \kappa^{<\omega}$ such that $x_G \in \Phi(V)$ for all $G \in \kappa^{<\omega}$ with $F \subseteq G$.

Note: Let $\langle x_F \rangle$ be a κ-net in X. The following terminology is used in these special cases.

$\Phi(V) = V$	q is a *cluster point* of $\langle x_F \rangle$ and $\langle x_F \rangle$ *converges* to q;
$\Phi(V) = V^-$	q is a θ-*cluster point* of $\langle x_F \rangle$ and $\langle x_F \rangle$ θ-*converges* to q;
$\Phi(V) = V^{-o}$	q is a δ-*cluster point* of $\langle x_F \rangle$ and $\langle x_F \rangle$ δ-*converges* to q.

Useful Construction: The following construction is a natural application of κ-nets. Let $q \in X$, let $\{V_\alpha : \alpha < \kappa\}$ be a local base for q, and let $\langle x_F \rangle$ be any κ-net in X such that $x_F \in \Phi(\cap_{\alpha \in F} V_\alpha)$ for each $F \in \kappa^{<\omega}$. Then $x_F \rightarrow_\Phi q$.

Lemma 2.1. *Let Φ be an operator, let X be a space, and let $q \in X$.*

(1) If $\langle x_F \rangle$ is a κ-net in X and $x_F \rightarrow_\Phi q$, then q is a Φ-cluster point of $\langle x_F \rangle$.

(2) If $\chi(X, q) \leq \kappa$ and $q \in A^\Phi$, then there is a κ-net $\langle x_F \rangle$ in A such that $x_F \rightarrow_\Phi q$ (see the Useful Construction).

Definition. Let ξ be a κ-net in X. A λ-*subnet* of ξ is obtained as follows: Let H be a function from $\lambda^{<\omega}$ into $\kappa^{<\omega}$ such that $F \cap \kappa \subseteq H(F)$ for all $F \in \lambda^{<\omega}$. Then $\xi \circ H$ is a λ-subnet of ξ and is written as $\langle x_{H(F)} : F \in \lambda^{<\omega} \rangle$ or just $\langle x_{H(F)} \rangle$, where $x_{H(F)} = (\xi \circ H)(F)$. Note that in the case where $\lambda \leq \kappa$, the condition on H reduces to $F \subseteq H(F)$.

Lemma 2.2. *Let $\langle x_F \rangle$ be a κ-net in X and let q be a Φ-cluster point of $\langle x_F \rangle$. Then there is a λ-subnet of $\langle x_F \rangle$ that Φ-converges to q.*

Proof. Let $\{V_\alpha : \alpha < \lambda\}$ be a local base for q. For each $F \in \lambda^{<\omega}$ there exists $H(F) \in \kappa^{<\omega}$ such that $F \cap \kappa \subseteq H(F)$ and $x_{H(F)} \in \Phi(\bigcap\{V_\alpha : \alpha \in F\})$. The Useful Construction now applies, $x_{H(F)} \rightarrow_\Phi q$, and $\langle x_{H(F)} : F \in \lambda^{<\omega} \rangle$ is the required subnet. □

Consider the following fundamental property of nets and subnets: if a net converges to q, then every subnet also converges to q. This is achieved as follows. A subnet of a net $\langle x_d : d \in D \rangle$ has the form $\langle x_{f(e)} : e \in E \rangle$, where E is a directed set and $f : E \rightarrow D$ is a function such that for all $d_0 \in D$, there exists $e_0 \in E$ such that if $e \geq e_0$, then $f(e) \geq d_0$. In the theory of κ-nets, this idea is captured by $F \cap \kappa \subseteq H(F)$ and $\kappa \leq \lambda$ as follows: let $F_0 \in \kappa^{<\omega}$; then $F_0 \in \lambda^{<\omega}$, and if $F_0 \subseteq G \in \lambda^{<\omega}$, then $F_0 \subseteq H(G)$ (proof: $F_0 = F_0 \cap \kappa \subseteq G \cap \kappa \subseteq H(G)$). With this observation, we have:

Lemma 2.3. *Let $\langle x_F \rangle$ be a κ-net in X and let $\langle x_{H(F)} \rangle$ be a λ-subnet of $\langle x_F \rangle$ with $\kappa \leq \lambda$.*

(1) *If $x_F \to_\Phi q$, then $x_{H(F)} \to_\Phi q$.*

(2) *If q is a Φ-cluster point of $\langle x_{H(F)} \rangle$, then q is a Φ-cluster point of $\langle x_F \rangle$.*

Definition. A κ-net $\langle x_F \rangle$ in X is *universal* if given any $A \subseteq X$, exactly one of the following holds:

- there exists $F \in \kappa^{<\omega}$ such that if $F \subseteq G$, then $x_G \in A$;
- there exists $F \in \kappa^{<\omega}$ such that if $F \subseteq G$, then $x_G \in X - A$.

Lemma 2.4. *Let $\langle x_F \rangle$ be a universal κ-net in X and let q be a Φ-cluster point of $\langle x_F \rangle$. Then $x_F \to_\Phi q$.*

Lemma 2.5. *Let $\langle x_F \rangle$ be a κ-net in X. Then there is a λ-subnet of $\langle x_F \rangle$ with $\lambda \geq \kappa$ that is universal.*

Proof. For each $F \in \kappa^{<\omega}$ let $A_F = \{x_G : G \in \kappa^{<\omega}$ and $F \subseteq G\}$. The collection $\{A_F : F \in \kappa^{<\omega}\}$ has the following properties: (1) $A_F \neq \varnothing$ for all $F \in \kappa^{<\omega}$; (2) $A_{F \cup G} = A_F \cap A_G$. It follows that $\{A_F : F \in \kappa^{<\omega}\}$ has the finite intersection property (FIP). Let $\{U_\alpha : \alpha < \lambda\}$ be an ultrafilter on X such that $\{A_F : F \in \kappa^{<\omega}\} \subseteq \{U_\alpha : \alpha < \lambda\}$ (may assume $\lambda \geq \kappa$). Define $H : \lambda^{<\omega} \to \kappa^{<\omega}$ as follows. Let $F \in \lambda^{<\omega}$; by the FIP, $A_{F \cap \kappa} \cap (\cap_{\alpha \in F} U_\alpha) \neq \varnothing$. Let

$$z \in A_{F \cap \kappa} \cap (\cap_{\alpha \in F} U_\alpha).$$

Since $z \in A_{F \cap \kappa}$, there exists $G \in \kappa^{<\omega}$ such that $z = x_G$ and $F \cap \kappa \subset G$. Let $H(F) = G$; we then have $F \cap \kappa \subseteq H(F)$ and thus $\langle x_{H(F)} : F \in \lambda^{<\omega} \rangle$ is a λ-subnet of $\langle x_F \rangle$ with $x_{H(F)} \in \cap_{\alpha \in F} U_\alpha$ for all $F \in \lambda^{<\omega}$. To check universality, let $A \subseteq X$, and assume that $A = U_\beta$. The required set is $\{\beta\}$. For, if $\{\beta\} \subseteq G$, then $x_{H(G)} \in (\cap_{\alpha \in G} U_\alpha) \subseteq U_\beta$ as required. □

Theorem 2.6. *Let X be a space, let $A \subseteq X$, and let Φ be an operator. The following are equivalent:*

(1) *if ξ is a κ-net in A, then there exists $q \in A$ such that q is a Φ-cluster point of ξ in X;*

(2) *if ξ is a κ-net in A that is universal in X, then there exists $q \in A$ such that $\xi \to_\Phi q$ in X.*

Proof. For (1) $\Rightarrow$ (2), use Lemma 2.4; for (2) $\Rightarrow$ (1), use Lemmas 2.5, 2.1(1), and 2.3(2). □

In [32], Tukey proves similar results in terms of phalanxes and ultraphalanxes.

Corollary 2.7. *Let X be a space and let Φ be an operator. The following are equivalent:*

(1) *every κ-net in X has a Φ-cluster point;*
(2) *every universal κ-net in X Φ-converges.*

3. κ-nets and nets with directed sets of cardinality at most κ; κ-Fréchet and κ-net spaces

In this section we consider the relationship between convergence and cluster point properties of sequences, nets, and κ-nets. These results show that we are justified in using κ-nets to extend Fréchet and sequential spaces to higher cardinality. We begin with the relationship between κ-sequences and κ-nets.

Definition. Let X be a space, let $q \in X$, let Φ be an operator, and let $\langle x_\alpha : \alpha < \kappa \rangle$ be a κ-sequence in X.

• The sequence $\langle x_\alpha : \alpha < \kappa \rangle$ Φ-*converges* to q, written $x_\alpha \to_\Phi q$, if given any open neighborhood V of q, there exists $\alpha \in \kappa$ such that for all $\beta \geq \alpha$, $x_\beta \in \Phi(V)$.

• The point q is a Φ-*cluster point* of $\langle x_\alpha : \alpha < \kappa \rangle$ if given any open neighborhood V of q and any $\alpha \in \kappa$, there exists $\beta \geq \alpha$ such that $x_\beta \in \Phi(V)$.

Theorem 3.1. *Let X be a space, let $q \in X$, and let Φ be an operator. Given any ω-net $\langle x_F : F \in \omega^{<\omega} \rangle$ in X, there is a sequence $\langle y_n : n \in \omega \rangle$ in X such that the following hold:*

(1) $\{y_n\} \subseteq \{x_F\}$;

(2) *if* $x_F \to_\Phi q$, *then* $y_n \to_\Phi q$;

(3) *if q is a Φ-cluster point of $\langle y_n \rangle$, then q is a Φ-cluster point of $\langle x_F \rangle$.*

Outline of proof. For each $n \in \omega$ let $y_n = x_{\{0,1,\ldots,n\}}$. □

Example. The converse of (2) and (3) do not hold. To see this, let $\Sigma\ a_n$ be an infinite series in $\mathbb{R}$ that converges and has sum S but is not absolutely convergent. For each $F \in \omega^{<\omega}$ let $x_F = \Sigma_{k \in F}$ a_k, and therefore

$$y_n = x_{\{0,1,\ldots,n\}} = a_0 + a_1 + \ldots + a_n.$$

Then

(1) $\langle y_n \rangle$ converges to S;

(2) every real number z is a cluster point of $\langle x_F : F \in \omega^{<\omega} \rangle$.

To check (2), let $\epsilon > 0$ and let $F \in \omega^{<\omega}$, say $F = \{i_1, \ldots, i_k\}$. Let $\sigma : \omega \to \omega$ be a permutation of ω such that $\Sigma\ a_{\sigma(n)} = z$. Let $\sigma(j_1) = i_1, \ldots, \sigma(j_k) = i_k$ and then choose n so large that $n > j_1, \ldots, j_k$ and $|z - (a_{\sigma(0)} + \cdots + a_{\sigma(n)})| < \epsilon$. Let $G = \{\sigma(0), \ldots, \sigma(n)\}$; $F \subseteq G$ and $|z - x_G| < \epsilon$. □

The construction in Theorem 3.1, which begins with an ω-net and gives a sequence, works only for the case $\kappa = \omega$; on the other hand, the construction in the opposite direction, which begins with a κ-sequence and gives a κ-net, works for all κ. For a further discussion of these ideas, see [32], p. 25.

Theorem 3.2. *Let X be a space, let $q \in X$, and let Φ be an operator. Given any κ-sequence $\langle x_\alpha : \alpha < \kappa \rangle$ in X, there is a κ-net $\langle y_F \rangle$ in X such that the following hold:*

(1) $\{y_F\} \subseteq \{x_\alpha\}$;

(2) $x_\alpha \to_\Phi q \Leftrightarrow y_F \to_\Phi q$;

(3) *q is a Φ-cluster point of $\langle x_\alpha \rangle \Leftrightarrow q$ is a Φ-cluster point of $\langle y_F \rangle$.*

Outline of proof. For each $F \in \kappa^{<\omega}$ take $y_F = x_\alpha$, where α is the largest element of F. □

Corollary 3.3. *For any space X and any operator Φ, the following are equivalent:*

(1) *every ω-net in X has a Φ-cluster point;*
(2) *every sequence $\langle x_n : n \in \omega\rangle$ in X has a Φ-cluster point.*

Corollary 3.4. *For any space X, the following are equivalent:*

(1) *X is a Fréchet space;*
(2) *if $q \in A^-$, then there is an ω-net $\langle x_F\rangle$ in A such that $x_F \to q$.*

Corollary 3.5. *For any space X, the following are equivalent:*

(1) *X is a sequential space;*
(2) *every subset A of X that satisfies the following condition is a closed set: if $\langle x_F\rangle$ is an ω-net in A and $x_F \to q$, then $q \in A$.*

We now investigate the extent to which nets can be restricted to κ-nets for a general theory of convergence and cluster points. Let $\langle D, \leq\rangle$ be a directed set. Recall that $\leq$ is reflexive, transitive, and directed (given $d_1, \dots, d_k \in D$, there exists $e \in D$ such that $d_j \leq e$ for $1 \leq j \leq k$). However, the relation $\leq$ need not be anti-symmetric. In this case, we can define an equivalence relation $\sim$ on $D \times D$ by

$$d \sim e \Leftrightarrow d \leq e \text{ and } e \leq d.$$

Then $\langle D/\sim, \leq^*\rangle$ (where $\leq^*$ is defined by $[d] \leq^* [e] \Leftrightarrow d \leq e$) is a directed set in which $\leq^*$ is anti-symmetric. Moreover, we have:

Lemma 3.6. *Let $\langle x_d : d \in D\rangle$ be a net in X and let $\langle x_{[d]} : [d] \in D/\sim\rangle$ be a net in X obtained as follows:*

$$x_{[d]} = x_e, \text{ where } e \in [d].$$

Then

(1) $\{x_{[d]}\} \subseteq \{x_d\}$;
(2) *if $x_d \to_\Phi q$, then $x_{[d]} \to_\Phi q$;*
(3) *if q is a Φ-cluster point of $\langle x_{[d]} : [d] \in D/\sim\rangle$, then q is a Φ-cluster point of $\langle x_d : d \in D\rangle$.*

Lemma 3.7. *Let $\langle x_d : d \in D\rangle$ be a net in X such that $|D| = \kappa$ and the order relation $\leq$ for D is anti-symmetric. Then there is a κ-net $\langle y_F\rangle$ in X such that the following hold:*

(1) $\{y_F\} \subseteq \{x_d\}$;
(2) $x_d \to_\Phi q \Leftrightarrow y_F \to_\Phi q$;
(3) *q is a Φ-cluster point of $\langle x_d\rangle \Leftrightarrow q$ is a Φ-cluster point of $\langle y_F\rangle$.*

Proof. Without loss of generality we may assume that $D = \kappa$. The required κ-net is obtained as follows: for $F \in \kappa^{<\omega}$, say $F = \{d_1, \ldots, d_k\}$, let

$$y_F = x_d,$$

where $d \in D$ is chosen as follows: $d_1 \leq_D d, \ldots, d_k \leq_D d$ and $d \in F$ if possible. Note the following: if $G = \{d_1, \ldots, d_k, e\}$ with $d_1 \leq_D e, \ldots, d_k \leq_D e$, then $y_G = x_e$ (anti-symmetry of $\leq_D$ used here).

To prove one direction of (2), assume that $y_F \to_\Phi q$ and let $q \in V$(open); we find $d \in D$ such that if $d \leq_D e$, then $x_e \in \Phi(V)$. There is $F = \{d_1, \ldots, d_k\}$ in $\kappa^{<\omega}$ such that if $F \subseteq G$, then $y_G \in \Phi(V)$. Choose $d \in D$ such that $d_1 \leq_D d, \ldots, d_k \leq_D d$. Now let $d \leq_D e$. Then $F \subseteq G = \{d_1, \ldots, d_k, e\}$ and therefore $y_G \in \Phi(V)$. But $y_G = x_e$ and therefore $x_e \in \Phi(V)$. □

Lemma 3.8. (***expansion property***) *Let $\langle x_F : F \in \lambda^{<\omega}\rangle$ be a λ-net in X. For each $\kappa \geq \lambda$, there is a κ-net $\langle y_F : F \in \kappa^{<\omega}\rangle$ in X such that*

(1) $\{y_F\} \subseteq \{x_F\}$;
(2) $x_F \to_\Phi q \Leftrightarrow y_F \to_\Phi q$;
(3) *q is a Φ-cluster point of the λ-net $\langle x_F\rangle \Leftrightarrow q$ is a Φ-cluster point of the κ-net $\langle y_F\rangle$.*

Outline of proof. The required κ-net is obtained as follows: for $F \in \kappa^{<\omega}$ let

$$y_F = x_{F\cap\lambda}.$$

To prove one direction of (2), assume that $y_F \to_\Phi q$ and let V be an open neighborhood of q. There is $F \in \kappa^{<\omega}$ such that if $F \subseteq G$, then $y_G \in \Phi(V)$. Let $F_0 = F \cap \lambda$ and let $F_0 \subseteq G$ with $G \in \lambda^{<\omega}$; we will show that $x_G \in \Phi(V)$. Let $L = F \cup G$, and note that $y_L \in \Phi(V)$. But $L \cap \lambda = G$, and therefore $y_L = x_G$ and $x_G \in \Phi(V)$ as required. □

As a consequence of Lemmas 3.6, 3.7, and 3.8 we have:

Theorem 3.9. *Let X be a space, let Φ be an operator, and let κ be an infinite cardinal. The following are equivalent:*

(1) *every κ-net in X has a Φ-cluster point;*

(2) *every λ-net in X with $\omega \leq \lambda \leq \kappa$ has a Φ-cluster point;*

(3) *every net $\langle x_d : d \in D \rangle$ in X with $|D| \leq \kappa$ has a Φ-cluster point;*

(4) *every net $\langle x_d : d \in D \rangle$ in X with $|D| = \kappa$ has a Φ-cluster point.*

Proof. For (1) $\Rightarrow$ (2), use Lemma 3.8. To prove (2) $\Rightarrow$ (3), let $\langle x_d : d \in D \rangle$ be a net in X with $|D| \leq \kappa$. Let $\langle x_{[d]} : [d] \in D/\sim \rangle$ be the net obtained from $\langle x_d : d \in D \rangle$ as described in Lemma 3.6, and then let $\langle y_F : F \in \lambda^{<\omega} \rangle$ be the net obtained from $\langle x_{[d]} : [d] \in D/\sim \rangle$ as described in Lemma 3.7. By (2), $\langle y_F \rangle$ has a Φ-cluster point q; it follows that q is Φ-cluster point of $\langle x_d : d \in D \rangle$ as required. □

For the remainder of this section we work with the operator $\Phi(V) = V$. We now offer the following definitions as extensions of Fréchet and sequential spaces to higher cardinality. This has already been done by Meyer in [20] with nets whose directed sets have cardinality at most κ; by Lemmas 3.6, 3.7, and 3.8, the two approaches are the same.

Definition. (also see Meyer [20]) Let X be a space.

• X is *κ-Fréchet* if the following holds: if $q \in A^-$, then there is a κ-net $\langle x_F \rangle$ in A such that $x_F \to q$.

• X is a *κ-net* space if every κ-net closed subset of X is a closed set (A is *κ-net closed* if it satisfies the following property: if $\langle x_F \rangle$ is a κ-net in A and $x_F \to q$, then $q \in A$).

The following hold:

• if X is λ-Fréchet (respectively, a λ-net space) and $\lambda \leq \kappa$, then X is κ-Fréchet (respectively, a κ-net space); see Lemma 3.8;

• for all κ: $\chi(X) \leq \kappa \Rightarrow X$ is κ-Fréchet $\Rightarrow X$ is a κ-net space $\Rightarrow t(X) \leq \kappa$; for each κ, the converses fail (examples at the end of this section);

• X is a Fréchet space $\Leftrightarrow$ X is an ω-Fréchet space (see Corollary 3.4);

• X is a sequential space $\Leftrightarrow$ X is an ω-net space (see Corollary 3.5).

Theorem 3.10. *Let X be a space and let κ be an infinite cardinal. The following are equivalent:*

(1) *X is κ-Fréchet;*

(2) *if $q \in A^-$, then there is a net $\langle x_d : d \in D\rangle$ in A with $|D| \leq \kappa$ such that $x_d \to q$ (Meyer's definition of κ-Fréchet).*

Outline of proof that (2) $\Rightarrow$ (1)*.* Use Lemma 3.6(2), then Lemma 3.7(2), and then Lemma 3.8(2). □

Theorem 3.11. *Let X be a space and let κ be an infinite cardinal. The following are equivalent:*

(1) *X is a κ-net space;*

(2) *every $A \subseteq X$ with the following property is a closed set: if $\langle x_d : d \in D\rangle$ is a net in A with $|D| \leq \kappa$ and $x_d \to q$, then $q \in A$ (Meyer's definition of a κ-net space).*

In [24], Nyikos outlines a general approach to convergence as follows. Let $\mathcal{P}$ be a class of directed sets. A space $\langle X, \tau\rangle$ is a $\mathcal{P}$-*net* space, or is $\mathcal{P}$-*pseudo-radial*, if every subset A of X with the following property is a closed set: if $\langle x_d : d \in D\rangle$ is a net in A with $D \in \mathcal{P}$, and $x_d \to q$, then $q \in A$. Likewise, $\langle X, \tau\rangle$ is a $\mathcal{P}$-*Fréchet* space, or is $\mathcal{P}$-*radial*, if given $q \in A^-$, there is a net $\langle x_d : d \in D\rangle$ in A with $D \in \mathcal{P}$ such that $x_d \to q$. Here are some possible choices for $\mathcal{P}$:

(1) $\mathcal{P}$ = class of all directed sets;
(2) $\mathcal{P} = \{\kappa\colon \kappa$ an infinite regular cardinal$\}$;
(3) $\mathcal{P}$ = class of all directed sets of cardinality at most κ;
(4) $\mathcal{P} = \{\kappa^{<\omega}\}$;
(5) $\mathcal{P} = \{\mathrm{D}\}$, where D a directed set;
(6) $\mathcal{P} = \{\omega\}$.
(7) $\mathcal{P} = \{\omega^{<\omega}\}$.

To elaborate: (1) is used by Kelley in [18]; (2) gives the class of radial and pseudoradial spaces; (3) is used by Meyer in his approach to κ-net spaces and κ-Fréchet spaces; (4) gives an alternate but equivalent approach to these two classes of spaces; (5) is discussed by Arens in [2]; either (6) or (7) gives the class of Fréchet and sequential spaces.

We end this section with several examples and folklore results on κ-net and κ-Fréchet spaces.

Definition. A space X is *κ-transitive* if the following holds for every κ-net $\langle x_F \rangle$ in X: if $x_F \to q$, and for each $F \in \kappa^{<\omega}$ there is a κ-net $\langle x_{F,G} : G \in \kappa^{<\omega} \rangle$ in X such that $x_{F,G} \to x_F$, then there is a κ-net in $\{x_{F,G} : F, G \in \kappa^{<\omega}\}$ that converges to q.

Clearly every κ-Fréchet space is κ-transitive. In fact:

Theorem 3.12. *For any space X, the following are equivalent:*

(1) *X is κ-Fréchet;*
(2) *X is a κ-net κ-transitive space.*

Proof. Assume (2), let $q \in A^-$, and let $L =$

$$\{p : p \in X \text{ and there exists a } \kappa\text{-net } \langle x_F \rangle \text{ in } A \text{ such that } x_F \to p\}.$$

Note that $A \subseteq L$. By κ-transitivity, the set L is a κ-net closed set; since X is a κ-net space, L is a closed set. We now have $q \in A^- \subseteq L$ as required. □

The next theorem is due to Birkhoff (for $\kappa = \omega$) with an acknowledgment to Baer (see [6]); also see Nyikos [24] and Meyer [21].

Theorem 3.13. *Let $\langle X, \tau \rangle$ be a topological space and let κ be an infinite cardinal. Then there is a topology ρ on X such that*

(1) *$\tau \subseteq \rho$;*

(2) *for every κ-net $\langle x_F \rangle$ in X, $x_F \to q$ for $\tau \Leftrightarrow x_F \to q$ for ρ;*

(3) *$\langle X, \rho \rangle$ is a κ-net space;*

(4) *if σ is any κ-net space topology on X such that $\tau \subseteq \sigma$, then $\rho \subseteq \sigma$.*

Outline of the proof. Let $\mathcal{C} = \{A : A$ is a κ-net closed subset of $\langle X, \tau\rangle\}$. Then $\mathcal{C}$ satisfies all of the axioms for the closed sets of a topology ρ on X. □

Corollary 3.14. *Let $\langle X, \tau\rangle$ be a topological space, let κ be an infinite cardinal, and assume that X is not κ-transitive. Then there is a topology ρ on X with $\tau \subseteq \rho$ and such that $\langle X, \rho\rangle$ is a κ-net space that is not κ-Fréchet.*

Proof. By Theorem 3.13, there is a κ-net space topology ρ on X with $\tau \subseteq \rho$ such that $\langle X, \tau\rangle$ and $\langle X, \rho\rangle$ have the same convergent κ-nets. Since $\langle X, \tau\rangle$ is not κ-transitive, $\langle X, \rho\rangle$ is also not κ-transitive and therefore is not κ-Fréchet. □

Definition. Let X be a topological space and let κ be an infinite cardinal. Define $cl_\kappa : P(X) \to P(X)$ by $cl_\kappa(A) = \bigcup\{A_\alpha : \alpha < \kappa^+\}$, where

$A_0 = A$;

$A_\alpha = \{q$: there is a κ-net $\langle x_F\rangle$ in $\bigcup\{A_\beta : \beta < \alpha\}$ such that $\{x_F \to q\}$.

The following properties hold:

- for all $A \subseteq X$, $A \subseteq cl_\kappa(A) \subseteq A^-$;
- $cl_\kappa(A)$ is a κ-net closed set;
- A is κ-net closed $\Leftrightarrow cl_\kappa(A) = A$;
- X is a κ-net space $\Leftrightarrow$ for all $A \subseteq X$, $cl_\kappa(A) = A^-$;
- X is κ-Fréchet $\Leftrightarrow$ for all $A \subseteq X$, $A_1 = A^-$.

Example 1. For each cardinal κ there is a Fréchet space of character κ. Let κ have the discrete topology and let $X = \kappa \cup \{q\}$ be the Alexandroff one-point compactification of κ; X is a Fréchet space with character κ. □

Example 2. For each cardinal κ there is a space of tightness $\leq \kappa$ that is not a κ-net space. Let p be a free ultrafilter on κ such that every base for p has cardinality $> \kappa$. (A collection $\mathcal{B} \subseteq p$ is a *base* for p if given any $U \in p$, there exists $B \in \mathcal{B}$ such that $B \subseteq U$.) Let $X = \kappa \cup \{p\}$; each point of κ is isolated, and a local base for p is $\{U \cup \{p\} : U \in p\}$. Clearly X has tightness κ. Moreover, κ is not closed in X, and thus the proof if complete if we can show that κ is a κ-net closed set. Let $\langle x_F \rangle$ be a κ-net in κ; we show that $\langle x_F \rangle$ cannot converge to p. For each $F \in \kappa^{<\omega}$ let $A_F = \{x_G : G \in \kappa^{<\omega}$ and $F \subseteq G\}$. Now $\{A_F : F \in \kappa^{<\omega}\}$ cannot be a base for p, and therefore there exists $U \in p$ such that $A_F \not\subseteq U$ for all $F \in \kappa^{<\omega}$. Let $x_{H(F)} \in A_F - U$. Then $\langle x_{H(F)} \rangle$ is a κ-subnet of $\langle x_F \rangle$ that does not converge to p. Thus $\langle x_F \rangle$ itself cannot converge to p as required. □

The following Lemma plays a key role in Example 3 below.

Lemma 3.15. *Let $X = \kappa \cup \{q\}$ have the following topology: each point of κ is isolated; a local base for q is $\{X - E : E \subset \kappa$ and $|E| < \kappa\}$. Let $A \subseteq \kappa$.*

(1) *If $|A| < \kappa$, then q is not a limit point of A;*

(2) *if $|A| = \kappa$, then there is a κ-net in A that converges to q;*

(3) *X is a κ-Fréchet space;*

(4) *if κ is singular, then there is no λ-sequence in κ with λ regular that converges to q (and therefore X is not a pseudoradial space);*

(5) *if κ is regular, then X is a radial space.*

Proof of (2). Since $|A| = \kappa$, we may assume that $A = \{x_F : F \in \kappa^{<\omega}\}$, where $x_F \neq x_G$ for $F \neq G$. Now let E be a subset of κ with $|E| < \kappa$; we find $F_0 \in \kappa^{<\omega}$ such that if $F_0 \subseteq G$, then $x_G \notin E$. Let $\mathcal{F} = \{F \in \kappa^{<\omega} : x_F \in E\}$, and note that $|\mathcal{F}| < \kappa$. It follows that $|\bigcup_{F \in \mathcal{F}} F| < \kappa$, and therefore there exists a non-empty $F_0 \in \kappa^{<\omega}$ such that $F_0 \cap F = \varnothing$ for all $F \in \mathcal{F}$. Now let $F_0 \subseteq G$; then $x_G \notin E$ as required. The proof of (4) will be given at the end of this section. □

Example 3. For each cardinal κ there is a κ-net space that is not κ-Fréchet. The required space is the higher cardinality version of an example due to Franklin [10]; also see Arens [2]. Let $X = [\kappa \times (\kappa \cup \{\kappa\})] \cup \{q\}$, for each $\alpha < \kappa$ let L_α be the column $\{\langle \alpha, \beta \rangle : 0 \leq \beta < \kappa\}$, and let $L_\alpha^+ = L_\alpha \cup \{\langle \alpha, \kappa \rangle\}$. The topology on X is described as follows: (1) each point $\langle \alpha, \beta \rangle$ $(0 \leq \beta < \kappa)$ is isolated; (2) a local base for $\langle \alpha, \kappa \rangle$ is the collection $\{L_\alpha^+ - E : E \subset L_\alpha$ and $|E| < \kappa\}$; (3) a basic open neighborhood of q is obtained as follows: given $E \subset \kappa$ with $|E| < \kappa$, remove the columns $\{L_\alpha^+ : \alpha \in E\}$ from X; for each column L_α with $\alpha \notin E$, remove a set of cardinality $< \kappa$.

First let us show that X is a κ-net space. Suppose by way of contradiction that there is a κ-net closed set A with a limit point $z \notin A$. Note that no κ-net in A converges to z.

Claim: For all $\alpha < \kappa$, if $\langle \alpha, \kappa \rangle \notin A$, then $\langle \alpha, \kappa \rangle$ is not a limit point of A. [Proof: Suppose that $|A \cap L_\alpha| = \kappa$. By Lemma 3.15 applied to the subspace L_α^+ with the subspace topology, there is a κ-net in A that converges to $\langle \alpha, \kappa \rangle$, a contradiction. Thus $|A \cap L_\alpha| < \kappa$ and there is an open neighborhood of $\langle \alpha, \kappa \rangle$ that misses A.]

We now consider possible cases for z. By the *Claim*, $z \neq \langle \alpha, \kappa \rangle$. Suppose that $z = q$. Let $E = A \cap (\kappa \times \{\kappa\})$. Then $|E| < \kappa$; otherwise, by Lemma 3.15, there is a κ-net in A that converges to q. For each $\langle \alpha, \kappa \rangle$ with $\langle \alpha, \kappa \rangle \notin E$, there is an open neighborhood of $\langle \alpha, \kappa \rangle$ that misses A (see the *Claim*). One can now construct a basic open neighborhood of q that misses A, a contradiction.

It remains to prove that X is not κ-Fréchet. Clearly q is a limit point of $\kappa \times \kappa$. Let $\langle x_F \rangle$ be a κ-net in $\kappa \times \kappa$; we show that $\langle x_F \rangle$ does not converge to q. There are two cases.

• There exists a non-empty $E \subset \kappa$ with $|E| < \kappa$ and $H : \kappa^{<\omega} \to \kappa^{<\omega}$ (with $F \subseteq H(F)$ for all $F \in \kappa^{<\omega}$) such that for all $F \in \kappa^{<\omega}$, $x_{H(F)} \in \bigcup_{\alpha \in E} L_\alpha$.

In this case $\langle x_{H(F)} \rangle$ is a κ-subnet of $\langle x_F \rangle$ that does not converge to q; it follows that $\langle x_F \rangle$ does not converge to q as required.

• Given any non-empty $E \subset \kappa$ with $|E| < \kappa$ and any $H : \kappa^{<\omega} \to \kappa^{<\omega}$ (with $F \subseteq H(F)$ for all $F \in \kappa^{<\omega}$), there exists $G \in \kappa^{<\omega}$ such that $x_{H(G)} \notin \bigcup_{\alpha \in E} L_\alpha$.

Let $\kappa^{<\omega} = \{F_\alpha : \alpha < \kappa\}$. We construct $\{G_\alpha : \alpha < \kappa\} \subseteq \kappa^{<\omega}$ and a set $\{\gamma_\alpha : \alpha < \kappa\}$ of distinct ordinals in κ such that for all α, $x_{F_\alpha \cup G_\alpha} \in L_{\gamma_\alpha}$. The construction is by transfinite induction. Given $E = \{0\}$ and $H(F) = F_0 \cup F$, there exists G_0 such that $x_{F_0 \cup G_0} \notin L_0$. Choose γ_0 such that $x_{F_0 \cup G_0} \in L_{\gamma_0}$. Now assume that $\alpha > 0$ and that $\{G_\beta : \beta < \alpha\}$ and $\{\gamma_\beta : \beta < \alpha\}$ have been constructed. Given $E = \{\gamma_\beta : \beta < \alpha\}$ and $H(F) = F_\alpha \cup F$, there exists G_α such that $x_{F_\alpha \cup G_\alpha} \notin \bigcup_{\gamma_\beta \in E} L_{\gamma_\beta}$. Now choose γ_α such that $x_{F_\alpha \cup G_\alpha} \in L_{\gamma_\alpha}$.

Finally, define H by $H(F) = F \cup G_\alpha$, where $F = F_\alpha$. The columns $\{L_{\gamma_\alpha} : \alpha < \kappa\}$ are pairwise disjoint and therefore q has a neighborhood that misses $\{x_{H(F)}\}$; it follows that $\langle x_F \rangle$ does not converge to q as required. □

We end this section with a few further comments on κ-sequences and κ-nets. Recall that X is *radial* if for every limit point q of A, there is a λ-sequence in A with λ regular that converges to q; also, X is *pseudoradial* if for every non-closed subset A of X there exists a point $q \notin A$ and a λ-sequence in A (λ regular) that converges to q. The following Lemma captures a key property of λ-sequences.

Lemma 3.16. *Let $A \subseteq X$ with $|A| \leq \kappa$, let q be a point of X such that $q \notin \{x\}^-$ for all $x \in A$, and let λ be a regular cardinal with $\lambda > \kappa$. Then no λ-sequence in A converges to q.*

Proof. Suppose by way of contradiction that there is a λ-sequence $\langle x_\alpha \rangle$ in A that converges to q with $\lambda > \kappa$ and regular. There is some $x \in A$ such that $\{\alpha : \alpha < \lambda \text{ and } x_\alpha = x\}$ is cofinal in λ. This contradicts $q \notin \{x\}^-$. □

Note that part (4) of Lemma 3.15 follows immediately from Lemma 3.16. In addition, we have the following higher cardinality version of Theorem 5.5 in [24].

Theorem 3.17. *Let X be a radial space with $t(X) \leq \kappa$. Then X is κ-Fréchet.*

Proof. Let q be a limit point of A; we may assume that $q \notin \{x\}^-$ for all $x \in A$ and that $|A| \leq \kappa$ (since $t(X) \leq \kappa$). Since X is radial, there is a λ-sequence in A with λ regular that converges to q. By Lemma 3.16, $\lambda \leq \kappa$; it now follows that there is a κ-net in A that converges to q as required. □

4. Convergence structures defined in terms of κ-nets; cryptomorphisms with κ-net and κ-Fréchet spaces

Throughout this section we will work with the operator $\Phi(V) = V$. The goal is to introduce a convergence structure on a set X that is defined in terms of κ-nets and then give a cryptomorphic description of κ-net spaces and κ-Fréchet spaces in terms of certain axioms for these convergence structures. These results closely parallel the classical study of sequential and Fréchet spaces in terms of sequential convergence structures and can be viewed as extending these results to higher cardinality. For a survey of the classical theory, see [11], [13], and [8].

We begin with a list of convergence properties of κ-nets that, except for (4), hold for any topology; these will serve as axioms for convergence.

Lemma 4.1. *Let X be a space and let κ be an infinite cardinal. The following convergence properties hold for κ-nets $\langle x_F \rangle$ in X.*

(1) *If $x_F = q$ for all $F \in \kappa^{<\omega}$, then $x_F \to q$.*

(2) *If $x_F \to q$, then every κ-subnet of $\langle x_F \rangle$ also converges to q.*

(3) *If $\langle x_F \rangle$ does not converge to q, then there is a κ-subnet $\langle x_{H(F)} \rangle$ of $\langle x_F \rangle$ such that $q \notin \{x_{H(F)} : F \in \kappa^{<\omega}\}^-$ and therefore no κ-net in $\{x_{H(F)} : F \in \kappa^{<\omega}\}$ converges to q.*

(4) *If X is κ-Fréchet, then X is κ-transitive.*

Property (3), essentially due to Arens [2], is a strengthening of a more traditional property of convergence: if $\langle x_F \rangle$ does not converges to q, then there is a κ-subnet $\langle x_{H(F)} \rangle$ of $\langle x_F \rangle$ such that no κ-subnet of $\langle x_{H(F)} \rangle$ converges to q.

Definition. Let X be a set and let κ be an infinite cardinal. A *κ-net convergence structure* on X is a collection $\mathcal{L}$ of pairs $\langle \xi, q \rangle$, where ξ is a κ-net in X and $q \in X$. We write

$$\xi \to q \text{ in } \mathcal{L}$$

or

$$x_F \to q \text{ in } \mathcal{L} \text{ (where } \xi(F) = x_F)$$

instead of the proper $\langle \xi, q \rangle \in \mathcal{L}$ and say that ξ *converges to q in $\mathcal{L}$* (or *$\mathcal{L}$-converges* to q). In our discussion of κ-net convergence structures, we will use the following ideas. Let $A \subseteq X$.

- The set A is *$\mathcal{L}$-closed* if it satisfies the following condition: if $\langle x_F \rangle$ is a κ-net in A and $x_F \to q$ in $\mathcal{L}$, then $q \in A$.

- The *$\mathcal{L}$-closure* of A, denoted $A^{\mathcal{L}}$, is defined by $A^{\mathcal{L}} = \{q : q \in X$ and there is a κ-net in A that $\mathcal{L}$-converges to $q\}$.

• The collection $\{A_\alpha : 0 \leq \alpha < \kappa^+\}$ is defined inductively as follows: $A_0 = A$; for $0 < \alpha < \kappa^+$, $A_\alpha = (\bigcup_{\beta<\alpha} A_\beta)^{\mathcal{L}}$. Informally, one starts with A and takes the $\mathcal{L}$-closure κ^+ times.

For each $A \subseteq X$, the following properties hold:

• A is $\mathcal{L}$-closed $\Leftrightarrow A^{\mathcal{L}} \subseteq A$;
• $\bigcup_{\alpha<\kappa^+} A_\alpha$ is an $\mathcal{L}$-closed set.

Let $\mathcal{L}$ be a κ-net convergence structure on a set X. The fundamental problem is to find axioms for $\mathcal{L}$ such that the following three properties hold:

(P1) the collection $\mathcal{C}$ of $\mathcal{L}$-closed subsets of X satisfy the axioms for the closed sets of a topology $\tau(\mathcal{L})$ on X;

(P2) if $\langle x_F \rangle$ is a κ-net in X such that $x_F \to q$ in $\mathcal{L}$, then $x_F \to q$ for the topology $\tau(\mathcal{L})$;

(P3) if $\langle x_F \rangle$ is a κ-net in X such that $x_F \to q$ for the topology $\tau(\mathcal{L})$, then $x_F \to q$ in $\mathcal{L}$.

For **(P1)** and **(P2)** to hold, we need just two axioms for $\mathcal{L}$:

(L1) If $x_F = q$ for all $F \in \kappa^{<\omega}$, then $x_F \to q$ in $\mathcal{L}$.

(L2) If a κ-net $\mathcal{L}$-converges to q, then every κ-subnet also $\mathcal{L}$-converges to q.

Here is a list of properties that follow from (L1) and (L2).

Lemma 4.2. *Let $\mathcal{L}$ be a κ-net convergence structure on X that satisfies (L1) and (L2).*

(1) *for all $A \subseteq X$, $A \subseteq A^{\mathcal{L}}$;*

(2) *the collection $\mathcal{C}$ of $\mathcal{L}$-closed subsets of X satisfy the axioms for the closed sets of a topology $\tau(\mathcal{L})$ on X;*

(3) *if $\langle x_F \rangle$ is a κ-net in X such that $x_F \to q$ in $\mathcal{L}$, then $x_F \to q$ for the topology $\tau(\mathcal{L})$;*

(4) *$\langle X, \tau(\mathcal{L}) \rangle$ is a κ-net space;*

(5) *for all $A \subseteq X$, $\bigcup_{\alpha<\kappa^+} A_\alpha = A^-$, where A^- is the closure of A in the topology $\tau(\mathcal{L})$.*

Proof. Use (L1) to prove (1); use (L2) to show that the union of two $\mathcal{L}$-closed sets is an $\mathcal{L}$-closed set; use (L2) and (2) to prove (3); use (3) to prove (4). To prove that $\bigcup_{\alpha<\kappa^+} A_\alpha \subseteq A^-$, use (3) to show that $A_\alpha \subseteq A^-$ for all α. For the other direction, we have: $A^- \subseteq (\bigcup_{\alpha<\kappa^+} A_\alpha)^-$; $\bigcup_{\alpha<\kappa^+} A_\alpha$ is an $\mathcal{L}$-closed set and therefore is a closed set. Thus $A^- \subseteq \bigcup_{\alpha<\kappa^+} A_\alpha$ as required. $\square$

To obtain property **(P3)**, we need additional axiom(s). One possibility is to choose these two.

(L3) If $\langle x_F \rangle$ is a κ-net in X that does not converge to q in $\mathcal{L}$, then there is a κ-subnet $\langle x_{H(F)} \rangle$ of $\langle x_F \rangle$ such that no κ-net in $\{x_{H(F)} : F \in \kappa^{<\omega}\}$ $\mathcal{L}$-converges to q (in other words, $q \notin (\{x_{H(F)} : F \in \kappa^{<\omega}\})^{\mathcal{L}}$).

(L4) If $\langle x_F \rangle$ is a κ-net in X such that $x_F \to q$ in $\mathcal{L}$, and for each $F \in \kappa^{<\omega}$ there is a κ-net $\langle x_{F,G} : G \in \kappa^{<\omega} \rangle$ in X such that $x_{F,G} \to x_F$ in $\mathcal{L}$, then there is a κ-net in $\{x_{F,G} : F, G \in \kappa^{<\omega}\}$ that $\mathcal{L}$-converges to q.

The choice of (L3) and (L4) is actually more than we need and together suffice to give a cryptomorphic description of the κ-Fréchet spaces. There is another possibility, a stronger version of (L3) that leads to a cryptomorphic description of the κ-net spaces.

$(\mathrm{L3})^+$ If $\langle x_F \rangle$ is a κ-net in X that does not converge to q in $\mathcal{L}$, then there is a κ-subnet $\langle x_{H(F)} \rangle$ of $\langle x_F \rangle$ such that $q \notin \bigcup_{\alpha<\kappa^+} \{x_{H(F)} : F \in \kappa^{<\omega}\}_\alpha$.

This axiom is inspired by Arens [2]; also compare with the condition (U_{ω_1}) in [17].

The three axioms (L3), $(\mathrm{L3})^+$, and (L4) are related as follows:

(1) $(\mathrm{L3})^+ \Rightarrow (\mathrm{L3})$;

(2) $(\mathrm{L3}) + (\mathrm{L4}) \Rightarrow (\mathrm{L3})^+$ [note that $(\mathrm{L1}) + (\mathrm{L4}) \Rightarrow \bigcup_{\alpha<\kappa^+} A_\alpha \subseteq A^{\mathcal{L}}$].

The choice of the axiom $(\mathrm{L3})^+$ is justified by the following:

Lemma 4.3. *Let $\mathcal{L}$ be a κ-net convergence structure on X that satisfies $(L1)$ and $(L2)$ and let $\tau(\mathcal{L})$ be the topology on X whose closed sets are the $\mathcal{L}$-closed subsets of X. The following are equivalent:*

(1) *if $\langle x_F \rangle$ is a κ-net in X such that $x_F \to q$ in the topology $\tau(\mathcal{L})$, then $x_F \to q$ in $\mathcal{L}$;*

(2) *$\mathcal{L}$ satisfies $(L3)^+$.*

Proof. Use the fact from Lemma 4.2(5) that $\{x_{H(F)} : F \in \kappa^{<\omega}\}^- = \bigcup_{\alpha<\kappa^+} \{x_{H(F)} : F \in \kappa^{<\omega}\}_\alpha$. □

We now give a cryptomorphic description of κ-net spaces and κ-Fréchet spaces in terms of κ-net convergence structures. The concept of a cryptomorphism is fairly simple, but unfortunately it is somewhat awkward to describe. Let X be a set and let α and β denote types of structures on X (for example, collections of subsets of X, binary relations on X, and so on). Structures of type α have axioms $\alpha(1), \ldots, \alpha(\mathrm{k})$ and structures of type β have axioms $\beta(1), \ldots, \beta(\mathrm{n})$. To establish a cryptomorphism between these two types of structures, we are required to prove two theorems that we denote by $\alpha \Rightarrow \beta(\alpha)$ and $\beta \Rightarrow \alpha(\beta)$.

In the proof of $\alpha \Rightarrow \beta(\alpha)$, we start with a structure of type α on X and then use it to construct a structure of type β on X. We also need to prove that this structure of type β satisfies the axioms $\beta(1), \ldots, \beta(\mathrm{n})$, where we are allowed to use the axioms $\alpha(1), \ldots, \alpha(\mathrm{k})$ in the proof. Similar remarks apply to the proof of $\beta \Rightarrow \alpha(\beta)$: begin with a structure of type β on X, use it to construct a structure of type α on X, and then verify the axioms $\alpha(1), \ldots, \alpha(\mathrm{k})$.

In addition to these two theorems, we are also required to verify two iterative properties that we denote by $\alpha(\beta(\alpha)) = \alpha$ and $\beta(\alpha(\beta)) = \beta$. To elaborate on $\alpha(\beta(\alpha)) = \alpha$: start with a structure of type α on X, call it A; apply the construction in the proof of $\alpha \Rightarrow \beta(\alpha)$ to A to obtain a structure of type β on X, call it $B(A)$; now use $B(A)$ and the construction in the proof of $\beta \Rightarrow \alpha(\beta)$ to obtain a structure of type α on X, call it $A(B(A))$; finally, show that the two structures A and $A(B(A))$, both of type α, are the same.

For the record, Bourbaki identifies at least three types of structures: algebraic structures (more precisely, various operations on a set), order structures, and topological structures (collections of subsets of a set).

We first establish a cryptomorphism between κ-net spaces and κ-net convergence structures that satisfy (L1), (L2), and (L3)$^+$.

Theorem 4.4. $[\tau \Rightarrow \mathcal{L}(\tau)]$ *Let $\langle X, \tau\rangle$ be a topological space. Then there is a κ-net convergence structure $\mathcal{L}(\tau)$ on X such that*

(1) *$\mathcal{L}(\tau)$ satisfies (L1), (L2), and (L3)$^+$;*

(2) *for every κ-net $\langle x_F\rangle$ in X, $x_F \to q$ in the topology $\tau \Leftrightarrow x_F \to q$ in $\mathcal{L}(\tau)$.*

Proof. Define $\mathcal{L}(\tau)$ as follows: for an arbitrary κ-net $\langle x_F\rangle$ in X,

$$x_F \to q \text{ in } \mathcal{L}(\tau) \Leftrightarrow x_F \to q \text{ in the topology } \tau.$$

By Lemma 4.1, (L1) and (L2) obviously hold. To prove (L3)$^+$, let $\langle x_F\rangle$ be a κ-net in X that does not $\mathcal{L}$-converge to q. Then $\langle x_F\rangle$ does not converge to q in τ, and therefore Lemma 4.1(3) applies and there is a κ-subnet $\langle x_{H(F)}\rangle$ of $\langle x_F\rangle$ such that $q \notin \{x_{H(F)} : F \in \kappa^{<\omega}\}^-$. Now for all $A \subseteq X$, $\bigcup_{\alpha<\kappa^+} A_\alpha \subseteq A^-$ (where $\{A_\alpha : \alpha < \kappa\}$ is defined in terms of $\mathcal{L}(\tau)$) and therefore $q \notin \bigcup_{\alpha<\kappa^+}\{x_{H(F)} : F \in \kappa^{<\omega}\}_\alpha$ as required. □

Note: In the proof of Theorem 4.4, we do not assume that τ is a κ-net space topology. However, this hypothesis will be needed to prove that $\tau = \tau(\mathcal{L}(\tau))$.

Theorem 4.5. $[\mathcal{L} \Rightarrow \tau(\mathcal{L})]$ *Let $\mathcal{L}$ be a κ-net convergence structure on X that satisfies (L1), (L2), and (L3)$^+$. Then there is a topology $\tau(\mathcal{L})$ on X such that*

(1) *$\langle X, \tau(\mathcal{L})\rangle$ is a κ-net space;*

(2) *for every κ-net $\langle x_F\rangle$ in X, $x_F \to q$ in $\mathcal{L} \Leftrightarrow x_F \to q$ in the topology $\tau(\mathcal{L})$.*

Proof. The collection $\mathcal{C} = \{A : A \subseteq X \text{ and } A^{\mathcal{L}} = A\}$ of $\mathcal{L}$-closed sets satisfies the axioms for the closed sets of the topology $\tau(\mathcal{L})$. Now use Lemmas 4.2 and 4.3. □

Theorem 4.6. *Let X be a set, let τ be a κ-net space topology on X, and let $\mathcal{L}$ be a κ-net convergence structure on X that satisfies (L1), (L2), and (L3)$^+$. Then $\mathcal{L} = \mathcal{L}(\tau(\mathcal{L}))$ and $\tau = \tau(\mathcal{L}(\tau))$.*

Proof. Let $\langle x_F \rangle$ be a κ-net in X. By Theorems 4.4 and 4.5:

(1) $x_F \to q$ in $\mathcal{L} \Leftrightarrow x_F \to q$ in the topology $\tau(\mathcal{L}) \Leftrightarrow x_F \to q$ in $\mathcal{L}(\tau(\mathcal{L}))$;

(2) $x_F \to q$ in the topology $\tau \Leftrightarrow x_F \to q$ in $\mathcal{L}(\tau) \Leftrightarrow x_F \to q$ in the topology $\tau(\mathcal{L}(\tau))$;

(3) $\tau(\mathcal{L}(\tau))$ is a κ-net space topology.

From (1) it follows that $\mathcal{L} = \tau(\mathcal{L}(\tau))$; to see that $\tau = \tau(\mathcal{L}(\tau))$, use (2), (3), and the Lemma below. □

Lemma 4.7. *Let X be a set and let τ_1 and τ_2 be κ-net space topologies on X such that for every κ-net ξ in X: $\xi \to q$ in the topology τ_1 if and only if $\xi \to q$ in the topology τ_2. Then $\tau_1 = \tau_2$.*

Next we establish a cryptomorphism between κ-Fréchet spaces and κ-net convergence structures that satisfy (L1), (L2), (L3), and (L4).

Theorem 4.8. $[\tau \Rightarrow \mathcal{L}(\tau)]$ *Let $\langle X, \tau \rangle$ be a κ-Fréchet space. Then there is a κ-net convergence structure $\mathcal{L}(\tau)$ on X such that*

(1) $\mathcal{L}(\tau)$ satisfies (L1), (L2), (L3), and (L4);

(2) for every κ-net $\langle x_F \rangle$ in X, $x_F \to q$ in the topology $\tau \Leftrightarrow x_F \to q$ in $\mathcal{L}(\tau)$.

Proof. Define $\mathcal{L}(\tau)$ as follows: for an arbitrary κ-net $\langle x_F \rangle$ in X,

$$x_F \to q \text{ in } \mathcal{L}(\tau) \Leftrightarrow x_F \to q \text{ in the topology } \tau.$$ □

Theorem 4.9. $[\mathcal{L} \Rightarrow \tau(\mathcal{L})]$ *Let $\mathcal{L}$ be a κ-net convergence structure on X that satisfies (L1), (L2), (L3), and (L4). Then there is a topology $\tau(\mathcal{L})$ on X such that*

(1) $\langle X, \tau(\mathcal{L}) \rangle$ is a κ-Fréchet space;

(2) for every κ-net $\langle x_F \rangle$ in X, $x_F \to q$ in $\mathcal{L} \Leftrightarrow x_F \to q$ in the topology $\tau(\mathcal{L})$.

Proof. By Theorem 4.5 and the fact that $(L_1) - (L_4) \Rightarrow (L3)^+$, there is a topology $\tau(\mathcal{L})$ on X such that $\langle X, \tau(\mathcal{L}) \rangle$ is a κ-net space that satisfies (2). By (L_4), $\langle X, \tau(\mathcal{L}) \rangle$ is also κ-transitive and therefore (1) also holds. □.

Theorem 4.10. *Let X be a set, let τ be a κ-Fréchet topology on X, and let $\mathcal{L}$ be a κ-net convergence structure on X that satisfies (L1), (L2), (L3), and (L4). Then $\mathcal{L} = \mathcal{L}(\tau(\mathcal{L}))$ and $\tau = \tau(\mathcal{L}(\tau))$.*

It is interesting to compare the results in this section with the ideas in Kelley (see p. 74 in [18]). A *convergence class* on X is a collection $\mathcal{C}$ of pairs $\langle \xi, q \rangle$, where ξ is a net in X, $q \in X$, and the following hold for each net ξ in X:

- if $\xi : D \to X$ with $\xi(d) = q$ for all $d \in D$, then $\xi \to q$ in $\mathcal{C}$;
- if $\xi \to q$ in $\mathcal{C}$, then every subnet of ξ also converges to q in $\mathcal{C}$;
- if ξ does not converge to q in $\mathcal{C}$, then there is a subnet ψ of ξ such that no subnet of ψ converges to q in $\mathcal{C}$;
- if $\xi \to q$ in $\mathcal{C}$, where $\xi : D \to X$, and for each $d \in D$ there is a net $\xi_d : E_d \to X$ such that $\xi_d \to \xi(d)$ in $\mathcal{C}$, then $\psi \to q$ in $\mathcal{C}$, where $\psi : D \times \prod_{d \in D} E_d \to X$ is the net defined by $\psi(d, f) = \xi_d(f(d))$.

Theorem 4.11. *Let $\mathcal{C}$ be a convergence class on X. Then there is a topology τ on X such that for every net $\langle x_d : d \in D \rangle$ in X: $x_d \to q$ in $\mathcal{C} \Leftrightarrow x_d \to q$ in τ.*

5. Φ-compactness and product theorems; (κ, ω)-nets and Lindelöf spaces

In this section we give characterizations of compact spaces, θ-compact spaces, and δ-compact spaces in terms of convergence of universal κ-nets. We will then use these results to give a unified approach to the following product theorems: any product of compact (θ-compact, δ-compact) spaces is compact (θ-compact, δ-compact). More generally, we also show that any product of H-sets is an H-set, thereby answering a question raised by Porter and Tikoo in [27]. The emphasis will be on the three operators $\Phi(V) = V$, $\Phi(V) = V^-$, and $\Phi(V) = V^{-\circ}$.

Definition. Let X be a space and let Φ be an operator. A subset A of X is Φ-*compact in X* if given any collection $\mathcal{V}$ of open sets in X that covers A, there is a finite subcollection of $\{\Phi(V) : V \in \mathcal{V}\}$ that covers A. For $\Phi(V) = V^-$ and $\Phi(V) = V^{-o}$, the following special terminology applies:

• A is Φ-compact in X and $\Phi(V) = V^-$	A is an H-*set in* X;
• X is Φ-compact and $\Phi(V) = V^-$	X is θ-*compact* (same as H-closed without the Hausdorff hypothesis);
• A is Φ-compact in X and $\Phi(V) = V^{-o}$	A is an N-*set in* X;
• X is Φ-compact and $\Phi(V) = V^{-o}$	X is δ-*compact*.

Here are some basic facts about these properties.

(1) For any space X, compact $\Rightarrow$ δ-compact $\Rightarrow$ θ-compact, but the converse does not hold (see the two well-known examples given below). For regular spaces, the three are equivalent.

(2) Every N-set in X is an H-set in X.

(3) If $A \subseteq X$ is θ-compact in the subspace topology, then A is an H-set in X.

(4) If X is Hausdorff, then every H-set in X is a closed set.

(5) If X is Urysohn, then every H-set in X is a θ-closed set.

(6) Every θ-closed subset of a θ-compact space X is an H-set in X.

(7) In a θ-compact Urysohn space X, every H-set in X is an N-set in X; in particular, every θ-compact Urysohn space is δ-compact (proof sketched below).

(8) Every δ-compact semiregular space is compact (X is *semiregular* if it has a base consisting of regular open sets).

For more details, see [25], [26], and [34].

Example 1 (a δ-compact space not compact). Let $\kappa\omega$ denote the Katětov extension of ω with the discrete topology. This space is described as follows: $\kappa\omega = \omega \cup T$, where T is a set of cardinality 2^{2^κ} that indexes the collection of all free ultrafilters on ω; for each $t \in T$, the collection $\{\{t\} \cup U : U \in \mathcal{U}_t\}$ is a local base for t, where $\mathcal{U}_t$ is the free ultrafilter indexed by t. The space $\kappa\omega$ has the following properties:

(a) Urysohn (therefore Hausdorff);
(b) δ-compact;
(c) not compact;
(d) there is a countable closed subset of $\kappa\omega$ that is not an H-set in $\kappa\omega$;

(e) $\kappa\omega - \omega$ is an H-set in $\kappa\omega$ that is not θ-compact in the subspace topology.

It is well known that $\kappa\omega$ is Urysohn, θ-compact, and not compact (see [25], p. 311); by (7) above, $\kappa\omega$ is δ-compact. To prove (d), let $\{U_n : n \in \omega\}$ be a partition of ω into a pairwise disjoint collection of infinite sets, and for each $n \in \omega$ let p_n be a free ultrafilter on ω with $U_n \in p_n$ and let t_n index the ultrafilter p_n; then $A = \{t_n : n \in \omega\}$ is the required subset of $\kappa\omega$. □

Example 2 (a θ-compact space not δ-compact; due to Urysohn). Let

$$X = Y^+ \cup Y^- \cup Z \cup \{p^+, p^-\},$$

where

$$\begin{aligned} Y^+ &= \{\langle 1/n, 1/m\rangle : m, n \in \mathbb{N}^+\}, \\ Y^- &= \{\langle 1/n, -1/m\rangle : m, n \in \mathbb{N}^+\}, \\ Z &= \{\langle 1/n, 0\rangle : n \in \mathbb{N}^+\}. \end{aligned}$$

Let $Y^+ \cup Y^- \cup Z$ have the subspace topology of $\mathbb{R}^2$. For each $k \geq 1$ let

$$\begin{aligned} V(p^+, k) &= \{\langle 1/n, 1/m\rangle : n \geq k, m \in \mathbb{N}^+\}, \\ V(p^-, k) &= \{\langle 1/n, -1/m\rangle : n \geq k, m \in \mathbb{N}^+\}. \end{aligned}$$

A countable local base for p^+ (respectively, p^-) is the collection $\{V(p^+, k) : k \in \mathbb{N}^+\}$ (respectively , $\{V(p^-, k) : k \in \mathbb{N}^+\}$). The space X has the following properties (for more details, see [25], p.300):

(a) Hausdorff but not Urysohn;

(b) θ-compact (note that $V(p^+, k)^- = V(p^+, k) \cup \{\langle 1/n, 0\rangle : n \geq k\}$);

(c) not δ-compact (note that $V(p^+, k)^{-o} = V(p^+, k)$). □

We now use the following definition to clarify the relationship between H-sets and N-sets.

Definition. (see [29]) A space X is *almost regular* if given $q \in X$ and an open neighborhood W of q, there exists an open neighborhood V of q such that $V^- \subseteq W^{-o}$.

This property can be characterized in a number of ways (see [29]): (1) $A^\delta = A^\theta$ for all $A \subseteq X$; (2) if H is a regular closed subset of X and $q \notin H$, then there exist disjoint open sets U and V in X such that $q \in U$ and $H \subseteq V$. Moreover, the following is easy to check.

Lemma 5.1. *Let X be almost regular and let $A \subseteq X$.*

(1) *if A is an H-set in X, then A is an N-set in X;*
(2) *if X is θ-compact, then X is δ-compact.*

Lemma 5.2. *Every θ-compact Urysohn space X is almost regular.*

Proof. Let W be an open neighborhood of $q \in X$. For each $x \in X - W$ let V_x and W_x be open sets such that $q \in V_x$, $x \in W_x$, and $V_x^- \cap W_x^- = \varnothing$. By θ-compactness,

$$X = W^- \cup {W_{x_1}}^- \cup \ldots \cup {W_{x_k}}^- \qquad (x_i \in X - W \text{ for } 1 \leq i \leq k).$$

Let $V = \bigcap \{V_{x_i} : 1 \leq i \leq k\}$ and let $N = X - ({W_{x_1}}^- \cup \ldots \cup {W_{x_k}}^-)$. We then have $V^- \subseteq N \subseteq W^-$. But N is an open set and therefore $V^- \subseteq W^{-o}$ as required. □

By Lemmas 5.1 and 5.2 we have:

Theorem 5.3. *Let X be a θ-compact Urysohn space and let $A \subseteq X$.*

(1) *If A is an H-set in X, then A is an N-set in X;*
(2) *X is δ-compact.*

For an alternate proof of Theorem 5.3, see exercises 4K(9) and 4N(11) in [25]. From Theorem 5.3 we have the following well-known result: X is compact Hausdorff if and only if it is Urysohn, θ-compact, and semi-regular.

We now turn to product theorems for these compactness properties. The following plays a key role in all of these results.

Theorem 5.4. *Let X be a space, let $A \subseteq X$, and let Φ be an operator. The following are equivalent:*

(1) *A is Φ-compact in X;*
(2) *if $\langle x_F \rangle$ is a κ-net in A, then there exists $q \in A$ such that q is a Φ-cluster point of $\langle x_F \rangle$ in X;*
(3) *if $\langle x_F \rangle$ is a κ-net in A that is universal in X, then there exists $q \in A$ such that $x_F \to_\Phi q$ in X.*

Proof. By Theorem 2.6, (2) $\Leftrightarrow$ (3). To prove that (1) $\Rightarrow$ (2), assume that $\langle x_F \rangle$ is a κ-net in A such that no point of A is a Φ-cluster point of $\langle x_F \rangle$. For each $q \in A$ let V_q be an open neighborhood of q and let $F_q \in \kappa^{<\omega}$ be such that for all G with $F_q \subseteq G$, $x_G \notin \Phi(V_q)$. Let $A \subseteq \Phi(V_{q_1}) \cup \ldots \cup \Phi(V_{q_k})$ and let $G_0 = F_{q_1} \cup \ldots \cup F_{q_k}$. The point x_{G_0} leads to a contradiction.

To prove that (2) $\Rightarrow$(1), let $\{V_\alpha : \alpha < \kappa\}$ be an open collection in X that covers A. Suppose by way of contradiction that no finite subcollection of $\{\Phi(V_\alpha) : \alpha < \kappa\}$ covers A. For each $F \in \kappa^{<\omega}$ let $x_F \in A - \bigcup_{\alpha \in F} \Phi(V_\alpha)$, let $q \in A$ be a Φ-cluster point of $\langle x_F \rangle$, and let $q \in V_\beta$. Given $\{\beta\} \in \kappa^{<\omega}$, there exists $F_0 \in \kappa^{<\omega}$ such that $\beta \in F_0$ and $x_{F_0} \in \Phi(V_\beta)$, a contradiction. □

Corollary 5.5. (*see* [14], [28]) *The following are equivalent for any space X:*

(1) *X is compact (θ-compact, δ-compact);*
(2) *every κ-net in X has a cluster point (θ-cluster point, δ-cluster point);*
(3) *every universal κ-net in X converges (θ-converges, δ-converges).*

We have used κ-nets to characterize various types of compactness (for example, X is compact if and only if every κ-net in X has a cluster point). With a suitable modification of a κ-net, we can also give similar characterizations of more general covering properties. We illustrate this idea with a characterization of Lindelöf spaces. Here is the required modification.

Definition. A (κ, ω)-*net* in X is a function ξ from $\kappa^{\leq\omega}$ into X; here $\kappa^{\leq\omega} = \{C : C \text{ a countable subset of } \kappa\}$ and is directed by $\subseteq$. We use the notation $\langle x_C \rangle$, where $C \in \kappa^{\leq\omega}$ and $x_C = \xi(C)$.

Theorem 5.6. *For any space X the following are equivalent:*
(1) *X is Lindelöf;*
(2) *every (κ,ω)-net in X has a cluster point.*

The proof is similar to (1) $\Leftrightarrow$ (2) of Theorem 5.4. For a considerably deeper discussion of these ideas, see the paper [33] by Vaughan; note especially the condition N$[\mathfrak{a},\mathfrak{b}]$.

In [35] (see p. 82), Willard states: "Filters are preferred to nets in dealing with convergence questions in topological spaces. The reason for this involves the difference that nets are, and must remain, essentially set-theoretic in nature, and hence passive, while filters can, with the addition of topological restrictions on their sets, become intimately involved with the structure of the space itself." (For example, one has open filters, z-filters, and so on; for additional examples, see p. 92 of [25].) While the claim has considerable merit, it should be noted that topological conditions can also be put on points; thus one has θ-cluster points, δ-cluster points, and so on. We now give an instance where the net approach may be the better choice.

Definition. An operator Φ is *productive* if it satisfies the following property with respect to the product topology: if $\{X_t : t \in T\}$ is a collection of topological spaces and $\prod_{t\in T} W_t$ is a basic canonical open subset of $\prod_{t\in T} X_t$, then $\prod_{t\in T} \Phi(W_t) \subseteq \Phi(\prod_{t\in T} W_t)$.

Lemma 5.7. *Each of the operators $\Phi(V) = V$, $\Phi(V) = V^-$, and $\Phi(V) = V^{-o}$ is productive.*

Proof. Let $\prod W_t$ be an open set with $W_t = X_t$ for all $t \notin \{t_1,\dots,t_k\}$. We want to show that

(1) $\prod {W_t}^- \subseteq (\prod W_t)^-$;
(2) $\prod {W_t}^{-o} \subseteq (\prod W_t)^{-o}$.

For (1), see [9], page 78. To prove (2):

$$\prod {W_t}^{-o} \subseteq \prod {W_t}^- \subseteq (\prod W_t)^-$$

and therefore

$$(\prod {W_t}^{-o})^o \subseteq (\prod W_t)^{-o}.$$

But $\prod W_t{}^{-o}$ is a canonical open set and therefore $(\prod W_t{}^{-o})^o = \prod W_t{}^{-o}$ and (2) holds as required. □

Theorem 5.8. *Let $\{X_t : t \in T\}$ be a collection of topological spaces, let Φ be a productive operator, and for each $t \in T$ let A_t be a Φ-compact set in X_t. Then $\prod A_t$ is a Φ-compact set in $\prod X_t$. In particular:*

(1) *if each X_t is compact, then $\prod X_t$ is compact (Tychonoff);*

(2) *if each X_t is θ-compact, then $\prod X_t$ is θ-compact (Chevally and Frink);*

(3) *if each X_t is δ-compact, then $\prod X_t$ is δ-compact;*

(4) *if each A_t is an H-set in X_t, then $\prod A_t$ is an H-set in $\prod X_t$;*

(5) *if each A_t is an N-set in X_t, then $\prod A_t$ is an N-set in $\prod X_t$.*

Proof. Let $\langle f_F : F \in \kappa^{<\omega}\rangle$ be a κ-net in $\prod A_t$ that is universal in $\prod X_t$. By Theorem 5.4, it suffices to prove that there exists $g \in \prod A_t$ such that $f_F \to_\Phi g$. For each $t \in T$, $\langle f_F(t) : F \in \kappa^{<\omega}\rangle$ is a κ-net in A_t that is universal in X_t, and therefore there exists $x_t \in A_t$ such that $f_F(t) \to_\Phi x_t$. Let $g(t) = x_t$, and let us show that $f_F \to_\Phi g$. Let $\prod W_t$ be a canonical open neighborhood of g with $W_t = X_t$ for all $t \notin \{t_1, \ldots, t_k\}$. We must find $F \in \kappa^{<\omega}$ such that for all $G \in \kappa^{<\omega}$ with $F \subseteq G$, $f_G \in \Phi(\prod W_t)$. Now Φ is productive, and therefore $\prod \Phi(W_t) \subseteq \Phi(\prod W_t)$. For $1 \le i \le k$, there exists $F_i \in \kappa^{<\omega}$ such that for all $G \in \kappa^{<\omega}$ with $F_i \subseteq G$, $f_G(t_i) \in \Phi(W_{t_i})$. Let $F = F_1 \cup \ldots \cup F_k$; then for all $G \in \kappa^{<\omega}$ with $F \subseteq G$, we have $f_G \in \prod \Phi(W_t) \subseteq \Phi(\prod W_t)$ as required. □

6. Properties related to Φ-compactness

This section is a systematic study of important ideas related to initial κ-compactness that go back to Alexandroff and Urysohn, with additional contributions by Smirnov, Noble, Vaughan, and others. For references and a summary of major results, see Theorem 2.2 of the paper by Stephenson [30].

Definition. Let κ be an infinite cardinal and let Φ be an operator. We define three compactness-like properties for an arbitrary space X:

NET(κ,Φ): Every κ-net in X has a Φ-cluster point.

SEQ(κ,Φ): Every λ-sequence $\langle x_\alpha : \alpha < \lambda \rangle$ in X with $\omega \leq \lambda \leq \kappa$ has a Φ-cluster point.

COVER(κ,Φ): If $\mathcal{V}$ is an open cover of X of cardinality at most κ, then there is a finite subcollection of $\{\Phi(V) : V \in \mathcal{V}\}$ that covers X.

Here is a quick summary of the relationship between these three properties. Let X be a space and let κ be an infinite cardinal.

- For any Φ — NET(κ,Φ) $\Rightarrow$ SEQ(κ,Φ); NET(κ,Φ) $\Rightarrow$ COVER(κ,Φ)
- For $\Phi(V) = V$ — NET(κ,Φ) $\Leftrightarrow$ SEQ(κ,Φ) $\Leftrightarrow$ COVER(κ,Φ)
- For $\Phi(V) = V^{-o}$ — NET(κ,Φ) $\Leftrightarrow$ SEQ(κ,Φ)
- For any Φ and $\kappa = \omega$ — NET(ω,Φ) $\Leftrightarrow$ SEQ(ω,Φ)
- For any Φ — NET(κ,Φ) for all κ $\Leftrightarrow$ COVER(κ,Φ) for all κ

On the other hand, for the operators $\Phi(V) = V^-$ and $\Phi(V) = V^{-o}$, there is a space X that satisfies COVER(ω,Φ) but not SEQ(ω,Φ) (the required example is given at the end of this section).

Question. Let X be a Hausdorff space such that every κ-sequence in X has a θ-cluster point. Is X θ-compact?

Many of these results are known and so proofs are often sketched or omitted. We are mainly interested in characterizing NET(κ,Φ) and SEQ(κ,Φ) in a number of ways so that the relationship between the three properties NET(κ,Φ), SEQ(κ,Φ), and COVER(κ,Φ) is transparent. We have already obtained several characterizations of NET(κ,Φ); see Theorem 3.9. In addition, we have:

Theorem 6.1. (*characterization of NET(κ,Φ)*) *Let X be a space, let κ be a cardinal, and let Φ be an operator. The following are equivalent:*

(1) *X satisfies NET(κ,Φ);*

(2) *if $\mathcal{F}$ is a collection of subsets of X with the FIP and $|\mathcal{F}| \leq \kappa$, then $\bigcap_{F \in \mathcal{F}} F^\Phi \neq \varnothing$.*

We now give characterizations of SEQ(κ,Φ). But first recall two simple examples related to cluster points of sequences.

Example. Let $W = \{\alpha : 0 \leq \alpha < \omega_1\}$ have the order topology. Then

(a) every sequence $\langle x_n : n \in \omega \rangle$ in W has a cluster point;

(b) there exist sequences $\langle x_\alpha : \alpha < \omega_1 \rangle$ in W with no cluster points.

Example. Let $\mathbb{R}$ have the usual topology. Then

(a) there exist sequences $\langle x_n : n \in \omega \rangle$ in $\mathbb{R}$ that have no cluster points;

(b) every sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ in $\mathbb{R}$ has a cluster point.

Theorem 6.2. (*characterizations of SEQ(κ,Φ)*) *Let X be a space, let κ be a cardinal, and let Φ be an operator. The following are equivalent:*

(1) *X satisfies SEQ(κ,Φ);*

(2) *every λ-sequence $\langle x_\alpha : \alpha < \lambda \rangle$ in X with $\omega \leq \lambda \leq \kappa$ and λ regular has a Φ-cluster point;*

(3) *if $\{F_\alpha : \alpha < \lambda\}$ is a decreasing sequence of non-empty sets in X with $\lambda \leq \kappa$, then $\bigcap_{\alpha<\lambda} F_\alpha^\Phi \neq \varnothing$;*

(4) *every infinite subset A of X with $|A| \leq \kappa$ and $|A|$ regular has a Φ-CAP (a point x in X such that $|A \cap \Phi(V)| = |A|$ for every open neighborhood V of x).*

Proof. The equivalence of (1), (2), and (3) is clear. To prove (2) $\Rightarrow$ (4), it suffices to prove: if q is a Φ-cluster point of $\langle x_\alpha : \alpha < \lambda \rangle$ in X with $\lambda \leq \kappa$ and λ regular, then q is a Φ-CAP of $A = \{x_\alpha : \alpha < \lambda\}$. If not, there is an open neighborhood V of q such that $|\Phi(V) \cap A| < \lambda$. By regularity of λ, there exists $\beta < \lambda$ such that $\Phi(V) \cap A \subseteq \{x_\alpha : \alpha < \beta\}$. This contradicts the fact that q is a Φ-cluster point of $\langle x_\alpha : \alpha < \lambda \rangle$.

It remains to prove that (4) $\Rightarrow$ (2). Let $\langle x_\alpha : \alpha < \lambda \rangle$ be a λ-sequence in X with $\lambda \leq \kappa$ and λ regular. There are two cases.

• $|\{x_\alpha : \alpha < \lambda\}| < \lambda$: In this case there is a $q \in X$ and a cofinal $A \subseteq \lambda$ such that $x_\alpha = q$ for all $\alpha \in A$ (λ regular used here); this q is a Φ-cluster point of $\langle x_\alpha : \alpha < \lambda \rangle$.

• $|\{x_\alpha : \alpha < \lambda\}| = \lambda$: In this case a Φ-CAP of $\{x_\alpha : \alpha < \lambda\}$ is also a Φ-cluster point of $\langle x_\alpha : \alpha < \lambda \rangle$. $\square$

For certain choices of Φ, we can show that NET(κ,Φ) and SEQ(κ,Φ) are equivalent and also give another characterization of SEQ(κ,Φ) in terms of Φ-CAP's. Call an operator Φ *special* if for every open set V: $\Phi(V)$ is open and $\Phi(\Phi(V)) \subseteq \Phi(V)$. Note that $\Phi(V) = V$ and $\Phi(V) = V^{-o}$ are special.

Theorem 6.3. *Let Φ be special. For any space X and any infinite cardinal κ, the following are equivalent:*

(1) *X satisfies NET(κ,Φ);*
(2) *X satisfies SEQ(κ,Φ);*
(3) *every infinite subset A of X with $|A| \leq \kappa$ has a Φ-CAP.*

Proof that (2) $\Rightarrow$ (1). Assume that X satisfies SEQ(κ,Φ) but not NET(κ,Φ). Choose the smallest infinite cardinal $\lambda \leq \kappa$ such that there is a collection $\mathcal{F}$ of subsets of X with the FIP, $|\mathcal{F}| = \lambda$, and $\bigcap_{F \in \mathcal{F}} F^{\Phi} = \varnothing$. Let $\mathcal{F} = \{F_\alpha : \alpha < \lambda\}$. For each $\alpha < \lambda$ let $\mathcal{H}_\alpha = \{F_\beta : \beta \leq \alpha\}$; $\mathcal{H}_\alpha$ is a collection of subsets of X with the FIP and of cardinality $< \lambda$ and therefore the set $H_\alpha = \bigcap_{\beta \leq \alpha} F^{\Phi}_\beta$ is non-empty. The collection $\{H_\alpha : \alpha < \lambda\}$ is a decreasing sequence of non-empty subsets of X of cardinality at most λ and therefore by SEQ(κ,Φ), there exists $q \in \bigcap_{\alpha<\lambda} H^{\Phi}_\alpha$. We now obtain a contradiction by showing that $q \in F^{\Phi}_\alpha$ for all $\alpha < \lambda$. Let V be an open neighborhood of q and let $z \in \Phi(V) \cap H_\alpha$. We have: $z \in F^{\Phi}_\alpha$ and $\Phi(V)$ is an open neighborhood of z, therefore there exists $x \in \Phi(\Phi(V)) \cap F_\alpha$. Thus $\Phi(V) \cap F_\alpha \neq \varnothing$ and $q \in F^{\Phi}_\alpha$ as required.

Proof that (2) $\Rightarrow$ (3). Let $|A| = \lambda$, where $\omega \leq \lambda \leq \kappa$ and λ is singular. Let $A = \{x_\alpha : 0 \leq \alpha < \lambda\}$ and let $\{\lambda_\beta : \beta < \theta\}$ be an increasing sequence of regular cardinals such that $\lambda = sup\{\lambda_\beta : \beta < \theta\}$ and θ is regular. For each $\beta < \theta$ let q_β be a Φ-CAP of $\{x_\alpha : 0 \leq \alpha < \lambda_\beta\}$ and let $T = \{q_\beta : 0 \leq \beta < \theta\}$. We consider two cases.

• $|T| < \theta$: In this case, there is a cofinal subset B of θ and some $q \in X$ such that $q_\beta = q$ for all $\beta \in B$ (θ regular used here). In this case q is a Φ-CAP of A (show that $|\Phi(V) \cap A| \geq \lambda_\beta$ for all $\beta \in B$).

• $|T| = \theta$: Let q be a Φ-CAP of T; then q is the required Φ-CAP of A (Φ special used here). □

The Alexandroff-Urysohn results are summarized by the following Corollary; also see Theorem 2.2 in Stephenson's paper.

Corollary 6.4. *Let X be a space and let κ be a cardinal. For $\Phi(V) = V$, the following are equivalent:*

(1) *Every κ-net in X has a cluster point;*

(2) *Every λ-sequence in X with $\omega \leq \lambda \leq \kappa$ has a cluster point;*

(3) *Every open cover of X of cardinality at most κ has a finite subcover.*

Proof. We have (1) $\Leftrightarrow$ (2) by Theorem 6.3, and the proof of (2) $\Rightarrow$ (1) in Theorem 5.4 shows that (1) $\Rightarrow$ (3). To prove that (3) $\Rightarrow$ (1), note that (3) implies condition (2) of Theorem 6.1. □

Corollary 6.5. *Let X be a space and let κ be a cardinal. For $\Phi(V) = V^{-o}$, the following are equivalent:*

(1) *Every κ-net in X has a δ-cluster point;*

(2) *Every λ-sequence in X, $\omega \leq \lambda \leq \kappa$, has a δ-cluster point.*

Example. There is a regular, extremally disconnected pseudocompact space X that is not countably compact. For $\Phi(V) = V^{-0}$ and $\Phi(V) = V^{-}$, X satisfies COVER(ω,Φ) but not SEQ(ω,Φ).

The proof we give was communicated to the author by R. Stephenson and seems to be a part of the folklore; also see Example 2.4 in [3]. Let $\{R_n : n \in \omega\}$ be a partition of ω into a pairwise disjoint collection of infinite subsets of ω, and for each $n \in \omega$ let p_n be a free ultrafilter on ω such that $R_n \in p_n$. The required space is

$$X = \beta\omega - \{\text{limit points of } \{p_n : n \in \omega\}\}.$$

(1) X is extremally disconnected ($\beta\omega$ is extremally disconnected and X is dense in $\beta\omega$);

(2) X is not countably compact (the infinite set $\{p_n : n \in \omega\}$ has no limit point);

(3) X is pseudocompact;

(4) X satisfies COVER(ω,Φ).

Proof of (3). It suffices to prove that every infinite subset of ω has a limit point in X (see [31]). Let $A \subseteq \omega$ be infinite, and consider two cases.

• There exists n such that $A \cap R_n$ is infinite: Let q be a free ultrafilter on ω such that $A \cap R_n \in q$; $q \in X$ and is a limit point of A.

• For all $n \in \omega$, $A \cap R_n$ is finite: Let q be a free ultrafilter on ω such that $A \in q$; $q \in X$ and is a limit point of A.

Proof of (4). Every pseudocompact space X has the following property: every countable open cover has a finite subcollection whose union is dense in X (see [31]). □

7. Applications of κ-nets to cardinal functions

In this section we outline some applications of κ-nets to cardinal functions. There are two results, both due to Gryzlov (see [12]): (1) if X is a compact T_1 space, then $|X| \leq 2^{\psi(X)}$; (2) if X is a θ-compact Hausdorff space with countable closed pseudocharacter, then $|X| \leq 2^{\omega}$. Each is a variation of Arhangel'skiĭ's equality, and each answers the original question of Alexandroff and Urysohn: Does every compact first-countable Hausdorff space have cardinality at most 2^{ω}? In the proof of (1), Gryzlov used complete accumulation points, and in the proof of (2) he used θ-complete accumulation points. However, his method of proof for (2) does not seem to extend to higher cardinality (see the Question in section 6), and in 1982 Dow and Porter [7] used a quite different attack to extend (2) beyond the countable case.

It turns out that a suitable modification of Gryzlov's original construction does extend to higher cardinality; the key is to replace θ-complete accumulation points of infinite sets with θ-cluster points of κ-nets. Below we use κ-nets to outline a proof of both (1) and the general case of (2); for more details, see [15]. Recall that $\psi(X)$ (respectively, $\psi_c(X)$) is the smallest infinite cardinal κ such that for every point $q \in X$, there is a collection $\{V_\alpha : \alpha < \kappa\}$ of open neighborhoods of q such that $\bigcap_{\alpha<\kappa} V_\alpha = \{q\}$ (respectively, $\bigcap_{\alpha<\kappa} V_\alpha^- = \{q\}$).

Lemma 7.1. *Let X be a compact T_1 space with $\psi(X) \leq \kappa$ and let A be a subset of X such that every κ-net in A has a cluster point in A. Then A is compact.*

Theorem 7.2. (*Gryzlov*) *Let X be a compact T_1 space. Then $|X| \leq 2^{\psi(X)}$.*

Proof. Let $\psi(X) = \kappa$, and for each $x \in X$ let $\mathcal{V}_x$ be a collection of open neighborhoods of x such that $|\mathcal{V}_x| \leq \kappa$ and $\bigcap\{V : V \in \mathcal{V}_x\} = \{x\}$. Construct a sequence $\{A_\alpha : \alpha < \kappa^+\}$ of subsets of X such that for $0 \leq \alpha < \kappa^+$:

(a) $|A_\alpha| \leq 2^\kappa$;
(b) if $\langle x_F \rangle$ is a κ-net in $\bigcup_{\beta<\alpha} A_\beta$, then some point of A_α is a cluster point of $\langle x_F \rangle$;
(c) if W is a finite union of elements of $\{V\colon\ V \in \mathcal{V}_x$ and $x \in \bigcup_{\beta<\alpha} A_\beta\}$ and $W \neq X$, then $A_\alpha - W \neq \varnothing$.

Let $A = \bigcup\{A_\alpha : \alpha < \kappa^+\}$; clearly $|A| \leq 2^\kappa$. Now check: A is compact (use (b) and Lemma 4.1); $A = X$ (use (c) and the compactness of A). □

We now turn to the Dow-Porter extension of Gryzlov's second theorem. The proof we give is a suitable modification of Gryzlov's original proof of (2) for the countable case.

Lemma 7.3. *Let X be a θ-compact Hausdorff space with $\psi_c(X) \leq \kappa$ and let A be a subset of X with the following property: if $\langle x_F \rangle$ is a κ-net in A, then there exists $q \in A$ such that given any open neighborhood R of q and any $F \in \kappa^{<\omega}$, there exists $G \in \kappa^{<\omega}$ such that $F \subseteq G$ and $x_G \in (R \cap A)^-$. Then A is an H-set.*

Lemma 7.4. *Let X be a θ-compact space with $\chi(X) \leq \kappa$ and let $\xi = \langle x_F \rangle$ be a κ-net in X. Then there is a subset $A(\xi)$ of X such that*

(1) *$\{x_F : F \in \kappa^{<\omega}\} \subseteq A(\xi)$;*
(2) *$|A(\xi)| \leq \kappa$;*
(3) *there exists $q \in A(\xi)$ such that if R is an open neighborhood of q and $F \in \kappa^{<\omega}$, then there exists $G \in \kappa^{<\omega}$ such that $F \subseteq G$ and $x_G \in (R \cap A(\xi))^-$.*

Theorem 7.5. (*Gryzlov for $\kappa = \omega$; Dow-Porter*) *Let X be a θ-compact Hausdorff space. Then $|X| \leq 2^{\chi(X)}$.*

Proof. Let $\chi(X) = \kappa$, and for each $x \in X$ let $\mathcal{V}_x$ be a local base for x such that $|\mathcal{V}_x| \leq \kappa$. For each κ-net $\xi = \langle x_F \rangle$ in X let $A(\xi)$ be a subset of X that satisfies (1)-(3) of Lemma 7.4. Construct a sequence $\{A_\alpha : \alpha < \kappa^+\}$ of subsets of X such that for $0 \leq \alpha < \kappa^+$:

(a) $|A_\alpha| \leq 2^\kappa$;
(b) if ξ is a κ-net in $\bigcup_{\beta<\alpha} A_\beta$, then $A(\xi) \subseteq A_\alpha$;
(c) if W is a finite union of elements of $\{V\colon\ V \in \mathcal{V}_x$ and $x \in \bigcup_{\beta<\alpha} A_\beta\}$ and $W^- \neq X$, then $A_\alpha - W^- \neq \varnothing$.

Let $A = \bigcup_{\alpha<\kappa^+} A_\alpha$; clearly $|A| \leq 2^\kappa$. By (b) and Lemma 7.3, A is an H-set; by (c), $A = X$. □

Let $\langle X, \tau \rangle$ be a θ-compact Hausdorff space and let $RO(X)$ be the collection of all regular open subsets of X. Then $RO(X)$ is a base for a coarser Hausdorff topology τ_S on X. Moreover, $\langle X, \tau_S \rangle$ is a θ-compact Hausdorff space and $\chi(\langle X, \tau_S \rangle) \leq \psi_c(\langle X, \tau \rangle)$. From these observations we have:

Corollary 7.6. (*Gryzlov for $\kappa = \omega$; Dow-Porter*) *Let X be a θ-compact Hausdorff space. Then $|X| \leq 2^{\psi_c(X)}$.*

8. κ-NETS IN ANALYSIS

McShane has written a very nice paper [19] on the use of nets in analysis. A significant number of his examples are κ-nets (suitably modified). Below we discuss several of these examples. McShane begins with two theorems on convergence of nets in complete metric spaces (see p. 4 of [19]); here are the κ-net versions.

Theorem 8.1. *Every Cauchy κ-net in a complete metric space converges.*

Theorem 8.2. *Let $\langle x_F \rangle$ be a κ-net in $\mathbb{R}$ that is increasing ($F \subseteq G \Rightarrow x_F \leq x_G$) and bounded above. Then $\langle x_F \rangle$ converges.*

Example 1 (theory of summability for $\mathbb{R}$). This is a topic for which κ-nets are tailor-made. A subset $\{a_\alpha : \alpha \in \kappa\}$ of $\mathbb{R}$ is *summable* if the κ-net $\langle x_F \rangle$ defined by $x_F = \sum_{\alpha \in F} a_\alpha$ converges.

In this case, the *sum* is the limit of this κ-net. Dirichlet proved that the series $\sum a_n$ is absolutely convergent if and only if $\{a_n : n \in \omega\}$ is summable.

Example 2 (arc length in the plane). Let $x = x(t)$, $y = y(t)$, $0 \leq t \leq 1$, be a simple rectifiable curve in $\mathbb{R}^2$. For each partition $P = \{t_0, \ldots, t_n\}$ of [0,1] with $0 = t_0 < \cdots < t_n = 1$ let

$$L(P) = \sqrt{\textstyle\sum_{1 \leq k \leq n}([x(t_k) - x(t_{k-1})]^2 + [y(t_k) - y(t_{k-1})]^2)}.$$

Now let $\mathcal{P} = \{P : P \text{ a partition of } [0,1]\}$. (In this example and the next we replace $(2^\omega)^{<\omega}$ with $\mathcal{P}$ ordered by $\subseteq$.) Then $\{L(P) : P \in \mathcal{P}\}$ is a 2^ω-net that is increasing and bounded above and therefore converges; the *length* of the curve is this limit.

Example 3 (Riemann integral, as defined by Darboux). Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $\mathcal{P} = \{P : P \text{ a partition of } [a, b]\}$. For each $P = \{a_0, a_1, \ldots, a_n\} \in \mathcal{P}$ with $a = a_0 < a_1 < \cdots < a_n = b$ let

$$U(f, P) = \textstyle\sum M_k \Delta x_k \qquad (M_k = lub\{f(c) : c \in [a_{k-1}, a_k]\});$$
$$L(f, P) = \textstyle\sum m_k \Delta x_k \qquad (m_k = glb\{f(c) : c \in [a_{k-1}, a_k]\}).$$

Then $\{U(f, P) : P \in \mathcal{P}\}$ is a 2^ω-net (except that $(2^\omega)^{<\omega}$ is replaced by $\mathcal{P}$) that is increasing and bounded above and therefore converges to $\int^- f$. Likewise, $\{L(f, P) : P \in \mathcal{P}\}$ is decreasing and bounded below and therefore converges to $\int_- f$. Finally, f is *Riemann integrable* on $[a, b]$ if $\int^- f = \int_- f$.

Acknowledgments. I am grateful to the members of the NC-SC Topology Seminar for many helpful comments. The seminar meets approximately once every two weeks at UNC-C, and occasionally at UNC-G. The original members are: A. Dow, UNC-C; R. Hodel, Duke; P. Nyikos and R. Stephenson, USC; J. Vaughan, UNC-G; recent additions are: G. Atim, USC-L; A. Iwasa, USC-B; L. Platkiewicz, UNC-P; U. Vyas, WSSU; L. Zsilinszky, UNC-P. I would also like to thank the referee for many helpful suggestions.

References

[1] A.V. Arhangel'skiĭ, *Some properties of radial spaces*, Math. Notes Russ. Acad. Sci. **27** (1980), 50-54.

[2] R. Arens, *Note on convergence in topology*, Math. Mag. **23** (1950), 229-234.

[3] H. Banilower, *Projecting C(S) onto C_0(S)*, Proc. Amer. Math. Soc. **33** (1972), 349-354.

[4] R.G. Bartle, *Nets and Filters in Topology*, Amer. Math. Monthly **62** (1955), 551-557.

[5] A. Bella and G. Tironi, *Pseudoradial spaces*, in: K. P. Hart, J. Nagata, J. Vaughan (Eds.), Encyclopedia of General Topology, 165-168, Elsevier, Amsterdam, 2004.

[6] G. Birkhoff, *Moore-Smith convergence in general topology*, Ann. of Math. **38** (1937), 39-56.

[7] A. Dow and J. Porter, *Cardinalities of H-closed spaces*, Topology Proceedings **7** (1982), 27-50.

[8] R. M. Dudley, *On sequential convergence*, Trans. Amer. Math. Soc. **112** (1964), 483-507.

[9] R. Engelking, *General Topology*, Heldermann, Berlin, 1989.

[10] S.P. Franklin, *Spaces in which sequences suffice*, Fund. Math. **57** (1965), 107-115.

[11] R. Frič, *History of Sequential Convergence Spaces*, in: C.E. Aull and R. Lowen (Eds.), Handbook of the History of General Topology, volume **1**, 343-355, Kluwer Academic Publishers, Boston, 1997.

[12] A. Gryzlov, *Two theorems on the cardinality of topological spaces*, Soviet Math. Dokl. **21** (1980), 506-509.

[13] H. Herrlich and G. Strecker, *Categorical Topology: its Origins, as exemplified by the Unfolding of the Theory of Topological Reflections and Coreflections before 1971*, in: C.E. Aull and R. Lowen (Eds.), Handbook of the History of General Topology, volume **1**, 255-341, Kluwer Academic Publishers, Boston, 1997.

[14] L.L. Herrington and P.E. Long, *Characterizations of H-closed spaces*, Proc. Amer. Math. Soc. **48** (1975), 469-475.

[15] R. Hodel, *Arhangel'skiĭ's solution to Alexandroff's problem: A survey*, Topology Appl. **153** (2006), 2199-2217.

[16] R. Hodel, *Cardinal Functions I*, in: K. Kunen and J. Vaughan (Eds.), Handbook of Set-Theoretic Topology, 1-61, North-Holland, Amsterdam, 1984.

[17] A. Kamiński, *Remarks on multivalued convergence*, in: J. Novak (Ed.), General Topology and its Relations to Modern Analysis and Algebra V, Proc. Fifth Prague Topol. Symp. 1981, 418-422, Heldermann Verlag, Berlin, 1982.

[18] J. L. Kelley, *General Topology*, Van Nostrand, Princeton, 1955.

[19] E. J. McShane, *Partial orderings and Moore-Smith limits*, Amer. Math. Monthly **59** (1952), 1-11.

[20] P. R. Meyer, *Sequential properties of ordered topological spaces*, Compositio Mathematicae **21** (1969), 102-106.

[21] P. R. Meyer, *Sequential space methods in general topological spaces*, Colloquium Mathematicum **22** (1971), 223-228.

[22] E.H. Moore and H.L. Smith, *A general theory of limits*, Amer. J. Math. **44** (1922), 102-121.

[23] T. Noiri, *Modified Open and Closed Sets*, in: K. P. Hart, J. Nagata, J. Vaughan (Eds.), Encyclopedia of General Topology, 8-10, Elsevier, Amsterdam, 2004.

[24] P. Nyikos, *Convergence in Topology*, in: M. Hušek and J. van Mill (Eds.), Recent Progress in General Topology, 537-570, North-Holland, Amsterdam, 1992.

[25] J. Porter and R. G. Woods, *Extensions and Absolutes of Hausdorff Spaces*, Springer-Verlag, New York, 1988.

[26] J. Porter and J. Vermeer, *H-closed spaces*, in: K. P. Hart, J. Nagata, J. Vaughan (Eds.), Encyclopedia of General Topology, 221-222, Elsevier, Amsterdam, 2004.

[27] J. Porter and M. Tikoo, *Separating H-sets with Open Sets*, to appear.

[28] A. Shivastava and S. Gupta, *On various properties of δ-compact spaces*, Bull. Calcutta Math. Soc. **97** (2005), 217-222.

[29] M. K. Singal and S. P. Arya, *On almost-regular spaces*, Glasnik Matematički **4** (1969), 89-99.

[30] R.M. Stephenson, Jr., *Initially κ-compact and related spaces*, in: K. Kunen and J. Vaughan (Eds.), Handbook of Set-Theoretic Topology, 603-632, North-Holland, Amsterdam, 1984.

[31] R.M. Stephenson, Jr., *Pseudocompact Spaces*, in: K. P. Hart, J. Nagata, J. Vaughan (Eds.), Encyclopedia of General Topology, 177-181, Elsevier, Amsterdam, 2004.

[32] J.W. Tukey, *Convergence and uniformity in topology*, Ann. of Math. Studies **2**, 1940.

[33] J.E. Vaughan, *Some properties related to* [a,b]-*compactness*, Fund. Math. **87** (1975), 251-260.

[34] N. V. Veličko, *H-closed topological spaces*, Amer. Math. Soc. Transl. **78** (1968), 103-118.

[35] S. Willard, *General Topology*, Addison-Wesley, Reading, MA, 1970.

Department of Mathematics, Duke University, Durham, NC 27708
E-mail address: `hodel@math.duke.edu`