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LIFTING CROOKED CIRCULAR CHAINS TO COVERING SPACES

KEVIN GAMMON

Dedicated to Tom Ingram on the occasion of his 70th birthday

Abstract. This paper examines the structure of the lifts of crooked circular chains to connected covering spaces. In particular, if $\{D_i\}$ is a sequence of circular chains satisfying minor conditions and p is a 2^k-fold covering map of the annulus onto itself, then $p^{-1}(D_{3^k+i})$ must be crooked inside of $p^{-1}(D_i)$.

1. Preliminary Information

A *continuum* is a compact connected nondegenerate metric space. All sets will be considered as subsets of a compact metric space. A continuum is *indecomposable* if it is not the union of two proper subcontinua. A continuum is *hereditarily indecomposable* if every subcontinuum is indecomposable.

A *chain* is a finite collection of open sets $U = \{u_0, u_1, \cdots u_m\}$ such that $u_j \cap u_k \neq \emptyset$ if and only if $|j - k| \leq 1$. The open sets contained in U are *links* of the chain U. The integer m is the *length* of U. An ϵ-*chain* is a chain in which each link has diameter less than ϵ. A continuum X is *chainable* if, given any $\epsilon > 0$, there exists an ϵ-chain covering X. If U is a chain, then the subchain of U consisting of the links $\{u_i, \cdots, u_k\}$ will be denoted by $U(i, k)$. The

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chain U is *contained inside of the chain* F if every link of U is a subset of a link in F.

A chain $E = \{e_0, e_1, \cdots, e_m\}$ is *crooked* inside of the chain $D = \{d_0, d_1, \cdots, d_m\}$ if

(1) every link of E is contained inside of a link of D and
(2) if e_j and e_k are links of E which intersect d_J and d_K, respectively, where $|J-K| \geq 3$, then the subchain $E(j,k)$ can be written as the union of three proper subchains $E(j,r)$, $E(r,s)$, and $E(s,k)$ where $(s-r)(k-j) > 0$ and e_r is a subset of the link of $D(J,K)$ adjacent to d_K, and e_s is a subset of the link of $D(J,K)$ adjacent to d_J.

The above definition is due to R. H. Bing [2].

Throughout the paper, let $\mathbb{Z}$ denote the set of integers. For any integers i and n, the non-standard notation i mod n will denote the unique integer j such that $0 \leq j \leq n-1$ and $i \equiv j \bmod n$.

A *circular chain* $U = \{u_i\}_{i \in \mathbb{Z}}$ is a collection of open sets so that for some positive $n \in \mathbb{Z}$, $u_i = u_j$ if and only if i mod $n = j$ mod n and $u_i \cap u_j \neq \emptyset$ if and only if there exists a $k \in \mathbb{Z}$ so that $u_i = u_k$ and $|k-j| \leq 1$. The integer n is called the *length* of the circular chain U. The circular chain U is *contained inside of the circular chain* F if every link of U is a subset of a link of F.

Let F be a circular chain contained inside of the circular chain U where U has length n. Suppose that F_1 is a proper subchain of F so that for a fixed integer j, F_1 contains a link that intersects u_j and if F_1 has a link which intersects u_m then j mod $n \leq m$ mod n. Next, suppose that F_1 intersects a link u_k such that if F_1 has a link which intersects u_l for some l this implies j mod $n \leq l$ mod $n \leq k$ mod n. If k is the least such integer greater than j which satisfies these conditions, then F_1 is said to have *span* $|k-j|$ inside of U.

The circular chain E is *crooked* inside the circular chain D if D contains E and for every proper subchain F of D, each subchain of E contained inside of F is crooked inside of F. This definition is also due to Bing [3].

A *pseudo-arc* is a hereditarily indecomposable, chainable continuum. A pseudo-arc was originally described by Bronisław Knaster [8] in 1922. At that time, it was not known that pseudo-arcs were topologically equivalent. Edwin E. Moise [9] constructed the pseudo-arc as a continuum which is homeomorphic to each of its nondegenerate subcontinua. In [3], Bing proved that all pseudo-arcs, including the examples of Knaster and Moise, were topologically equivalent. In terms of chains contained in a metric space, Bing [2] described the pseudo-arc as the intersection of a sequence of sets $\{U_i\}$ where U_i is the union of the chain D_i and the sequence $\{D_i\}$ satisfies the following conditions:

(1) D_{i+1} is crooked inside of D_i;
(2) the closure of each element of D_{i+1} is contained in an element of D_i;
(3) the first link of D_{i+1} is contained in the first link of D_i and the last link of D_{i+1} is contained in the last link of D_i;
(4) D_i is an ϵ_i-chain;
(5) $\lim_{i\to\infty} \epsilon_i = 0$.

P will be used to denote the pseudo-arc.

A *pseudo-circle* is a hereditarily indecomposable, circularly chainable, non-chainable continuum which is embeddable in the plane. The pseudo-circle was described by Bing [3] as a hereditarily indecomposable continuum which separates the plane. In terms of circular chains, Bing described this space as the intersection of a sequence of sets $\{U_i\}$, where U_i is the union of all links of the circular chain D_i and the sequence $\{D_i\}$ satisfies the following conditions:

(1) D_{i+1} is crooked inside of D_i;
(2) the closure of each element of D_{i+1} is contained in an element of D_i;
(3) D_{i+1} has winding number ± 1 in D_i;
(4) D_i is an ϵ_i-chain;
(5) $\lim_{i\to\infty} \epsilon_i = 0$.

In this paper, it will be assumed that the circular chain D_{i+1} has winding number 1 inside of the circular chain D_i. The collection of circular chains $\{D_i\}$ which satisfies the above conditions will be said to define a pseudo-circle. Lawrence Fearnley proved that all pseudo-circles are topologically equivalent ([4] and [6]). It has also been shown by Fearnley [5] and by James T. Rogers, Jr., [10] that the pseudo-circle is not homogeneous. C will be used to denote the pseudo-circle.

Covering spaces have provided a useful tool to prove results related to hereditarily indecomposable continua such as the result due to Rogers in [11] which states that a homogeneous continuum which separates the plane must be decomposable. It has been shown by Jo W. Heath [7] that for any integer k, the pseudo-circle is a k-fold covering space of itself. However, this proof focused on properties of confluent mappings and did not use the structure of crooked circular chains. This paper will examine the structure of crooked circular chains. This can be used to provide an alternative proof of Heath's result.

2. The connected k-fold covering space of a pseudo-circle

Throughout this paper, $\{D_i\}_{i\geq 0}$ will be a collection of circular chains $D_i = \{d^i_j\}_{j\in\mathbb{Z}}$, contained in a planar annulus, which consists of connected open sets satisfying the following conditions:

(1) D_0 contains at least 6 links;
(2) D_{i+1} is crooked inside of D_i;
(3) the closure of each element of D_{i+1} is contained in an element of D_i;
(4) D_{i+1} has winding number 1 inside of D_i;
(5) $d^{(i+1)}_0$ is contained in d^i_0.

The first condition is used to avoid trivialities. The second and third conditions are typical when describing a pseudo-circle. The fourth condition is used to ease notation in the following proofs. The length of D_i will be denoted by $n(i)$.

The annulus A is a connected 2-fold covering space of itself. Let p denote the covering map. Denote $p^{-1}(D_i)$ by $F_i = \{f^i_j\}_{j\in\mathbb{Z}}$ and assume that F_i is enumerated so that $p(f^i_j) = d^i_j$. Then F_i is a circular chain of length $2n(i)$ where $p(f^i_j) = p(f^i_k)$ if and only if $j \bmod n(i) = k \bmod n(i)$. It will be shown that in the sequence $\{D_i\}_{i\geq 0}$, as n grows without bound, the span of proper subchains of D_{i+n} becomes so large inside of D_i that for some N, the inverse image of D_{i+N} must be crooked inside of the inverse image of D_i.

When considering the inverse image of D_{i+1} inside of the inverse image of D_i, there is a minimum number of links in $p^{-1}(D_i)$ that one subchain $U \subset p^{-1}(D_{i+1})$ of length $n(i+1)$ must intersect. The following two lemmas find this number by constructing a specific proper subchain of D_{i+1} which has a large span inside of D_i.

Lemma 2.1. *There is a subchain* $V = \{v_1, v_2, \cdots, v_m\}$ *of* F_{i+1} *such that*

1. V *contains the link* f_0^{i+1},
2. $p(V) = \{p(v_1), p(v_2), \cdots, p(v_m)\}$ *is a proper subchain of* D_{i+1},
3. $p(v_i) = p(v_j)$ *if and only if* $i = j$, *and*
4. V *has span at least* $2n(i) - 3$ *inside of* F_i.

Proof: Since D_{i+1} has winding number 1 inside of D_i, there exists a subchain $F_{i+1}(j, m)$ so that $0 < j < m < n(i+1) - 1$, f_m^{i+1} intersects $f_{n(i)-1}^i$, f_j^{i+1} intersects f_1^i, and $F_{i+1}(j, m)$ is contained inside of $F_i(1, n(i) - 1)$. Since the chain $p(F_i(1, n(i) - 1)$ is a proper subchain of D_i and $F_{i+1}(j, m)$ is contained inside of $F_i(1, n(i) - 1)$, the chain $F_{i+1}(j, m)$ must be crooked inside of $F_i(1, n(i) - 1)$. This implies that $F_{i+1}(j, m)$ can be written as the union of three subchains:

1. $F_{i+1}(j, k)$ where $f_j^{i+1} \cap f_1^i \neq \emptyset$ and $f_k^{i+1} \subset f_{n(i)-2}^i$,
2. $F_{i+1}(k, l)$ where f_k^{i+1} is as above and $f_l^{i+1} \subset f_2^i$, and
3. $F_{i+1}(l, m)$ where f_l^{i+1} is as above and $f_m^{i+1} \cap f_{n(i)-1}^i \neq \emptyset$

where $0 < j < k < l < m$. Let r be an integer such that $-n(i+1) < r < 0$ and $r \bmod n(i+1) = l \bmod n(i+1) = l$.

The chain V consists of the links $F_{i+1}(r, k)$. The chain $p(V)$ is proper because it does not contain each link of $p(F_{i+1}(k, l))$.

A chain of F_i which contains $F_{i+1}(r, -1)$ must contain at least $n(i) - 2$ links. Likewise, a chain of F_i which contains $F_{i+1}(0, k)$ must contain at least $n(i) - 1$ links. Therefore, V intersects every link of a subchain of F_i which contains at least $2n(i) - 3$ links. □

The chain V mentioned in the above proof has an additional property that will be used in subsequent proofs. As mentioned previously, the lift of $p(V)$ consists of two distinct, disjoint chains, each of which intersects all but at most three links of F_i. Since F_i contains at least 12 distinct links, there must be at least 6 links which both of these chains intersect. In particular, the following lemma is true.

Lemma 2.2. *Let V be the chain described in Lemma 2.1. Then there exists a subchain G of D_i consisting of three adjacent links so that, for each link g of $p^{-1}(G)$, both chains of $p^{-1}(p(V))$ have a link contained inside of g.*

Lemma 2.3. *For any $l \in \mathbb{Z}$, there is a proper subchain $V = \{v_1, v_2, \cdots, v_m\}$ of F_{i+1} such that*

1. *V contains the link f_l^{i+1},*
2. *$p(V) = \{p(v_1), p(v_2), \cdots, p(v_m)\}$ is a proper subchain of D_{i+1},*
3. *$p(v_j) = p(v_k)$ if and only if $j = k$, and*
4. *V has span at least $2n(i) - 3$ inside of F_i.*

Proof: The chains D_i, D_{i+1}, F_i, and F_{i+1} can be renumbered so that Lemma 2.1 can be applied. □

The next two lemmas show that one proper subchain of length $n(i+2)$ in the inverse image of D_{i+2} must intersect every link in the inverse image of D_i. This is done by applying the previous lemma to the circular chains D_{i+1} and D_{i+2}.

Lemma 2.4. *There is a subchain V of F_{i+2} containing the link f_0^{i+2} such that*

1. *V intersects each element of F_i,*
2. *$p(V) = \{p(v_1), p(v_2), \cdots, p(v_m)\}$ is a proper subchain of D_{i+2}, and*
3. *$p(v_j) = p(v_k)$ if and only if $j = k$.*

Proof: Let V_1 be a subchain of F_{i+1} as described in Lemma 2.3 chosen in such a way that d_0^i is the middle link of a chain G as described in Lemma 2.2. Next, apply Lemma 2.1 to the link f_0^{i+2} and the circular chain F_{i+1} to obtain a chain V which intersects all but at most three elements of F_{i+1}.

Notice that since $d_0^{i+2} \subset d_0^i$ and d_0^i is the middle link of the chain G, the three links which V may not intersect in F_{i+1} must be contained inside of $p^{-1}(G)$. However, since V must intersect the other links of both chains of $p^{-1}(p(V_1))$, it follows that V must still intersect every element of F_i. □

Lemma 2.5. *For $l \in \mathbb{Z}$, there is a subchain V of F_{i+2} containing the link f_l^{i+2} such that*

1. *V intersects each element of F_i,*

(2) $p(V) = \{p(v_1), p(v_2), \cdots, p(v_m)\}$ *is a proper subchain of* D_{i+2}, *and*
(3) $p(v_j) = p(v_k)$ *if and only if* $j = k$.

Proof: The circular chains D_{i+1}, D_{i+2}, F_{i+1}, and F_{i+2} can be renumbered so that Lemma 2.4 can be applied. □

The following theorem uses the large span of proper subchains in D_{i+2} to show that the inverse image of D_{i+3} must be crooked inside of the inverse image of D_i.

Theorem 2.6. F_{i+3} *is crooked inside of the circular chain* F_i.

Proof: Let E be a proper subchain of F_i and let G be a subchain of F_{i+3} which is contained inside of E. Let H be a subchain of F_{i+2} which contains G. From Lemma 2.5, H is contained inside of a chain in the lift of a proper subchain of D_{i+2} which intersects each element of F_i. Hence, G must be crooked inside of H and therefore also crooked inside of E. □

Theorem 2.7. *The sequence of circular chains* $\{F_{3(i)}\}_{i\geq 0}$ *defines a pseudo-circle. In particular, the connected 2-fold cover of the pseudo-circle is a pseudo-circle.*

Proof: This is a consequence of Theorem 2.6. □

The remaining theorems in this section are used to extend the previous result to n-fold covering spaces for $n > 2$.

Theorem 2.8. *If* $p : A \to A$ *denotes the* 2^k*-fold covering of the annulus onto itself, then the sequence of circular chains* $\{F_{3^k(i)}\}$ *defines a pseudo-circle. In particular, the connected* 2^k*-fold covering of the pseudo-circle is a pseudo-circle.*

Proof: This follows from the fact that the 2^k-fold covering space is a 2-fold covering space of the $2^{(k-1)}$-fold covering space. □

This leads to the following alternative proof of Heath's result originally presented in [7].

Corollary 2.9. *Let* p *be a* j*-fold covering map of the annulus to itself, where* $2^k < j \leq 2^{k+1}$ *for some* k*. Then for each* i*, there exists an* n *such that* $3^k(i) < n \leq 3^{k+1}(i)$ *and* F_n *is crooked inside of* F_i*. In particular, the* j*-fold connected covering space of the pseudo-circle is a pseudo-circle.*

3. The infinite, connected covering space of a pseudo-circle

The methods of this proof can also be used to provide more insight into a result due to David P. Bellamy and Wayne Lewis [1], which states that the Hausdorff two point compactification of the infinite, connected covering space of the pseudo-circle is a pseudo-arc. The proof provided by Bellamy and Lewis uses a specific construction of the pseudo-circle which controls the span of the proper subchains of D_{i+1} inside of D_i. While the underlying idea of the following proof is similar to that in [1], the author utilizes the methods developed in section 2 to avoid a specific construction of the pseudo-circle and to provide more detail to the proof introduced by Bellamy and Lewis.

In the following, let $\widetilde{A}$ denote the universal covering space of the annulus with covering map p and $\hat{A}$ denote the two point compactification of $\widetilde{A}$ obtained by adding points a and b. Then $\widetilde{A}$ contains an infinite, connected covering space of the pseudo-circle. Let $\{D_i\}_{i\geq 0}$ be a sequence of circular chains defining a pseudo-circle satisfying the four conditions listed in section 2.

Theorem 3.1. *The two point compactification of the infinite, connected covering space of the pseudo-circle is a pseudo-arc.*

Proof: For each i, $p^{-1}(D_i)$ is an infinite chain consisting of infinitely many copies of D_i. Assume, without loss of generality, that proceeding through the links of $p^{-1}(D_i)$ in the direction of a corresponds to traveling through $D(i)$ with a negative orientation.

Arbitrarily select a point $x \in p^{-1}(C)$ such that d_1^0 contains $p(x)$ and select a copy of D_0 in $p^{-1}(D_0)$ which contains x in the first link. Denote this copy by E_0^0. Then E_{-1}^0 will consist of the copy of D_0 that intersects E_0^0 and travels towards a, and E_1^0 will consist of the copy of D_0 that intersects E_0^0 and travels towards b. In general, number the copies of D_0 inductively by subtracting one while moving toward point a and adding one while moving toward point b.

Let F_0 be the chain from a to b whose links consist of the links of E_0^0 except the first link and the links of E_1^0 except the last two links (see Figure 1.) The neighborhood of a will consist of the union of the elements of those chains E_i^0 where $i < 0$ and the first link of

E_0^0. The neighborhood of b will consist of the union of those copies of E_i^0 where $i > 1$ and the last two links of E_1^0. Then this chain has length $2n(0) - 1$, which is one less than the length of the 2-fold cover of D_0.

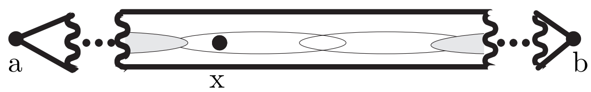

FIGURE 1. Two copies of D_0 provide the middle links of F_0. The rest of the lift of D_0 is a neighborhood either of a or of b.

In a similar fashion, let E_0^1 be a copy of D_3 contained inside of $p^{-1}(D_3)$ whose first link is contained inside of the first link of E_0^0. The copies of D_3 will be enumerated inductively in a fashion similar to the copies of D_0. As illustrated in Figure 2, let F_1 be a chain from a to b whose links consist of the links of E_{-1}^1 except the first link, the links of E_0^1, the links of E_1^1, and the links of E_2^1 except the last two links. The links containing a and b are defined in a similar fashion to those in F_0. Applying the proof of Theorem 2.6, F_1 is crooked inside of the chain F_0. Notice that F_1 has length $4n(3) - 1$ which is one less than the length of the 4-fold covering of D_3.

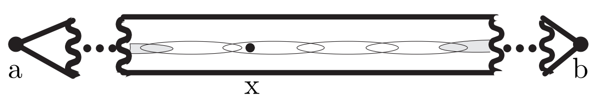

FIGURE 2. Four copies of D_3 provide the middle links of F_1. The rest of the lift of D_4 is a neighborhood either of a or of b.

In general, if F_i has already been constructed using $p^{-1}(D_j)$ for some j, then F_{i+1} consists of 2^{i+1} copies of $D_{(3^i+j)}$ selected in a similar fashion to those in F_1. Neighborhoods of a and b are also constructed similarly. Again, by the proof of Theorem 2.6, F_{i+1} is crooked inside of the chain F_i. Notice that F_{i+1} will have length $2^{(i+1)}n(3^i + j) - 1$ which is one less than the length of the $2^{(i+1)}$-fold cover of $D_{(3^i+j)}$.

Since the mesh of the links of F_i goes to zero as i increases without bound, if follows that $\cap F_i$ is a pseudo-arc. $\square$

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