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NEW IRREDUCIBLE PLANE CONTINUA WITHOUT THE FIXED POINT PROPERTY

VERÓNICA MARTÍNEZ-DE-LA-VEGA

Abstract. In *On spirals and fixed point property* [Fund. Math. **144** (1994), no. 1, 1–9], Roman Mańka constructs a Young spiral which does not have the fixed point property (see §5 Example, p. 5). In this paper, using Marwan M. Awartani's paper *An uncountable collection of mutually incomparable chainable continua* [Proc. Amer. Math. Soc. **118** (1993), no. 1, 239–245] and the example from Mańka's paper noted above, we generalize this result showing an uncountable family of incomparable Young spirals, each without the fixed point property. As a consequence we show an uncountable family of rational irreducible plane continua that admit fixed point free maps with the condition that all of their tranches have the fixed point property. The techniques used in this paper are different and each member of this family is different from those used by Charles L. Hagopian and Mańka in *Rational irreducible plane continua without the fixed-point property* [Proc. Amer. Math. Soc. **133** (2005), no. 2, 617–625].

1. Introduction, notation, and preliminaries

We will use notation from [5]. For the convenience of the reader we reproduce some definitions here.

A *continuum is* a non-void compact connected metric space. According to Marwan M. Awartani [1], an *Elsa continuum* is a compactification of the ray $(0,1]$ with a closed arc as remainder. It

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follows from [11, Lemma 11] that each Elsa continuum can be embedded in a rectangle where one of its sides is the limiting arc. In [2] it is shown that there is an uncountable family of *incomparable* Elsa continua (i.e., there is not a continuous surjective map between any two members of the family). A *double Warsaw circle* is a union of two Elsa continua, each continuum limits on an initial arc of the other. Also there exists uncountably many topological types of double Warsaw circles (see [1] for stronger results). Again by [11, Lemma 11], each double Warsaw circle is planar.

By a *map*, we mean a continuous function.

By a *Young spiral*, we mean a continuum that is a ray limiting on a double Warsaw circle.

A continuum has the *fixed point property* if each map from it into itself has a fixed point.

In [10, §5 Example], Roman Mańka constructed a Young spiral which does not have the fixed point property. In this paper we construct an uncountable family of Young spirals each one with the same property.

A continuum is *irreducible* between two of its points if none of its proper subcontinua contains both of these points. A continuum is *rational* if each of its points admits a base of neighborhoods with countable boundaries.

Rational continua are hereditarily decomposable [8, Theorem 5, p. 285]. Each rational irreducible continuum C is of *type* λ, i.e., of the order type of the unit interval $[0, 1]$. Thus, by [7, §3, pp. 248-262], C admits a uniquely determined monotone upper semicontinuous decomposition to an *arc* (i.e., homeomorphic image of $[0, 1]$) with the property that each element of the decomposition has void interior relative to C [8, Theorem 2, p. 215; Theorem 3, p. 216]. The continua that are elements of this decomposition are called the *tranches of C*.

In [4], Charles L. Hagopian defined a non-planar continuum $\mathcal{M}$ such that each tranche of $\mathcal{M}$ has the fixed point property and $\mathcal{M}$ admits a fixed point free map. In [4, Question 1], he asked if there is a continuum with these properties in the plane. A positive answer was given in [5] by Hagopian and Roman Mańka. Here a new family of such continua is constructed.

The construction involves two Young spirals. The closure of one spiral has the fixed point property though the complete continuum can be retracted to a double Warsaw circle which does not have the fixed point property.

We use $\mathbb{N}$ and $\mathbb{R}$ to denote the positive integers and the real numbers, respectively. For $A \subset X$, we denote $\mathrm{cl}_X(A)$, $\mathrm{bd}_X(A)$, and $\mathrm{int}_X(A)$ for the closure, the boundary, and the interior of A in X, correspondingly. We will omit the subscript X if the meaning of the space X is clear.

Given two sets A and B, $A - B$ denotes the set theoretical difference of A minus B. For a map $f\colon A \to B$, $\mathrm{Im}\,(f) = \{f(x) \in B : x \in A\}$.

If p and q are points lying in the plane, then $\overline{pq}$ stands for the straight line segment joining p and q. Also $\pi_1(p)$ is the projection of p into the first coordinate and $\pi_2(p)$ its projection into the second coordinate.

In this paper a *ray* is a topological half-line, i.e., a homeomorphic image of $[0, 1)$. Every ray R will be considered with its natural order $<_R$ inherited from $[0, 1)$. If R is a ray contained in a continuum, then $\mathrm{cl}(R)$ is a continuum irreducible between the origin (*initial point*) of R and each limit point of R, which is a limit of an increasing convergent sequence of points of R [10, Proposition 2.1]. The continuum $\mathrm{cl}(R)$ is rational if and only if the continuum $L(R)$ consisting of all the limit points of R is rational. If $L(R)$ is rational, then $R \cap L(R) = \emptyset$ and $R \cup L(R) = \mathrm{cl}(R)$. A continuum C of the form $C = \mathrm{cl}(R) \cup \mathrm{cl}(S)$, where R and S are rays with the origins p and q, respectively, is irreducible between p and q if $R\cap \mathrm{cl}(S) = \emptyset = S\cap \mathrm{cl}(R)$ (see [9, Remark 1, p. 125] for a general statement).

A space is *(uniquely) arcwise connected* if every two of its points are end points of a (unique) arc in the space.

2. An uncountable family of Young spirals which do not have the fixed point property

In this section we will generalize the result of [10, §5 Example]. This construction is done in the Euclidean space $\mathbb{R}^2$. Given a set $Z \subset \mathbb{R}^2$, its antipodal set is

$$Z' = \left\{(-x, -y) \in \mathbb{R}^2 : (x, y) \in Z\right\}.$$

Theorem 2.1. *There exists an uncountable family of incomparable Young spirals which do not have the fixed point property.*

Proof: In [2] an uncountable family of incomparable Elsa continua is constructed. In [12, Lemma 3.1], it is shown that if E is an Elsa continuum, then E can be embedded in the plane in such a way that the remainder is the interval $[-1,1]$ on the y-axis and the rest of the continuum is the graph of a continuous function f_E from $(0,1]$ to $[-1,1]$.

Let E be an Elsa continuum, $E = (\{0\} \times [-1,1]) \cup \{(x, f_E(x)) : x \in (0,1]\} = \mathrm{cl}_{\mathbb{R}^2}\{(x, f_E(x)) : x \in (0,1]\}$.

Let $g_E \colon (0,1] \to [1,3]$ be the map given by $g_E(x) = 2 + f_E(x)$.

Let Γ_E denote the graph of g_E, i.e., $\Gamma_E = \{(x, g_E(x)) : x \in (0,1]\}$. Set $a = (0,-3)$, $b = (2,0)$, and $c_E = (1, g_E(1))$. Then $A_E = \overline{ab} \cup \overline{bc_E} \cup \Gamma_E \cup (\{0\} \times [1,3])$ is an Elsa continuum with initial point $0_{A_E} = a$. Then the set $A_E \cup A'_E$ is a double Warsaw circle and the *antipodal map* $\alpha \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\alpha((x,y)) = (-x,-y)$ determines a fixed point free map on it.

Let $g_n^E \colon [0,1] \to [1,4]$, $n = 1,2,....$ be defined by

$$g_n^E(x) = \begin{cases} 3 + \frac{1}{n}, & \text{if } x = 0, \\ 3 + \frac{1}{n} - (1 - f_E(x))x^{\frac{1}{n}}, & \text{if } x > 0. \end{cases}$$

Then the maps g_n^E satisfy the following conditions.

(1) $g_n^E(0) = 3 + \frac{1}{n}$, $g_n^E(1) = g_E(1) + \frac{1}{n}$,
(2) $g_1^E(x) > g_2^E(x) > g_3^E(x) > g_4^E(x)...$ for every $x \in [0,1]$,
(3) $g_n^E(x) \to g_E(x)$ as $n \to \infty$ for $x > 0$,
(4) $\sup_{x\in[0,1]} \left|g_n^E(x) - g_{n+1}^E(x)\right| \to 0$ as $n \to \infty$.

Denote by Γ_n^E the graph of g_n^E.

Set $a_n^E = (0, -3 - \frac{1}{n})$, $b_n^E = (2 + \frac{1}{n}, 0)$, and $c_n^E = (1, g_E(1) + \frac{1}{n})$, and let S_E denote the set

$$\bigcup_{n=1}^{\infty} \left\{ \overline{a_n^E b_n^E} \cup \overline{b_n^E c_n^E} \cup \Gamma_n^E \cup \overline{(a_n^E)\,(b_n^E)'} \cup \overline{(b_n^E)\,\left(c_{n+1}^E\right)'} \cup \left(\Gamma_{n+1}^E\right)' \right\}.$$

Then S_E is a topological half-line and $Y_E = \mathrm{cl}_{\mathbb{R}^2}(S^E)$ is a Young spiral with remainder $A_E \cup A'_E$.

Now we show that there is a fixed point free map $h_E \colon Y_E \to Y_E$ defined as follows.

$$h_E(p) = \begin{cases} \alpha(p) & \text{for } p \in A_E \cup A_E{}' \cup \overline{a_n^E b_n^E} \cup \left(\Gamma_{n+1}^E\right)'; \\ (1-t)\left(b_n^E\right)' + t\left(c_{n+1}^E\right)' & \text{for } p = (1-t)b_n^E + tc_n^E \in \overline{b_n^E c_n^E}, \\ & \quad 0 \le t \le 1; \\ (1-t)a_{n+1} + tb_{n+1} & \text{for } p = (1-t)\left(a_n^E\right)' + t\left(b_n^E\right)' \in \\ & \quad \overline{(a_n^E)'(b_n^E)'},\ 0 \le t \le 1; \\ (1-t)b_{n+1} + tc_{n+1} & \text{for } p = (1-t)\left(b_n^E\right)' + t\left(c_{n+1}^E\right)' \in \\ & \quad \overline{(b_n^E)'\left(c_{n+1}^E\right)'},\ 0 \le t \le 1; \\ (-x, -g_{n+1}^E(x)) & \text{for } p = (x, g_n^E(x)) \in \Gamma_n^E. \end{cases}$$

Following the same argument as in [10, §5 Example], one can easily check that h_E is a well defined fixed point free map and extends the antipodal map α, restricted to $A_E \cup A_E{}'$.

By [2, 4.7 Theorem], there is an uncountable family $\mathcal{D}$ of incomparable Elsa continua. Thus, the family $\{Y_E : E \in \mathcal{D}\}$ is an uncountable family of incomparable Young spirals, each of which admits a fixed point free map h_E. □

3. Construction of the family

Let E_1 be an Elsa continuum embedded in the rectangle $[0,1] \times [1,2]$ with initial point $(1,1)$ and remainder $\{0\} \times [1,2]$ and let $E_2 = \{(x, -y+5) : (x,y) \in E_1\}$ be the reflection of E_1 over the line $y = \frac{5}{2}$.

Since E_1 and E_2 are Elsa continua, by [12, Lemma 31], E_1 can be embedded in the plane in such a way that the remainder is the interval $\{0\} \times [1,2]$. The rest of the continuum E_1 is the graph of a continuous function $f_{E_1} \colon (0,1] \to [1,2]$, with the additional condition that $f_{E_1}(1) = 1$. E_2 is the graph of the continuous function $f_{E_2} \colon (0,1] \to [3,4]$ such that $f_{E_2}(x) = -f_{E_1}(x) + 5$, together with the interval $\{0\} \times [3,4]$.

Let q be the point $(2,0)$ and define the following lines:

$L_1 = \overline{q(1,4)}$, $L_2 = \overline{q(1,1)}$, $L_3 = \{(x,-y) : (x,y) \in L_1\}$, and $L_4 = \{(x,-y) : (x,y) \in L_2\}$, the reflections of L_1 and L_2, respectively, over the line $y = 0$.

$L_5 = \overline{(1,-4)(0,-3)}$ and $L_6 = \overline{(1,-1)(0,-2)}$.

Using the notation of §2, for every set $A \subset \mathbb{R}^2$, A' is the antipodal set of A.

Let $E = E_2 \cup L_1 \cup L_3 \cup L_5$. We define our exterior double Warsaw circle as $H = E \cup E'$.

Let D be an Elsa continuum incomparable to E_1 embedded in the rectangle $[-1,0] \times [1,2]$ with initial point $(-1,1)$, remainder $\{0\} \times [1,2]$ and that has a point in its ray with first coordinate $x_0 < -\frac{1}{2}$ and second coordinate equal to 2. Define $I_R = E_1 \cup L_2 \cup L_4 \cup L_6$.

Now we define two different interior continua, a double Warsaw circle $I_0 = I_R \cup I'_R$ and a continuum $I = (I_0 - L'_6) \cup D$.

Let $Y_0 = H \cup I_0$ and $Y = H \cup I$.

Since D is an Elsa continuum, by [12, Lemma 31] it can be embedded in the plane in such a way that the remainder is the interval $\{0\} \times [1,2]$ and the rest of the continuum D is the graph of a continuous function $f_D\colon [-1,0) \to [1,2]$ with the additional condition that $f_D(-1) = 1$.

To define our ray approaching I, consider M the complementary domain of Y that contains the origin.

Set $a = (0,-2)$, $b = (1,-1)$, $c = (2,0)$, $d = (1,1)$, and $e = (0,1)$.

Define $f_n^{E_1}\colon [0,1] \to [0,2]$ and $f_n^D\colon [-1,0] \to [0,2]$ by

$$f_n^{E_1}(x) = \begin{cases} 1 - \frac{1}{n}, & \text{if } x = 0, \\ 1 - \frac{1}{n} - (1 - f_{E_1}(x))x^{\frac{1}{n}}, & \text{if } x \neq 0. \end{cases} \quad \text{and}$$

$$f_n^D(x) = \begin{cases} 1 - \frac{1}{n}, & \text{if } x = 0, \\ 1 - \frac{1}{n} - (1 - f_D(x))\,|x|^{\frac{1}{n}}, & \text{if } x \neq 0. \end{cases}$$

Then the maps $f_n^{E_1}$ and f_n^D satisfy the following conditions:

(1) $f_n^{E_1}(0) = 1 - \frac{1}{n}$, $f_n^{E_1}(1) = 1 - \frac{1}{n}$ and
$f_n^D(0) = 1 - \frac{1}{n}$, $f_n^D(-1) = 1 - \frac{1}{n}$,

(2) $f_1^{E_1}(x) < f_2^{E_1}(x) < f_3^{E_1}(x) < f_4^{E_1}(x) \dots$ and
$f_1^D(x) < f_2^D(x) < f_3^D(x) < f_4^D(x) \dots$ for every $x \in [0,1]$,

(3) as $n \to \infty$, $f_n^{E_1}(x) \to f_{E_1}(x)$ for $x > 0$, and
$f_n^D(x) \to f_D(x)$ for $x < 0$,

(4) $\sup_{x\in[0,1]} \left| f_n^{E_1}(x) - f_{n+1}^{E_1}(x) \right| \to 0$, and
$\sup_{x\in[-1,0]} \left| f_n^D(x) - f_{n+1}^D(x) \right| \to 0$ as $n \to \infty$.

Let $\{w_n\}_{n=2}^{\infty}$ be a sequence of points in the line $\overline{(-\frac{1}{2},1)(0,2)}$ converging to $(0,2)$ where the second coordinate of w_n is equal to $f_n^D(x_0)$.

Set the points $a_n = w_n'$, $b_n = (1, -f_n^D(-1))$, $c_n = (2 - \frac{1}{n}, 0)$, $d_n = (1, f_n^{E_1}(1))$, and $e_n = (0, f_n^{E_1}(0)) = (0, f_n^D(0))$.

Also, define E_1^n and D_n as the graphs of the maps $f_n^{E_1}$ and f_n^D, respectively.

Let J_n denote the union of the following sets: $\overline{a_n b_n}$, $\overline{b_n c_n}$, $\overline{c_n d_n}$, E_1^n, D_n, $\overline{b_n' c_n'}$, $\overline{c_n' d_{n+1}'}$, $\left(E_1^{n+1}\right)'$, and $\overline{e_{n+1}' a_{n+1}}$. Now we are ready to define the ray $S_2 = \bigcup_{n=2}^{\infty} J_n$. Observe that S_2 is a ray in M with initial point $\Theta_2 = a_2$ limiting on I.

Let M_0 be the complementary domain of Y_0 that contains the origin. To define the ray S_{2_*} in M_0 limiting on I_0, we will consider a sequence of maps.

Definition 3.1. Let l_0 be the arc in D that joins $(-1, 1)$ and $(x_0, f_D(x_0))$. Define $r_0 \colon D \to L_6' \cup (\{0\} \times [1, 2])$ as follows:

$$r_0(z) = \begin{cases} \text{the horizontal projection of } z \text{ onto } L_6', & \text{if } z \in l_0; \\ \text{the horizontal projection of } z \text{ onto } \{0\} \times [1,2], & \text{if } z \in D - l_0. \end{cases}$$

Definition 3.2. Let l_n be the arc in D_n that joins the points $b_n' = (-1, f_n^D(-1))$ and $(x_0, f_n^D(x_0))$. Define $r_n \colon D_n \to \overline{b_n' w_n} \cup \overline{w_n e_n}$ as follows:

$$r_n(z) = \begin{cases} \text{the horizontal projection of } z \text{ onto } \overline{b_n' w_n}, & \text{if } z \in l_n; \\ \text{the horizontal projection of } z \text{ onto } \overline{w_n e_n}, & \text{if } z \in D_n - l_n. \end{cases}$$

Let K_n denote the union of the following sets: $\overline{a_n b_n}$, $\overline{b_n c_n}$, $\overline{c_n d_n}$, E_1^n, $\overline{e_n w_n}$, $\overline{w_n b_n'}$, $\overline{b_n' c_n'}$, $\overline{c_n' d_{n+1}'}$, $\left(E_1^{n+1}\right)'$, and $\overline{e_{n+1}' a_{n+1}}$. Now we are ready to define the ray $S_{2_*} = \bigcup_{n=2}^{\infty} K_n$, with initial point $\Theta_{2_*} = a_2$. By the construction, S_{2_*} is a ray in M_0 limiting on I_0.

Consider N the unbounded complementary domain of H and we will construct a ray S_3 in N limiting on H. To do so, we define two homeomorphisms.

Claim 1. *I_0 and H are homeomorphic.*

Proof: Let h_0 be the homeomorphism from I_R onto E ($h_0 \colon I_R \to E$) that sends L_2 onto L_1, E_1 onto E_2, L_4 onto L_3, and L_6 onto L_5 as follows.

$$h_0(x,y) = \begin{cases} (1-y)(2,0)+y(1,4), & \text{if } (x,y)\in L_2,\\ (x,-y+5), & \text{if } (x,y)\in E_1,\\ (1+y)(2,0)-y(1,-4), & \text{if } (x,y)\in L_4,\\ x(1,-4)+(1-x)(0,-3), & \text{if } (x,y)\in L_6. \end{cases}$$

Now we define a homeomorphism $\widehat{h_0}$ from I_0 onto H ($\widehat{h_0}\colon I_0 \to H$) as follows:

$$\widehat{h_0}(x,y) = \begin{cases} h_0(x,y), & \text{if } (x,y)\in I_R,\\ -h_0(-x,-y), & \text{if } (x,y)\in I'_R. \end{cases}$$

It is very easy to check that $\widehat{h_0}$ is a homeomorphism, and the claim is proved. □

By Claim 1, I_0 and H are homeomorphic and, in fact, $N \cup H$ and $(M_0-(0,0))\cup I_0$ are homeomorphic.

Definition 3.3. Let $\widehat{h_1}\colon (M_0-(0,0))\cup I_0 \to N\cup H$ be a homeomorphism such that $\widehat{h_1} \mid I_0 = \widehat{h_0}$.

Define S_3 as the image of S_{2_*} under $\widehat{h_1}$. That is, $S_3 = \widehat{h_1}(S_{2_*})$ is a ray in N limiting on H, with initial point $\Theta_3 = \widehat{h_1}(\Theta_{2_*})$.

Consider M_1 the complementary domain of $Y \cup S_2$ that contains the origin and let S_1 be a ray in M_1 limiting on $S_2 \cup I$, with initial point Θ_1. Define a retraction ρ as follows.

Definition 3.4. Let $\rho\colon S_1\cup S_2\cup I \to S_2\cup I$ be a retraction such that $\rho(\Theta_1)=\Theta_2$.

Now we are ready to construct our example.

Example 3.5. Let $X = S_1\cup S_2\cup S_3\cup Y$.

Now we will prove that X has the required properties.

Theorem 3.6. *X is a rational irreducible planar continuum that admits a fixed point free map.*

Proof: Recall that if R is a ray, then the continuum $\mathrm{cl}(R)$ is rational if and only if the continuum $L(R)$ consisting of all the limit points of R is rational. Thus, arcs, Elsa continua, and finite unions of rational continua are rational continua.

Therefore, Y and each of its subcontinua are is rational. Also $\mathrm{cl}(S_3) = S_3 \cup H$ and $\mathrm{cl}(S_2) = S_2\cup I$ are rational. Hence, $\mathrm{cl}(S_1) = S_1\cup S_2\cup I$ is rational. Therefore, $\mathrm{cl}(S_1)\cup \mathrm{cl}(S_3) = S_1\cup S_2\cup S_3\cup Y = X$ is rational.

Notice that by construction, X is planar. Also recall that a continuum C of the form $C = \mathrm{cl}(R) \cup \mathrm{cl}(S)$, where R and S are rays with the origins p and p', respectively, is irreducible between p and p' if $R \cap \mathrm{cl}(S) = \emptyset = S \cap \mathrm{cl}(R)$ (see [9, Remark 1, p. 125] for a general statement). Since $X = \mathrm{cl}(S_1) \cup \mathrm{cl}(S_3)$, where $S_1 \cap \mathrm{cl}(S_3) = \emptyset = \mathrm{cl}(S_1) \cap S_3$, we obtain that X is irreducible between Θ_1 and Θ_3. To prove that X admits a fixed point free map, define

$f_1 \colon X \to S_2 \cup Y \cup S_3$ the retraction given by

$$f_1(x) = \begin{cases} x, & \text{if } x \in S_2 \cup Y \cup S_3 \\ \rho(x), & \text{if } x \in S_1 \end{cases}$$

where ρ is the retraction of Definition 3.4,

$f_2 \colon S_2 \cup Y \cup S_3 \to S_{2_*} \cup Y_0 \cup S_3$ given by

$$f_2(x) = \begin{cases} x, & \text{if } \; x \in \overline{a_n b_n} \cup \overline{b_n c_n} \cup \overline{c_n d_n} \cup E_1^n; \\ r_n(x), & \text{if } \; x \in D_n; \\ x, & \text{if } \; x \in \overline{b_n' c_n'} \cup \overline{c_n' d_{n+1}'} \cup \left(E_1^{n+1}\right)' \cup \overline{e_{n+1}' a_{n+1}} \cup \\ & \qquad \cup (Y - D) \cup S_3; \\ r_0(x), & \text{if } \; x \in D, \end{cases}$$

where r_0 and r_n are the maps defined in definitions 3.1 and 3.2, respectively, and

$f_3 \colon S_{2_*} \cup Y_0 \cup S_3 \to S_3 \cup H$ by

$$f_3(x) = \begin{cases} x, & \text{if } x \in S_3 \cup H, \\ \widehat{h_1}(x), & \text{if } x \in S_{2_*} \cup I_0 \end{cases}$$

where $\widehat{h_1}$ is the map of Definition 3.3.

Thus, $\phi \colon X \to H \cup S_3$, given by $\phi(x) = f_3 \circ f_2 \circ f_1(x)$, is a retraction from X onto a Young spiral $Y_3 = H \cup S_3$. Since Y_3 is homeomorphic to $Y_2 = S_{2_*} \cup I_0$, by Theorem 2.1 and the methods of §2, it follows that Y_2 does not have the fixed point property. Therefore, X does not have the fixed point property. □

Theorem 3.7. *All of the tranches of X have the fixed point property.*

Proof: Consider the homeomorphisms $\xi_1 \colon S_3 \to [0, \frac{1}{2})$ and $\xi_2 \colon S_1 \to (\frac{1}{2}, 1]$ such that $\xi_1(\Theta_3) = 0$ and $\xi_2(\Theta_1) = 1$, and define $\xi \colon X \to [0,1]$ by

$$\xi(x) = \begin{cases} \xi_1(x), & \text{if } x \in S_3, \\ \frac{1}{2}, & \text{if } x \in S_2 \cup Y, \\ \xi_2(x), & \text{if } x \in S_1. \end{cases}$$

It is easy to prove that ξ is a monotone map and that $\mathcal{D} = \{\xi^{-1}(p) : p \in [0,1]\}$ is an upper semi-continuous decomposition of X to the interval, so we need only to prove that $\xi^{-1}(p)$ has the fixed point property for each $p \in [0,1]$. For each $p \in [0,1] - \{\frac{1}{2}\}$, $\xi^{-1}(p)$ is degenerate, so it has the fixed point property. Now it remains only to prove that $\xi^{-1}(\frac{1}{2}) = S_2 \cup Y$ has the fixed point property.

Notice that $Z = S_2 \cup Y$ has four arc components: $A_0 = \{0\} \times [1,2]$, A_q the arc component of Z that contains q, $A_{q'}$ the arc component of Z that contains q', and S_2. So $Z = A_0 \cup A_q \cup A_{q'} \cup S_2$. We suppose to the contrary that Z admits a fixed point free map f and we obtain a contradiction.

Claim 1. $A_0 \cap \mathrm{Im}\,(f) = \emptyset$.

Let $K \in \{A_0, A_q, A_{q'}, S_2\}$, and notice that for each K, $A_0 \subset \mathrm{cl}_Z(K)$. Assume by way of contradiction that there exists $x \in Z$ such that $f(x) \in A_0$. Since arc components are mapped to arc components, there exists K such that $f(K) \subset A_0$; thus, $f(\mathrm{cl}_Z(K)) \subset \mathrm{cl}_Z(A_0) = A_0$ which implies that $f \mid A_0 \colon A_0 \to A_0$ has a fixed point. This is a contradiction. Therefore, $A_0 \cap \mathrm{Im}\,(f) = \emptyset$.

By Claim 1, $A_0 \cap \mathrm{Im}\,(f) = \emptyset$. Since $\mathrm{Im}\,(f)$ is a compact subset of Z, there exists two disjoint open sets U and V of Z such that $A_0 \subset U$ and $\mathrm{Im}\,(f) \subset V$. We may assume, without loss of generality, that $[A_0 \cup \mathrm{int}_Y(D) \cup \mathrm{int}_Y(E_1)] \subset U$.

Claim 2. $\mathrm{Im}\,(f) \subset A_{q'} \cup A_q - (A_0 \cup \mathrm{int}_Y(D) \cup \mathrm{int}_Y(E_1))$.

Since $\mathrm{Im}\,(f)$ is connected, $\mathrm{Im}\,(f)$ is contained in a component of $Z - U$. Since $A_0 \subset U$ and $A_0 \subset \mathrm{cl}_Z(S_2)$, it follows that $S_2 \cap U \neq \emptyset$. Thus, the components of $Z - U$ are subarcs of the ray S_2 or connected subsets of $A_{q'} \cup A_q - (A_0 \cup \mathrm{int}_Y(D) \cup \mathrm{int}_Y(E_1))$.

In the case that $\mathrm{Im}\,(f) \subset S_2$ by compactness there exists two points $a \leq b \in S_2$ such that $\mathrm{Im}\,(f)$ is the arc $[a,b]$ of S_2; hence, $f \mid [a,b] \colon [a,b] \to [a,b]$ has a fixed point. Therefore, $\mathrm{Im}\,(f) \subset A_{q'} \cup A_q - (A_0 \cup \mathrm{int}_Y(D) \cup \mathrm{int}_Y(E_1))$, and the claim is proved.

Claim 3. $f(A_{q'}) \subset A_q$ and $f(A_q) \subset A_{q'}$.

Since arc components are mapped to arc components and by Claim 2, $\mathrm{Im}\,(f) \subset A_{q'} \cup A_q - (A_0 \cup \mathrm{int}_Y(D) \cup \mathrm{int}_Y(E_1))$, we have that

$f(A_{q'}) \subset A_{q'}$ or $f(A_{q'}) \subset A_q$ (similarly, $f(A_q) \subset A_{q'}$ or $f(A_q) \subset A_q$).

Assume by way of contradiction that $f(A_{q'}) \subset A_{q'}$ (similarly, $f(A_q) \subset A_q$). Using a dead-end argument (see [3, Theorem 13]), it is easy to see that if $f(q') \in A_{q'}$ (similarly, $f(q) \in A_q$), then the dog chases and gets the rabbit, and we obtain a fixed point. Hence, $f(A_{q'}) \subset A_q$, $f(A_q) \subset A_{q'}$, and the claim is proved.

Claim 4. $f(A_0) \cap A_{q'} = \emptyset$.

Suppose to the contrary that $f(A_0) \cap A_{q'} \neq \emptyset$. Then, by Claim 2, $f(A_0) \subset A_{q'} - \operatorname{int}_Y D$. If $f(A_0) \cap \operatorname{int}_Z(A_{q'}) \neq \emptyset$, since every point of A_0 is a limit point of $D - A_0$, we obtain that $f(D) \cap \operatorname{int}_Z(A_{q'}) \neq \emptyset$. Hence, $f(A_{q'}) \subset A_{q'}$ which is a contradiction to Claim 3

Thus, by Claim 2 and the assumption $f(A_0) \subset \operatorname{bd}_Z(A_{q'}) \cap (A_{q'} - \operatorname{int}_Y(D)) = \{0\} \times [3,4] \cup L'_2 \cup L'_4 \cup (E'_1 - (\{0\} \times [-2,-1]))$.

By the continuity of f, we have that $f(A_0) \subset \{0\} \times [3,4]$ or $f(A_0) \subset L'_2 \cup L'_4 \cup (E'_1 - (\{0\} \times [-2,-1]))$. Since D limits in A_0, we obtain that either

(a) $f(D)$ limits in $\{0\} \times [3,4]$, or

(b) $f(D)$ limits in $L'_2 \cup L'_4 \cup (E'_1 - (\{0\} \times [-2,-1]))$.

If (a) occurs, we obtain by Claim 3 that $f(D) \subset A_q \cup \{0\} \times [3,4]$, which implies that $f(D)$ is mapped onto a copy of E_2 which is a contradiction since D and E_2 are incomparable.

If (b) occurs, we obtain that $f(D)$ is mapped into $L'_2 \cup L'_4 \cup (E'_1 - (\{0\} \times [-2,-1])) \subset A_{q'}$, which is a contradiction to Claim 3.

Therefore, $f(A_0) \cap A_{q'} = \emptyset$, and the claim is proved.

Claim 5. $f(A_0) \subset \{0\} \times [-4,-3]$ or $f(A_0) \subset \{0\} \times [-2,-1]$.

By claims 2 and 4, $f(A_0) \subset A_q$. Notice that if $f(A_0) \cap \operatorname{int}_Z(A_q) \neq \emptyset$, since every point of A_0 is a limit point of $E_1 - A_0$, we obtain that $f(E_1) \cap \operatorname{int}_Z(A_q) \neq \emptyset$. Hence, $f(A_q) \subset A_q$, which is a contradiction to Claim 3. Thus, by Claim 2, $f(A_0) \subset \operatorname{bd}_Z(A_q) \cap (A_q - \operatorname{int}_Y E_1)$ $= (\{0\} \times [-4,-3]) \cup L_2 \cup L_4 \cup L_6 \cup (\{0\} \times [-2,-1])$. By the continuity of f, we have that either

(a) $f(A_0) \subset L_2 \cup L_4 \cup L_6 \cup (\{0\} \times [-2,-1])$, or

(b) $f(A_0) \subset \{0\} \times [-4,-3]$.

If (a) occurs, since E_1 limits in A_0, we obtain by Claim 3 that $f(E_1 - A_0) \subset A_{q'}$ and that $f(E_1) \subset A_{q'} \cup L_2 \cup L_4 \cup L_6 \cup (\{0\} \times [-2,-1])$, which implies that $f(E_1)$ is mapped onto a copy of E'_1, $f(A_0) \subset \{0\} \times [-2,-1]$, and the claim follows.

Claim 6. $f(S_2) \subset A_{q'}$.

By Claim 2, we may assume by way of contradiction, that $f(S_2) \subset A_q - \mathrm{int}_Y(E_1)$. Then $f(\mathrm{cl}_Z(S_2)) \subset \mathrm{cl}_Z(f(S_2)) \subset \mathrm{cl}_Z(A_q - \mathrm{int}_Y(E_1)) = (A_q - \mathrm{int}_Y(E_1)) \cup (\{0\} \times [3,4])$.

Since $I_R - A_0 \subset A_q$, by Claim 3, $f(I_R - A_0) \subset A_{q'}$. Thus, $f(I_R) = f(\ \mathrm{cl}_Z(I_R - A_0)) \subset \mathrm{cl}_Z(f(I_R - A_0)) \subset \{0\} \times [3,4]$.

Also notice that $A_0 \subset I_R$. Hence, $f(A_0) \subset \{0\} \times [3,4] \subset A_{q'}$, which contradicts Claim 4.

Therefore, $f(S_2) \subset A_{q'}$, and the claim is proved.

Claim 7. $f((I'_R - L'_6) \cup D) \subset \{0\} \times [-4,-3]$ or $f((I'_R - L'_6) \cup D) \subset \{0\} \times [-2,-1]$.

Notice that $f((I'_R - L'_6) \cup D) \subset f(\mathrm{cl}_Z(S_2)) \subset \mathrm{cl}_Z(f(S_2)) \subset \mathrm{cl}_Z(A_{q'} - \mathrm{int}_Y(D)) = A_{q'} - \mathrm{int}_Y(D) \cup (\{0\} \times [-4,-3] \cup \{0\} \times [-2,-1])$. By the connectedness of $(I'_R - L'_6) \cup D$ and Claim 3, the claim follows.

Now we are ready to obtain our final contradiction to the assumption that f is a fixed point free map.

Notice that $\{0\} \times [-2,-1] \subset ((I'_R - L'_6) \cup D) \cap A_q$. Hence, by Claim 7, $f(\{0\} \times [-2,-1]) \subset (\{0\} \times [-4,-3] \cup \{0\} \times [-2,-1]) \subset A_q$. Thus, $f(A_q) \not\subseteq A_{q'}$, which is a contradiction to Claim 3.

Hence, $Z = S_2 \cup Y = \xi^{-1}(\frac{1}{2})$ has the fixed point property and the theorem is proved. □

Theorem 3.8. *There exists an uncountable family of incomparable rational irreducible continua in the plane, each of which admits a fixed point free map, with the condition that all of their tranches have the fixed point property.*

Proof: Notice that in our construction we used two incomparable Elsa continua E_1 and D. By [2, 4.7 Theorem] there is an uncountable family $\mathcal{D}$ of incomparable Elsa continua. We may assume that E_1 and D are in $\mathcal{D}$. For each $D_\alpha \in \mathcal{D}$ ($D_\alpha \neq D$, E_1) construct X_α, as we constructed X, replacing D_α for D. Thus, $\mathcal{X} = \{X_\alpha : D_\alpha \in \mathcal{D} - \{D, E_1\}\}$ is an uncountable family.

First we prove that the elements of $\mathcal{X}$ are incomparable. Notice that for each $X_\alpha \in \mathcal{X}$, X_α has exactly six arc components: S_1^α, S_2^α, S_3^α, A_0^α, A_q^α, and $A_{q'}^\alpha$.

Let X_α and X_β be two different elements of $\mathcal{X}$, where D_α is incomparable to D_β. Suppose to the contrary that there exists a surjective map $f\colon X_\alpha \to X_\beta$.

Claim 1. If K is an arc component of X_α, there exists exactly one arc component L of X_β such that $f(K) = L$.

Let L be the arc component of X_β such that $f(K) \cap L \neq \emptyset$. Clearly, $f(K) \subset L$, since X_α and X_β have exactly six arc components and f is onto. By the pigeon hole principle, the claim follows.

Claim 2. Let K be an arc component of X_α, let $K_1, \ldots, K_m$ be the arc components of X_α such that $K_i \cap \mathrm{cl}_{X_\alpha}(K) \neq \emptyset$, let L be the arc component of X_β such that $f(K) = L$, and let $L_1, \ldots, L_n$ be the arc components of X_β such that $L_j \cap \mathrm{cl}_{X_\beta}(L) \neq \emptyset$, then $f(K_1 \cup \ldots \cup K_m) \subset L_1 \cup \ldots \cup L_n$ and $n \geq m$.

By continuity, $f(\mathrm{cl}_{X_\alpha} K) \subset \mathrm{cl}_{X_\beta} f(K) \subset L_1 \cup \ldots \cup L_n$. Therefore, for each $i \in \{1, \ldots, m\}$, $f(K_i) \cap (L_1 \cup \ldots \cup L_n) \neq \emptyset$, and by Claim 1, there exists exactly one element j_i of $\{1, \ldots, n\}$ such that $f(K_i) = L_{j_i}$. Therefore, $f(K_1 \cup \ldots \cup K_m) = L_{j_1} \cup \ldots \cup L_{j_m} \subset L_1 \cup \ldots \cup L_n$ which implies that $n \geq m$.

Claim 3. $f(A_0^\alpha) = A_0^\beta$.

By Claim 1, $f(A_0^\alpha)$ is a compact arc component of X_β. Therefore, $f(A_0^\alpha) = A_0^\beta$.

Claim 4. $f(S_1^\alpha) = S_1^\beta$, $f(S_2^\alpha) = S_2^\beta$, and $f(S_3^\alpha) = S_3^\beta$.

By claims 1, 2, and 3, we have that $f(S_1^\alpha) = L$, where $L \in \{S_2^\beta, S_3^\beta, A_q^\beta, A_{q'}^\beta\}$ or $f(S_1^\alpha) = S_1^\beta$. In the first case, notice that $\mathrm{cl}_{X_\beta}(L)$ intersects at most four arc components and $\mathrm{cl}_{X_\alpha}(S_1^\alpha)$ intersects five arc components. By Claim 2, we conclude that $f(S_1^\alpha) = S_1^\beta$.

Proceeding as before, we have that either

- $f(S_2^\alpha) = L$, where $L \in \{S_3^\beta, A_q^\beta, A_{q'}^\beta\}$, or
- $f(S_2^\alpha) = S_2^\beta$.

In the first case, notice that $\mathrm{cl}_{X_\beta}(L)$ intersects three arc components and $\mathrm{cl}_{X_\alpha}(S_2^\alpha)$ intersects four arc components. By Claim 2, we conclude that $f(S_2^\alpha) = S_2^\beta$.

As before, notice that $f(S_3^\alpha) = L$, where $L \in \{A_q^\beta, A_{q'}^\beta\}$ or $f(S_3^\alpha) = S_3^\beta$. In the first case, the arc components of X_α that intersect $\mathrm{cl}_{X_\alpha}(S_3^\alpha)$ are S_3^α, A_q^α, and $A_{q'}^\alpha$, and the arc components of X_B that intersect $\mathrm{cl}_{X_\beta}(L)$ are A_q^β, $A_{q'}^\beta$, and A_0^β. By Claim 2, we have that $f(S_3^\alpha \cup A_q^\alpha \cup A_{q'}^\alpha) \subset A_q^\beta \cup A_{q'}^\beta \cup A_0^\beta$, and by claims 1 and 3, we have that $f(S_3^\alpha \cup A_q^\alpha \cup A_{q'}^\alpha) \subset A_q^\beta \cup A_{q'}^\beta$, which is a contradiction to Claim 1. Hence, $f(S_3^\alpha) = S_3^\beta$, and the claim is proved.

Now we are ready to obtain our final contradiction to the assumption that $f\colon X_\alpha \to X_\beta$ is an onto map.

By claims 1, 2, 3, and 4, we have that either $f(A_q^\alpha) = A_q^\beta$ or $f(A_q^\alpha) = A_{q'}^\beta$.

Case 1: $f(A_q^\alpha) = A_q^\beta$.

By Claim 1, this implies that $f(A_{q'}^\alpha) = A_{q'}^\beta$. Let U be an open set of X_β such that $A_0^\beta \subset U$ and $U \cap (A_{q'}^\beta \cup A_0^\beta) \subset D_\beta$. By continuity and Claim 3, there exists an open set V of X_α, such that $A_0^\alpha \subset V$ and $f(V) \subset U$. Notice that there exists a homeomorphic copy G_α of D_α, such that $G_\alpha \subset V$. By our choice of U, Claim 3, and our assumption that $f(A_{q'}^\alpha) = A_{q'}^\beta$, we have that $f(G_\alpha) = G_\beta$, a homeomorphic copy of D_β. This is a contradiction, since D_α and D_β are incomparable. Hence, Case 1 is impossible.

Case 2: $f(A_q^\alpha) = A_{q'}^\beta$.

Following a similar argument as the one used in Case 1, we obtain that there exists G_α and F_1 homeomorphic copies of D_α and E_1, respectively, such that $f(G_\alpha) = F_1$, which contradicts the fact that D_α and E_1 are incomparable. Therefore, Case 2 is also impossible.

So, we have that both cases are impossible, a contradiction to the assumption that there exists an onto map $f\colon X_\alpha \to X_\beta$. Hence, X_α and X_β are incomparable.

The rest of the theorem follows since by Theorem 3.6 and Theorem 3.7, each element X_α of the family $\mathcal{X}$ has the required properties. $\square$

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