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by

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EXTENSION THEORY AND THE FIRST UNCOUNTABLE ORDINAL SPACE

LEONARD R. RUBIN

Abstract. We shall examine the extension theory of products $Y = Z \times [0, \Omega)$ where Z is a compact metrizable space and Ω is the first uncountable ordinal. Our main result is that if a CW-complex K is an absolute extensor for Z, then K is an absolute extensor for Y. This implies, as a corollary, the classical fact that Y is normal. We shall also examine the extension theory of pseudo-compact spaces and will prove that if X is a normal, Hausdorff, pseudo-compact space, and K is an absolute extensor for X, then it is also an absolute extensor for the Stone-Čech compactification of X. From this we will be able to deduce that for the preceding space Y, K is an absolute extensor for $\beta(Y)$.

1. Introduction

Let X and K be spaces; suppose that for all closed subsets A of X and for every map $f : A \to K$ there exists a map $F : X \to K$ such that $F|A = f$. Then we write $X\tau K$ and say either that X is an *absolute co-extensor* for K or K is an *absolute extensor* for X. This is the fundamental notion of *extension theory* (see [1] or [4]) where usually K is a CW-complex. It then follows that X is a normal space if and only if $X\tau\,\mathbb{R}$.

Let Ω designate the first uncountable ordinal. Then $[0, \Omega)$ will denote the set of ordinals less than Ω with the order topology,

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often called the *first uncountable ordinal space* [6]. An important tool in the study of $[0, \Omega)$ is its pseudo-compactness. We shall examine this property in section 2. In Proposition 2.4 we exhibit the extension-theoretic relation between a pseudo-compactum X and its Stone-Čech compactification $\beta(X)$.

In section 3 we study the extension theory of products $Z \times [0, \Omega)$ where Z is compact and metrizable. Our main theorem, Theorem 3.11, states that if Z is a compact metrizable space, K is a CW-complex, $Z\tau K$, and $Y = Z \times [0, \Omega)$, then both $Y\tau K$ and $\beta(Y)\tau K$.

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2. Extension theory and pseudo-compacta

A space X is called *pseudo-compact* if for each map $f : X \to \mathbb{R}$, $f(X)$ is contained in a compact subset of $\mathbb{R}$, or, equivalently, $f(X)$ is a compact subset of $\mathbb{R}$. A good source of facts about such spaces can be found in [3]. Here is some information about pseudo-compact spaces.

Lemma 2.1. *Let X be a pseudo-compact space.*

(1) *If Y is a compact space, then $X \times Y$ is pseudo-compact.*
(2) *If X is normal and A is closed in X, then A is pseudo-compact.*

Lemma 2.2. *Let X be a space. The following are equivalent.*

(1) *X is pseudo-compact.*
(2) *For each* CW*-complex K and map $f : X \to K$, $f(X)$ is contained in a compact subset of K.*
(3) *For each* CW*-complex K and map $f : X \to K$, $f(X)$ is a compact subset of K.*

Proof: (1) $\Rightarrow$ (2). Suppose that $f(X)$ is not contained in a compact subset of K. Then there exists a countably infinite closed discrete subspace A of K such that $A \subset f(X)$. Let $g : A \to \mathbb{R}$ be a function such that $g(A) = \mathbb{N}$. Then g is a map, and since K is normal, there exists a map $h : K \to \mathbb{R}$ such that $h|A = g$. Define $F = h \circ f : X \to \mathbb{R}$. Then F is a map of X to $\mathbb{R}$. But $\mathbb{N} \subset F(X)$, so $F(X)$ is not contained in a compact subset of $\mathbb{R}$, a contradiction.

(2) $\Rightarrow$ (3). Let L be the minimum subcomplex such that $f(X) \subset L$. Then L is compact and metrizable. Suppose that $f(X)$ is not compact; then $f(X)$ is not closed in L. Let $p \in L \setminus f(X)$. Since L is minimal, p cannot be isolated. It then follows that there is a map $h : L \setminus \{p\} \to \mathbb{R}$ such that for all $n \in \mathbb{N}$, there exists $x \in L \setminus \{p\}$ with $h(x) > n$. Treating $f : X \to L \setminus \{p\}$, put $F = h \circ f : X \to \mathbb{R}$. Then F is a map of X to $\mathbb{R}$. But $F(X)$ is not contained in a compact subset of $\mathbb{R}$, a contradiction.

(3) $\Rightarrow$ (1). This follows from the fact that $\mathbb{R}$ may be given the structure of a CW-complex. $\square$

Now we investigate the relation between pseudo-compact spaces and extension theory.

In the proof of Proposition 2.4 we shall use Lemma 2.11 of [7], which we state here for the convenience of the reader.

Lemma 2.3. *Let Y be a compact Hausdorff space and X a dense subset of Y. Then for each closed subset A of Y and neighborhood G of A, there exists a closed neighborhood N of A such that $N \subset G$ and $\overline{N \cap X} = N$.* $\square$

If X is a Tychonoff space, then $\beta(X)$ will denote its Stone-Čech compactification. Here is the principal extension-theoretic fact about pseudo-compacta.

Proposition 2.4. *Let X be a normal, Hausdorff, pseudo-compact space and K a* CW*-complex. Suppose that $X \tau K$. Then $\beta(X) \tau K$.*

Proof: Of course X is a Tychonoff space. Let A be a closed subset of $\beta(X)$ and $f : A \to K$ a map. Using the fact that K is an absolute neighborhood extensor for $\beta(X)$ along with Lemma 2.3, we may as well assume that $\overline{A \cap X} = A$. Since $X \tau K$, there is a map $f_0 : X \to K$ such that $f_0|A \cap X = f|A \cap X$. By Lemma 2.2, the image of f_0 lies in a compact subset of K. Hence, there is a map $F : \beta(X) \to K$ such that $F|X = f_0|X$. Since $A \cap X$ is dense in A and $F|A \cap X = f_0|A \cap X = f|A \cap X$, then $F|A = f$, and our proof is complete. $\square$

3. Extension theory and $[0, \Omega)$

It is well known that for each compact metrizable space Z, $Z \times [0, \Omega)$ is normal. We plan to present a proof of this fact that will lend itself to a generalization into extension theory.

We shall provide a proof of the following known fact.

Theorem 3.1. *Let K be a* CW*-complex, X be a compact Hausdorff space with $X\tau K$, and Y be a compact Hausdorff space with $Y\tau S^0$. Then,*

$$(Y \times X)\tau K.$$

Note that for a compact Hausdorff space Y, $Y\tau S^0$ is equivalent to $\dim Y \leq 0$.

Before presenting our proof of Theorem 3.1, let us introduce two lemmas. The first is the "tube" lemma (see [2, XI.2.6, p. 228] or [6, 3.26.8, p. 168]).

Lemma 3.2. *Let X and Y be spaces, Y be compact, A be a subset of X, and U be a neighborhood of $A \times Y$ in $X \times Y$. Then there exists a neighborhood V of A in X such that $V \times Y \subset U$.* □

Lemma 3.3. *Let X be a space such that $X\tau S^0$, A be a closed subset of X, and U be a neighborhood of A in X. Then there exists an open and closed neighborhood V of A in X such that $V \subset U$.*

Proof: Let $f : A \cup (X \setminus U) \to S^0$ be the map with $f(A) \subset \{0\}$ and $f(X \setminus U) \subset \{1\}$. Since $X\tau S^0$, there exists a map $F : X \to S^0$ that extends f. Let $V = F^{-1}(\{0\})$. It is easy to check that $A \subset V \subset U$ and that V is open and closed. □

Now we give our proof of Theorem 3.1.

Proof: We may as well assume that $Y \neq \emptyset$. Let A be a closed subset of $Y \times X$ and $f : A \to K$ a map. Fix $y \in Y$ and consider the closed subspace $P_y = A \cup (\{y\} \times X) \subset Y \times X$. Since $X\tau K$, there exists a map $f_y : P_y \to K$ such that $f_y|A = f$.

Now $Y \times X$ is compact and Hausdorff; hence, K is an absolute neighborhood extensor for $Y \times X$. So there exists a neighborhood U_y of P_y in $Y \times X$ and a map $G_y : U_y \to K$ extending f_y. Using Lemma 3.2 and Lemma 3.3, select an open and closed neighborhood V_y of y in Y such that $V_y \times X \subset U_y$.

There exists a finite subset $\mathcal{F} \subset Y$ such that $\{V_y \,|\, y \in \mathcal{F}\}$ covers the compact space Y. Write $\mathcal{F} = \{y_1, \ldots, y_n\}$ where $n = \operatorname{card} \mathcal{F}$.

Put $W_1 = V_{y_1}$, and for $1 < k \leq n$, $W_k = V_{y_k} \setminus \bigcup\{V_{y_i} \,|\, 1 \leq i < k\}$. Then $\{W_k \,|\, 1 \leq k \leq n\}$ is an open and closed cover of Y, for each $1 \leq k \leq n$, $W_k \subset V_{y_k}$, and if $1 \leq j < k \leq n$, then $W_j \cap W_k = \emptyset$.

Define $F_k = G_{y_k}|(W_k \times X) : W_k \times X \to K$. One may now check that

$$F = \bigcup\{F_k \,|\, 1 \leq k \leq n\} : Y \times X \to K$$

is a map that extends f. □

The next is a list of well-known facts about the first uncountable ordinal space.

Lemma 3.4. *Let $X = [0, \Omega)$.*

(1) *X is a normal Hausdorff space.*
(2) *Let $0 \leq \lambda < \Omega$; then $[0, \lambda]$ is a compact, 0-dimensional metrizable subspace of X.*
(3) *Let P be a closed nonempty subset of X and $f : P \to \mathbb{R}$ be a map. Then there exists $\sigma \in P$ such that whenever μ,$\beta \in P$ and $\sigma \leq \mu < \beta$, then $f(\mu) = f(\beta)$.*
(4) *X is pseudo-compact.* □

We now present our main lemma.

Lemma 3.5. *Let Z be a nonempty compact metrizable space, A be a closed subset of $Y = Z \times [0, \Omega)$ with $Z \times \{0\} \subset A$, $f : A \to \mathbb{R}$ be a map, and $\epsilon > 0$. For each $z \in Z$, define $S_\epsilon(z) = \{\sigma \in [0, \Omega) \,|\, \exists\, \mu,\, \beta \in [0, \Omega),\, \sigma \leq \mu < \beta,\, (z, \mu),\, (z, \beta) \in A,\, |f(z, \mu) - f(z, \beta)| \geq \epsilon\}$. Then*

($*$) *there exists $0 \leq \lambda < \Omega$ such that $\bigcup\{S_\epsilon(z) \,|\, z \in Z\} \subset [0, \lambda]$.*

Proof: Fix $z \in Z$ and define $A_z = A \cap (\{z\} \times [0, \Omega))$. Then A_z is a nonempty closed subspace of $\{z\} \times [0, \Omega)$, the latter being a copy of $[0, \Omega)$. Consider $f|A_z : A_z \to \mathbb{R}$, and apply Lemma 3.4(3) to this map. Accordingly, there is a first element $l(z) \in [0, \Omega)$ such that if $l(z) \leq \mu < \beta$, and (z, μ) and $(z, \beta) \in A$, then $|f(z, \mu) - f(z, \beta)| < \epsilon$. It then follows that,

(F1) $S_\epsilon(z) \subset [0, l(z)]$,
(F2) $l(z) \notin S_\epsilon(z)$, and
(F3) if $0 \leq \lambda < l(z)$, then $\lambda \in S_\epsilon(z)$.

Having defined $l(z) \in [0, \Omega)$ for each $z \in Z$ satisfying (F1)–(F3), let us put $T = \{l(z) \,|\, z \in Z, S_\epsilon(z) \neq \emptyset\}$. Suppose we can find $\alpha \in [0, \Omega)$ so that $T \subset [0, \alpha]$. Let $z \in Z$. We claim that $S_\epsilon(z) \subset [0, \alpha]$. By (F1), $S_\epsilon(z) \subset [0, l(z)]$. Since $l(z) \in T$, then $l(z) \in [0, \alpha]$, so $S_\epsilon(z) \subset [0, l(z)] \subset [0, \alpha]$. Hence, (F4) $(*)$ is true if there exists $\alpha \in [0, \Omega)$ such that $T \subset [0, \alpha]$.

In case $T = \emptyset$, then define $\alpha = 0$. If $T \neq \emptyset$ and T is countable, then put $\alpha = \sup(T)$. In either case, $T \subset [0, \alpha]$, so by (F4), $(*)$ is true. Hence, we shall assume that T is uncountable; we choose an uncountable subset $Z_0 \subset Z$ so that the function $l|Z_0 : Z_0 \to T$ is a bijection. To reach a contradiction, suppose that there is no $\alpha \in [0, \Omega)$ with $T \subset [0, \alpha]$. This along with (F3) means that

(F5) for all $\alpha \in [0, \Omega)$, there exists $z \in Z_0$ with $\alpha < l(z)$, and

(F6) for all $\alpha \in [0, \Omega)$ and $z \in Z_0$ with $\alpha < l(z)$, there are $\alpha \leq \mu < \beta$ with (z, μ), $(z, \beta) \in A$, and $|f(z, \mu) - f(z, \beta)| \geq \epsilon$.

Employing the well-ordering of $T \subset [0, \Omega)$, we shall treat Z_0 as a well-ordered set induced by the bijection $l|Z_0 : Z_0 \to T$. Let us write $<_0$ for the ordering in Z_0.

Let z_0 be the first element of Z_0. Applying (F5) and (F6) with $\alpha = l(z_0)$, there exist a first element $a(z_0) \in Z_0$ such that $l(z_0) < l(a(z_0))$, and $l(z_0) \leq h(z_0) < g(z_0)$, such that $q(z_0) = (a(z_0), h(z_0)) \in A$, $r(z_0) = (a(z_0), g(z_0)) \in A$, and $|f(q(z_0)) - f(r(z_0))| \geq \epsilon$.

We proceed with a transfinite construction. Let $\overline{z} \in Z_0 \setminus \{z_0\}$ and suppose that for all $z \in Z_0$ with $z <_0 \overline{z}$, we have chosen $a(z) \in Z_0$ as well as $l(z) \leq h(z) < g(z)$ such that if $z <_0 z^* \leq_0 z' <_0 \overline{z}$, then the following inductive statements are true.

(I1) $a(z) <_0 a(z^*)$,

(I2) $g(z) < h(z^*)$ and $h(z^*) < g(z')$ if $z^* <_0 z'$,

(I3) $q(z) = (a(z), h(z)) \in A$, $r(z) = (a(z), g(z)) \in A$, and

(I4) $|f(q(z)) - f(r(z))| \geq \epsilon$.

Let $E = \{z \in Z_0 \,|\, z <_0 \overline{z}\}$. Then E is a countable subset of Z_0. Put $B = \{l(a(z)) \,|\, z \in E\}$, and $M = \{g(z) \,|\, z \in E\}$. Each of these sets is a nonempty and countable subset of $[0, \Omega)$. Hence, $\gamma = \sup(B \cup M)$ exists in $[0, \Omega)$.

Note that $a(E)$ is a countable subset of Z_0. Let F be the subset of Z_0 consisting of those elements u with $a(z) < u$ for all $z \in E$. Then F is an uncountable subset of Z_0. Using this and (F5), there exists $u \in F$ such that $\alpha = \max\{l(\overline{z}), \gamma + 1\} < l(u)$. Define $a(\overline{z})$ to be the first element of F such that $\alpha < l(a(\overline{z}))$.

Applying (F6) to α and $z = a(\overline{z})$, there are $\alpha \le h(\overline{z}) < g(\overline{z})$ such that $q(\overline{z}) = (a(\overline{z}), h(\overline{z})) \in A$, $r(\overline{z}) = (a(\overline{z}), g(\overline{z})) \in A$, and $|f(q(\overline{z})) - f(r(\overline{z}))| \ge \epsilon$.

Let $z \in E$. Then $a(z) <_0 a(\overline{z})$ since $a(\overline{z}) \in F$. Surely $\gamma < h(\overline{z})$. Thus, $h(z) < g(z) < h(\overline{z})$. This ends our inductive construction.

We have defined functions $a : Z_0 \to Z_0$, $h : Z_0 \to [0, \Omega)$, $g : Z_0 \to [0, \Omega)$ such that whenever z, z^*, $z' \in Z_0$ with $z <_0 z^* \le_0 z'$, statements (I1)–(I4) hold true.

It follows from (I1) that $Q = \{a(z) \,|\, z \in Z_0\}$ is an uncountable subset of the second countable space Z_0. Hence, there exists a point $z_0 \in Z_0$ so that $a(z_0)$ is a limit point of Q. So there is a sequence (y_n) in $Q \setminus \{a(z_0)\}$ that converges to $a(z_0)$. Since Z_0 is well ordered, so is Q, and we may assume that (y_n) is increasing. Let us write $y_n = a(z_n)$ for each $n \in \mathbb{N}$. Because of (I1), (z_n) is increasing.

Applying (I2), one sees that both $(h(z_n))$ and $(g(z_n))$ are increasing sequences in $[0, \Omega)$. So they both converge in $[0, \Omega)$. From (I2) one may conclude that when $m < n < v$ in $\mathbb{N}$, then $g(z_m) < h(z_n) < g(z_v)$. Hence, the sequences $(h(z_n))$ and $(g(z_n))$ have the same limit, say $\rho \in [0, \Omega)$.

For each $n \in \mathbb{N}$, let $q_n = (a(z_n), h(z_n))$ and $r_n = (a(z_n), g(z_n))$. By (I3), both (q_n) and (r_n) are sequences in A. They converge to $(a(z_0), \rho)$ in Y. Since A is closed in Y, then $(a(z_0), \rho) \in A$. But $f : A \to \mathbb{R}$ is a map, so each of the sequences $(f(q_n))$ and $(f(r_n))$ converges to the same element $f(a(z_0), \rho)$ of $\mathbb{R}$. This leads to a contradiction, because by (I4), $|f(q_n) - f(r_n)| \ge \epsilon$ for all $n \in \mathbb{N}$. Our proof is complete. □

Before we can successfully apply Lemma 3.5, we need some additional facts.

Lemma 3.6. *Let Z be a compact metrizable space. Then the coordinate projection $\pi_Z : Z \times [0, \Omega) \to Z$ is a closed map.*

Proof: Let $A \subset Z \times [0, \Omega)$ be closed and suppose that $\pi_Z(A)$ is not closed in Z. Then there is a sequence (a_n) in A and $z \in Z \setminus \pi_Z(A)$ such that $(\pi_Z(a_n))$ converges to z.

For some $0 \le \lambda < \Omega$, $a_n \in Z \times [0, \lambda]$ for all $n \in \mathbb{N}$. Hence, (a_n) is a sequence in the compact metrizable space $A \cap (Z \times [0, \lambda])$. Passing to a subsequence if necessary, we may assume that (a_n) converges to $a \in A$. Therefore, $\pi_Z(a) = z \in \pi_Z(A)$, a contradiction. □

Lemma 3.7. *Let Z be a compact metrizable space and K be a* CW*-complex such that $Z\tau K$. Suppose that A is a closed subset of $Y = Z \times [0, \Omega)$ and $f : A \to K$ is a map. Suppose further that*

$(*)$ there exists $0 \leq \lambda < \Omega$ such that if $\lambda \leq \mu < \beta < \Omega$ and (a, μ), $(a, \beta) \in A$, then $f(a, \mu) = f(a, \beta)$.

Then f extends to a map of Y to K.

Proof: Since $(Z \times \{0\})\tau K$ and $Z \times \{0\}$ is closed in Y, we may as well assume that $Z \times \{0\} \subset A$. Let $\pi_\lambda : Z \times [\lambda, \Omega) \to Z \times \{\lambda\}$ be the map given by $\pi_\lambda(z, \alpha) = (z, \lambda)$. Noting that $[\lambda, \Omega)$ is homeomorphic to $[0, \Omega)$, one may apply Lemma 3.6 to see that π_λ is a closed map. Let $A^\# = A \cap (Z \times [\lambda, \Omega))$, $\pi^\# = \pi_\lambda | A^\# : A^\# \to Z \times \{\lambda\}$, and $A_\lambda = \pi^\#(A^\#)$. Then, of course, A_λ is a closed subset of both $Z \times \{\lambda\}$ and $Z \times [0, \lambda]$.

Now $\pi^\# : A^\# \to A_\lambda$ is a closed map and hence is a quotient map. As a result of this and $(*)$, there is a map $f_\lambda : A_\lambda \to K$ such that if $(z, \lambda) \in A_\lambda$ and $(z, \lambda) = \pi^\#(z, \alpha)$ with $(z, \alpha) \in A^\#$, then $f_\lambda(z, \lambda) = f(z, \alpha)$.

Put $h_0 = f|(A \cap (Z \times [0, \lambda]))$. Then $h_0|(A \cap A_\lambda) = f_\lambda|(A \cap A_\lambda)$. This shows that there is a map $h_1 : (A \cap (Z \times [0, \lambda])) \cup A_\lambda \to K$ such that $h_1|(A \cap (Z \times [0, \lambda])) = f|(A \cap (Z \times [0, \lambda]))$ and $h_1|A_\lambda = f_\lambda$.

Applying Lemma 3.4(2) and Theorem 3.1, one sees that $(Z \times [0, \lambda])\tau K$. Since $(A \cap (Z \times [0, \lambda])) \cup A_\lambda$ is closed in $Z \times [0, \lambda]$, then there is a map $F : Z \times [0, \lambda] \to K$ having the property that $F(t) = h_1(t)$ for all $t \in (A \cap (Z \times [0, \lambda])) \cup A_\lambda$. Let $q : [0, \Omega) \to [0, \lambda]$ be the unique retraction sending μ to λ for all $\mu > \lambda$ and $r = \mathrm{id}_Z \times q : Y \to Z \times [0, \lambda]$. Then for each $z \in Z$ and $\lambda \leq \mu < \Omega$, $r(z, \mu) = (z, \lambda)$. Thus, $F \circ r : Y \to K$ is a map, and in consideration of the preceding construction, one can check that $(F \circ r)|A = f$. $\square$

Lemma 3.8. *Let Z be a compact metrizable space. Then $Y = Z \times [0, \Omega)$ is normal.*

Proof: Let $A \subset Y$ be a closed subset and $f : A \to \mathbb{R}$ a map. For each $n \in \mathbb{N}$, apply Lemma 3.5 to find $\lambda_n \in [0, \Omega)$ so that

$(*)_n$ if $\lambda_n \leq \mu < \beta < \Omega$ and (z, μ), $(z, \beta) \in A$, then $|f(z, \mu) - f(z, \beta)| < \frac{1}{n}$.

Put $\lambda = \sup\{\lambda_n \,|\, n \in \mathbb{N}\}$. Then $\lambda \in [0, \Omega)$, and

$(*)$ if $\lambda \leq \mu < \beta < \Omega$ and (z, μ), $(z, \beta) \in A$, then $|f(z, \mu) - f(z, \beta)| = 0$.

By Lemma 3.7 the map f extends to a map of Y to $\mathbb{R}$. □

Corollary 3.9. *For every compact metrizable space Z, $Z \times [0, \Omega)$ is binormal.* □

In [5, Proposition 4.4], the authors prove that if X is a binormal pseudo-compact space, then X has the homotopy extension property with respect to CW-complexes. Applying Lemma 2.1(1) and Corollary 3.9, we see the following corollary.

Corollary 3.10. *Let Z be a compact metrizable space. Then $Z \times [0, \Omega)$ has the homotopy extension property with respect to* CW-*complexes.* □

Now we have our main theorem.

Theorem 3.11. *Let Z be a compact metrizable space and $Y = Z \times [0, \Omega)$.*

(1) *Then Y is pseudo-compact, Hausdorff, and binormal, and*
(2) *if K a* CW-*complex and $Z \tau K$, then both $Y \tau K$ and $\beta(Y) \tau K$.*

Proof: (1) Certainly Y is Hausdorff. Its pseudo-compactness follows from Lemma 3.4(4) and Lemma 2.1(1), while Corollary 3.9 yields its binormality.

(2) Let A be closed in Y and $f : A \to K$ be a map. Since Y is normal and A is closed, Lemma 2.1(2) yields that A is pseudo-compact. Applying Lemma 2.2, we find a finite subcomplex L of K such that $f(A) \subset L$. Since L is a finite dimensional metrizable compactum, we may assume that $L \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$. Let $1 \leq k \leq n$ and $\pi_k : \mathbb{R}^n \to \mathbb{R}$ be the k-coordinate projection. Applying Lemma 3.5, we find $\lambda_k \in [0, \Omega)$ such that if $\lambda_k \leq \mu < \beta < \Omega$, and (a, μ), $(a, \beta) \in A$, then $\pi_k \circ f(a, \mu) = \pi_k \circ f(a, \beta)$. Let $\lambda = \max\{\lambda_k \,|\, 1 \leq k \leq n\}$. Then $\lambda \in [0, \Omega)$. We deduce from the preceding that $(*)$ of Lemma 3.7 is in effect for this choice of λ. Therefore, f extends to a map of Y to K. The final statement follows from this and Proposition 2.4. □

It is well known that $\beta([0, \Omega)) \cong [0, \Omega]$. Here is our final result.

Corollary 3.12. *Let K be a nonempty* CW-*complex. Then both $[0, \Omega) \tau K$ and $[0, \Omega] \tau K$.*

Proof: Let $Z = \{0\}$, and apply Theorem 3.11(2). □

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