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I. V. Protasov and Anastasiia Tsvietkova

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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DECOMPOSITION OF CELLULAR BALLEANS

I. V. PROTASOV AND ANASTASIIA TSVIETKOVA

ABSTRACT. A ballean is a set endowed with some family of its subsets which are called balls. We postulate the properties of the family of balls in such a way that balleans can be considered as the asymptotic counterparts of uniform topological spaces. The isomorphisms in the category of balleans are called asymorphisms. Every metric space can be considered as a ballean. The ultrametric spaces are prototypes for cellular balleans. We prove some general theorem about decomposition of a homogeneous cellular ballean in a direct product of a pointed family of sets. Applying this theorem, we show that the balleans of two uncountable groups of the same regular cardinality are asymorphic.

A *ball structure* is a triple $\mathcal{B} = (X, P, B)$ where X and P are non-empty sets, and for all $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. The set X is called the *support* of $\mathcal{B}$ and P is called the *set of radii*.

Given any $x \in X, A \subseteq X, \alpha \in P$, we put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\},$$

$$B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha), \tag{1}$$

$$B^*(A, \alpha) = \bigcup_{a \in A} B^*(a, \alpha). \tag{2}$$

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A ball structure $\mathcal{B} = (X, P, B)$ is called a *ballean* (or a *coarse structure*) if

- $\forall \alpha, \beta \in P\ \exists \alpha', \beta' \in P$ such that $\forall x \in X$
$$B(x, \alpha) \subseteq B^*(x, \alpha'),\ B^*(x, \beta) \subseteq B(x, \beta');$$
- $\forall \alpha, \beta \in P\ \exists \gamma \in P$ such that $\forall x \in X$
$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma).$$

Let $\mathcal{B}_1 = (X_1, P_1, B_1)$ and $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans.

A mapping $f : X_1 \to X_2$ is called a $\prec$-*mapping* if for all $\alpha \in P_1$ there exists $\beta \in P_2$ such that

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta).$$

A bijection $f : X_1 \to X_2$ is called an *asymorphism* between $\mathcal{B}_1$ and $\mathcal{B}_2$ if f and f^{-1} are $\prec$-mappings. In this case $\mathcal{B}_1$ and $\mathcal{B}_2$ are called *asymorphic*.

If $X_1 = X_2$ and the identity mapping id $X_1 \to X_2$ is an asymorphism, we identify $\mathcal{B}_1$ and $\mathcal{B}_2$ and write $\mathcal{B}_1 = \mathcal{B}_2$.

For motivation to study balleans, see [1], [2], [3], and [4].

Every metric space (X, d) determines the *metric ballean* $\mathcal{B}(X, d) = (X, \mathbb{R}^+, B_d)$, where $\mathbb{R}^+$ is the set of non-negative real numbers,

$$B_d(x, r) = \{y \in X : d(x, y) \leq r\}.$$

A ballean $\mathcal{B}$ is called *metrizable* if $\mathcal{B}$ is asymorphic to $\mathcal{B}(X, d)$ for some metric ballean. By [3, Theorem 2.1], a ballean $\mathcal{B}$ is metrizable if and only if $\mathcal{B}$ is connected and the cofinality $\mathrm{cf}(\mathcal{B}) \leq \aleph_0$. A ballean $\mathcal{B} = (X, P, B)$ is *connected* if, for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, \alpha)$. To define $\mathrm{cf}(\mathcal{B})$, we use the natural preordering on P: $\alpha \leq \beta$ if and only if $B(x, \alpha) \subseteq B(x, \beta)$ for every $x \in X$. A subset P' is *cofinal* in P if, for every $\alpha \in P$, there exists $\alpha' \in P'$ such that $\alpha \leq \alpha'$, so $\mathrm{cf}(\mathcal{B})$ is the minimal cardinality of cofinal subsets of P.

Given an arbitrary ballean $\mathcal{B} = (X, P, B)$, $x, y \in X$, and $\alpha \in P$, we say that x, y are *α-path connected* if there exists a finite sequence $x_0, x_1, ..., x_n$, $x_0 = x$, $x_n = y$ such that $x_{i+1} \in B(x_i, \alpha)$, for every $i \in \{0, 1, ..., n-1\}$. For any $x \in X$ and $\alpha \in P$, we put

$$B^{\square}(x, \alpha) = \{y \in X : x, y \,\text{are}\, \alpha\text{-path connected}.$$

The ballean $\mathcal{B}^{\Box} = (X, P, B^{\Box})$ is called the *cellularization* of $\mathcal{B}$. A ballean $\mathcal{B}$ is called *cellular* if $\mathcal{B}^{\Box} = \mathcal{B}$. For characterizations of cellular balleans see [3, Ch. 3].

Example 1. A metric d on a set X is called an *ultrametric* if

$$d(x, y) \leq \max\{d(x, z), d(y, z)\}$$

for all $x, y, z \in X$. If (X,d) is an ultrametric space then the ballean $\mathcal{B}(X, d)$ is cellular. Moreover, by [3, Theorem 3.1], a ballean $\mathcal{B}$ is metrizable and cellular if and only if $\mathcal{B}$ is asymorphic to the metric ballean $\mathcal{B}(X, d)$ of some ultrametric space (X, d).

Example 2. Let G be an infinite group with the identity e, κ be an infinite cardinal such that $\kappa \leq |G|$, $\mathcal{F}(G, \kappa) = \{A \subseteq G : e \in A, |A| < \kappa\}$. Given any $g \in G$ and $A \in F(G, \kappa)$, we put $B(g, A) = gA$ and get the ballean $\mathcal{B}(G, \kappa) = (G, F(G, \kappa), B)$. In the case $\kappa = |G|$, we write $\mathcal{B}(G)$ instead of $\mathcal{B}(G, \kappa)$. A ballean $\mathcal{B}(G, \kappa)$ is cellular if and only if either $\kappa > \aleph_0$ or $\kappa = \aleph_0$ and G is locally finite (i.e., every finite subset of G is contained in some finite subgroup).

Example 3. A family of subsets of a group G is called a *Boolean group ideal* if

- $A, B \in \Im \Rightarrow A \cup B \in \Im$;
- $A \in \Im, A' \subset A \Rightarrow A' \in \Im$;
- $A, B \in \Im \Rightarrow AB \in \Im, A^{-1} \in \Im$;
- $F \in \Im$ for every finite subset F of G.

Every Boolean group ideal $\Im$ on G determines the ballean $\mathcal{B}(G, \Im) = (G, \Im, B)$, where $B(g, A) = gA$ for all $g \in G, A \in \Im$. The balleans on groups determined by the Boolean group ideals can be considered (see [3, Ch. 6]) as the asymptotic counterparts of the group topologies. A ballean $\mathcal{B}(G, \Im)$ is cellular if and only if $\Im$ has a base consisting of the subgroups of G.

A connected ballean $\mathcal{B} = (X, P, B)$ is called *ordinal* if there exists a cofinal well-ordered (by $\leq$) subset of P. Clearly, every metrizable ballean is ordinal.

Theorem 1. *Let $\mathcal{B} = (X, P, B)$ be an ordinal ballean. Then $\mathcal{B}$ is either metrizable or cellular.*

Proof: If $\mathrm{cf}(\mathcal{B}) \leq \aleph_0$, then $\mathcal{B}$ is metrizable by Theorem 2.1 from [3]. Assume that $\mathrm{cf}(\mathcal{B}) > \aleph_0$. Given an arbitrary $\alpha \in P$, we choose inductively a sequence $(\alpha_n)_{n\in\omega}$ in P such that $\alpha_0 = \alpha$ and $B(B(x, \alpha_n), \alpha) \subseteq B(x, \alpha_{n+1})$ for every $x \in X$. Since $\mathrm{cf}(\mathcal{B}) > \aleph_0$, we can pick $\beta \in P$ such that $\beta \geq \alpha_n$ for every $n \in \omega$. Then $B^{\square}(x, \alpha) \subseteq B(x, \beta)$ for every $x \in X$, so $\mathcal{B}^{\square} = \mathcal{B}$. □

Let γ be an ordinal, $\{Z_\lambda : \lambda < \gamma\}$ be a family of non-empty sets. For every $\lambda < \gamma$, we fix some element $e_\lambda \in Z_\lambda$ and say that the family $\{(Z_\lambda, e_\lambda) : \lambda < \gamma\}$ is *pointed*. A *direct product* $Z = \otimes_{\lambda<\gamma}(Z_\lambda, e_\lambda)$ is the set of all functions $f : \{\lambda : \lambda < \gamma\} \to \cup_{\lambda<\gamma} Z_\lambda$ such that $f(\lambda) \in Z_\lambda$ and $f(\lambda) = e_\lambda$ for all but finitely many $\lambda < \gamma$. We consider the ball structure $\mathcal{B}(Z) = (Z, \{\lambda : \lambda < \gamma\}, B)$, where $B(f, \lambda) = \{g \in Z : f(\lambda') = g(\lambda') \text{ for all } \lambda' \geq \lambda\}$ It is easy to verify that $\mathcal{B}(Z)$ is a cellular ballean.

We say that a ballean $\mathcal{B}$ is *decomposable in a direct product* if $\mathcal{B}$ is asymorphic to $\mathcal{B}(Z)$ for some direct product Z.

Theorem 2. *Let γ be a limit ordinal, $\mathcal{B} = (Z, \{\lambda : \lambda < \gamma\}, B)$ be a ballean such that*

(i) *$B^{\square}(x, \alpha) = B(x, \alpha)$ for all $x \in X, \alpha \in P$;*
(ii) *if $\alpha < \beta < \gamma$, then $B(x, \alpha) \subset B(x, \beta)$ for each $x \in X$;*
(iii) *if β is a limit ordinal and $\beta < \gamma$, then $B(x, \beta) = \cup_{\alpha<\beta} B(x, \alpha)$ for each $x \in X$;*
(iv) *there exists a cardinal κ_0 such that $B(x, 0) = \kappa_0$ for each $x \in X$;*
(v) *for every $\alpha < \gamma$ there exists a cardinal κ_α such that every ball of radius $\alpha + 1$ is a disjoint union of κ_α-many balls of radius α.*

Then $\mathcal{B}$ is decomposable in a direct product.

Proof: We fix some set Z_0 of cardinality κ_0 and define inductively a family of sets $\{Z_\alpha, \alpha < \gamma\}$. If α is a limit ordinal, we take Z_α to be a singleton. If $\alpha = \beta + 1$, we take a set Z_α of cardinality κ_β. For every $\alpha < \gamma$, we choose some element $e_\alpha \in Z_\alpha$, put $Z = \otimes_{\lambda<\gamma}(Z_\lambda, e_\lambda)$, and show that $\mathcal{B}$ is asymorphic to $\mathcal{B}(Z)$. To this end, we fix some element $x_0 \in X$ and, for every $\alpha < \gamma$, we define a mapping $f_\alpha : B(x_0, \alpha) \to \otimes_{\beta\leq\alpha}(Z_\beta, e_\beta)$ such that for all $\beta < \alpha < \gamma, f_\alpha |_{B(x_0,\beta)} = f_\beta$ and the inductive limit f of the family $\{f_\alpha :$

$\alpha < \gamma\}$ is an asymorphism between $\mathcal{B}$ and $\mathcal{B}(Z)$. Here we identify $\otimes_{\beta\le\alpha}(Z_\beta, e_\beta)$ with the corresponding subset of $\otimes_{\beta<\gamma}(Z_\beta, e_\beta)$.

At the first step we fix some bijection $f_0 : B(x_0, 0) \to Z_0$ such that $f_0(x_0) = e_0$. Let us assume that for some $\alpha < \gamma$, we have defined the mappings $\{f_\beta : \beta < \alpha\}$. If α is a limit ordinal, we put $f_\alpha : B(x_0, \alpha) \to \otimes_{\beta<\alpha}(Z_\beta, e_\beta)$ to be an inductive limit of the family $\{f_\beta : \beta < \alpha\}$. Since $Z_\alpha = \{e_\alpha\}$, we can identify $\otimes_{\beta<\alpha}(Z_\beta, e_\beta)$ with $\otimes_{\beta\le\alpha}(Z_\beta, e_\beta)$, so $f_\alpha : B(x_0, \alpha) \to \otimes_{\beta\le\alpha}(Z_\beta, e_\beta)$. If $\alpha = \beta + 1$, by cellularity of $\mathcal{B}$, there exists a subset $Y \subseteq B(x_0, \alpha), x_0 \in Y$ such that $B(x_0, \alpha)$ is a disjoint union of the family $\{B(y, \beta) : y \in Y\}$. For every $y \in Y$, we can repeat the inductive procedure of construction of $f_\alpha : B(x_0, \beta) \to \otimes_{\lambda\le\beta}(Z_\lambda, e_\lambda)$ to define a mapping $f'_{\beta,y}B(y, \beta) \to \otimes_{\lambda\le\beta}(Z_\lambda, e_\lambda)$. Thus, we fix some bijection $h : Y \to Z_\alpha, h(x_0) = e_\alpha$ and put $f_{\beta,y}(x) = (f'_{\beta,y}(x), h(y)), x \in B(y, \beta)$. At last, given any $x \in B(x_0, \alpha)$, we choose $y \in Y$ such that $x \in B(y, \beta)$ and put $f_\alpha(x) = f_{\beta,y}(x)$. By the construction of f as an inductive limit of the family $\{f_\alpha : \alpha < \gamma\}$, given any $x \in X$ and $\alpha < \gamma$, we have $f(B(x, \alpha)) = B(f(x), \alpha)$, so f is an asymorphism. □

In the next two corollaries and Theorem 3, $\mathcal{B}(G)$ is a ballean defined in Example 2.

Corollary 1. *Let G be a countable locally finite group. Then $\mathcal{B}(G)$ is decomposable in a direct product of finite sets.*

Proof: We write G as a union $G = \cup_{n<\omega} G_n$ of an increasing chain of finite groups. Clearly, $\mathcal{B}(G)$ is asymorphic to the ballean $\mathcal{B} = (G, \omega, B)$ where $B(g, n) = gG_n$. We put $\kappa_0 = |G_0|$, $\kappa_{n+1} = |G_{n+1} : G_n|$ and apply Theorem 2. □

Corollary 2. *Let G be an uncountable group of regular cardinality γ. Then $\mathcal{B}(G)$ is decomposable in a direct product.*

Proof: We write G as a union $G = \cup_{\alpha<\gamma} G_\alpha$ of an increasing chain of subgroups such that $|G_0| = \aleph_0$, $|G_\alpha| < \gamma$ and $G_\alpha = \cup_{\beta<\alpha} G_\beta$ for every limit ordinal α. Since γ is regular, every subset $F \subset G$, $|F| < |G|$ is contained in some subgroup G_α. It follows that $\mathcal{B}(G)$ is asymorphic to the ballean $\mathcal{B} = (G, \gamma, B)$, where $B(g, \alpha) = gG_\alpha$. Apply Theorem 2. □

Theorem 3. *Let G and H be two uncountable groups of the same regular cardinality γ. Then $\mathcal{B}(G)$ and $\mathcal{B}(H)$ are asymorphic.*

Proof: We consider two cases.

Case 1: γ is a limit cardinal. We choose an increasing family $\{G_\alpha : \alpha < \gamma\}$ of subgroups of G such that $G = \cup_{\alpha<\gamma} G_\alpha$, $|G_0| = \aleph_0$, $|G_{\alpha+1}| = |G_\alpha|^+$, and $G_\beta = \cup_{\alpha<\beta} G_\alpha$ for every limit ordinal β. Put $\kappa_\alpha = |G_\alpha|^+, \alpha < \gamma$. By Theorem 2, $\mathcal{B}(G)$ is asymorphic to $\mathcal{B}(Z)$ where the direct product Z is defined by the family of cardinals $\{\kappa_\alpha : \alpha < \gamma\}$. Since H admits a filtration $H = \cup_{\alpha<\gamma} H_\alpha$ with the same family $\{\kappa_\alpha : \alpha < \gamma\}$ of parameters, $\mathcal{B}(H)$ is also asymorphic to $\mathcal{B}(Z)$.

Case 2: $\gamma = \lambda^+$ for some cardinal λ. We write G as a union $G = \cup_{\alpha<\gamma} G_\alpha$ of an increasing family of subgroups such that $|G_\alpha| = \lambda$, $|G_{\alpha+1} : G_\alpha| = \lambda$ for every $\alpha < \gamma$, and $G_\beta = \cup_{\alpha<\beta} G_\alpha$ for every limit ordinal β. Put $\kappa_\alpha = \lambda$ for every $\alpha < \gamma$. By Theorem 2, $\mathcal{B}(G)$ is asymorphic to $\mathcal{B}(Z)$, where Z is defined by the family of parameters $\{\kappa_\alpha : \alpha < \gamma\}$. Since H admits a filtration with the same family of parameters, $\mathcal{B}(H)$ is also asymorphic to $\mathcal{B}(Z)$.

This completes the proof. □

It should be mentioned that Theorem 3 does not hold for countable groups. By [2, Theorem 10.6], there exists a family $\mathcal{F}$ of countable locally finite groups such that any two groups from $\mathcal{F}$ are non-asymorphic and $|\mathcal{F}| = 2^{\aleph_0}$.

We do not know if Corollary 2 and Theorem 3 are true for groups of singular cardinalities.

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(Protasov) Department of Cybernetics; Kyiv National University; Volodimirska 64; Kiev 01033, Ukraine
E-mail address: protasov@unicyb.kiev.ua

(Tsvietkova) Department of Mathematics; 121 Ayres Hall; University of Tennessee; Knoxville, Tennessee 37996-1300, USA
E-mail address: tsvietkova@math.utk.edu