TOPOLOGY PROCEEDINGS
Volume 36, 2010
Pages 123–129
http://topology.auburn.edu/tp/

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Electronically published on February 12, 2010

Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

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Volume 36 (2010)
Pages 123-129

E-Published on February 12, 2010

CARDINALITY BOUNDS OF H-SETS IN URYSOHN SPACES

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Abstract. A Urysohn space X is constructed which has an H-set A with $|A| > 2^{\bar{\psi}(X)}$, where $\bar{\psi}(X)$ is the closed-pseudocharacter of the space X. The space provides a counterexample to Alessandro Fedeli's question in ωH-*sets and cardinal invariants* [Comment. Math. Univ. Carolin. **39** (1998), no. 2, 367–370]. In addition, it is demonstrated that there is no θ-continuous map from a compact Hausdorff space to the space X with the H-set A as the image, giving a Urysohn counterexample to Johannes Vermeer's conjecture in *Closed subspaces of H-closed spaces* [Pacific J. Math. **118** (1985), no. 1, 229–247]. Finally, it is shown that the cardinality of an H-set in a Urysohn space X is bounded by $2^{\chi(X_s)}$, where $\chi(X)$ is the character of X and X_s is the semiregularization of X. This refines Angelo Bella's result in *A couple of questions concerning cardinal invariants* [Questions Answers Gen. Topology **14** (1996), no. 2, 139–143] that the cardinality of such an H-set is bounded by $2^{\chi(X)}$.

Introduction

Recall that a space X is H-closed if for every family of open sets covering X there is a finite subfamily whose union is dense in X. More generally, an H-set of a space X is a subset A for which every family of open sets of X covering A has a finite subfamily whose

2010 *Mathematics Subject Classification.* 54A25, 54D35.
Key words and phrases. cardinal functions, H-set.

union is dense in A. Also recall the definitions of the pseudocharacter, closed-pseudocharacter, and character of a space. We will use the notation $\psi(X)$, $\bar{\psi}(X)$, and $\chi(X)$, respectively, for these three quantities. Finally also note that the semiregularization of a space X, denoted X_s, is the set X with the topology generated by the regular open sets of X, $RO(X) = \{\text{int cl } U : U \text{ is open in } X\}$.

It is well known for a given space X that $\psi(X) \leq \bar{\psi}(X) \leq \chi(X)$ (see [7]), in fact $\psi(X) \leq \bar{\psi}(X_s) = \bar{\psi}(X) \leq \chi(X_s) \leq \chi(X)$. In particular, it follows that $2^{\psi(X)} \leq 2^{\bar{\psi}(X_s)} = 2^{\bar{\psi}(X)} \leq 2^{\chi(X_s)} \leq 2^{\chi(X)}$. If a statement P is true for all H-closed spaces, a natural question to ask is whether P also holds for H-sets, or perhaps for H-sets embedded in a space with a particular separation property. For example, Alan Dow and Jack R. Porter show in [5] that if X is an H-closed space, then $|X| \leq 2^{\chi(X)}$ (in fact, they show $|X| \leq 2^{\bar{\psi}(X)}$), and later, in [2], Angelo Bella shows that if X is Urysohn and A is an H-set of X, then $|A| \leq 2^{\chi(X)}$. On the other hand, Bella and I. V. Yaschenko show in [4] that an H-set A of a Hausdorff space X may have cardinality larger than $2^{\chi(X)}$. Similar results are those of Bella and Porter in [3] showing if X is H-closed, then $|X|$ may be larger than $2^{\psi(X)}$; in fact, one example of this has $X = \kappa\omega$, a Urysohn space. Here a space X is constructed demonstrating that the cardinality of an H-set of a Urysohn space is not always bounded by $2^{\bar{\psi}(X)}$. It is also proven that the cardinality of an H-set is bounded by $2^{\chi(X_s)}$. These two results constrain the maximum cardinality of an H-set of a Urysohn space as much as is possible with the inequalities listed above.

In [11], J. Vermeer conjectures that a subset A of X is an H-set if and only if there is a compact Hausdorff space K and a θ-continuous map $f : K \to X$ with $f[K] = A$. Bella and Yaschenko [4] provide a counterexample with countable character with their construction. A counterexample to Vermeer's conjecture, similar in construction to Bella and Yaschenko's, but which is Urysohn and has countable closed-pseudocharacter, is provided in this paper.

Bella and Yaschenko in [4] reintroduce, under the name of *relatively H-closed*, a concept first considered by P. Th. Lambrinos in [8] under the name of *H-bounded* and investigated further by Douglas D. Mooney in [9]. Given a space X, a subspace A is called relatively H-closed if for every open cover $\mathcal{U}$ of X there is a finite subfamily of $\mathcal{U}$ whose union is dense in A. The following notation is introduced for the purpose of this paper.

Notation 1. Let $X \subseteq Y \subseteq Z$ be spaces. We write $H(X;Y;Z)$ if every cover $\mathcal{U}$ of Y with open sets of Z has a finite subfamily $\mathcal{F} \subseteq \mathcal{U}$ for which $X \subseteq \bigcup_{U \in \mathcal{F}} \mathrm{cl}_Z U$.

When considering the space κX for a given space X, the author will use the notation $o(U) = U \cup \{p \in \kappa X \setminus X : U \in p\}$. The following lemma will also prove useful (see [10]).

Lemma 2. *Let X be a space. If $\mathcal{B}$ is an open neighborhood base of $x \in X$, then $\bigcap\{o(B) : B \in \mathcal{B}\} = \{x\}$.*

Finally given a space X, we say an open filter $\mathcal{G} \subseteq \tau(X)$ has the *weak countable intersection property* if for every countable subset $\mathcal{G}'$ of $\mathcal{G}$ we have $\bigcap\{\mathrm{cl}_X U : U \in \mathcal{G}'\} \neq \varnothing$. We then call a space X *weakly realcompact* if every open ultrafilter $\mathcal{U} \subseteq \tau(X)$ with the weak countable intersection property has nonempty adherence.

1. Constructions

The first construction is a basic space from which the counterexample to Alessandro Fedeli's question (see [6]) will be built. We modify a construction given by Bella and Yaschenko in [4].

Construction 3. Let X be a weakly realcompact space with countable closed-pseudocharacter and κX Urysohn, e.g., $X = \omega$, and let $\widehat{X} = \kappa X \setminus X$. Define

$$Z = X \cup (X \times \omega \times \widehat{X}) \cup \widehat{X}$$

with the following topology. If $U \in \tau(X)$ and $n \in \omega$, then

$$U(n) = U \cup (U \times [n, \omega) \times \widehat{X}) \in \tau(Z)$$

is a basic open neighborhood of $x \in X$, and if $p \in \widehat{X} = \kappa X \setminus X$ and $U \in p$, then a basic open neighborhood of p is

$$U(p) = (U \times \omega \times \{p\}) \cup \{p\} \in \tau(Z).$$

Finally, the points of $X \times \omega \times \widehat{X}$ are isolated.

Fact 4. (1) *Z is Urysohn.*

(2) *X is relatively H-closed in Z, in other words $H(X;Z;Z)$.*

(3) *$H(X; X \cup \widehat{X}; Z)$.*

(4) *Z has countable closed-pseudocharacter.*

Proof: (1) If $x, y \in X$, then there exist U and V open neighborhoods, in X, of x and y, respectively, with $\mathrm{cl}_X U \cap \mathrm{cl}_X V = \varnothing$. Hence, $\mathrm{cl}_Z U(0) \cap \mathrm{cl}_Z V(0) = \varnothing$, and $U(0)$, and $V(0)$ are neighborhoods of x and y in Z.

If $p, q \in \widehat{X}$, then there exist $U \in p$ and $V \in q$ such that $\mathrm{cl}_X U \cap \mathrm{cl}_X V = \varnothing$. Thus, $p \in U(p)$, $q \in V(q)$, and $\mathrm{cl}_Z U(p) \cap \mathrm{cl}_Z V(q) = \varnothing$.

If $x \in X$ and $p \in \widehat{X}$, then there exist U an open neighborhood of x in X and $V \in p$ with $\mathrm{cl}_X U \cap \mathrm{cl}_X V = \varnothing$. Therefore, $x \in U(0)$, $p \in V(p)$, and $\mathrm{cl}_Z U(0) \cap \mathrm{cl}_Z V(p) = \varnothing$.

Finally, the points of $X \times \omega \times \widehat{X}$ are isolated.

(2) Let $\mathcal{C}$ be an open cover of Z. Without loss of generality, for each $x \in X$ we can assume there exists an $U_x \in \tau(X)$ and an $n_x \in \omega$ with $x \in U_x(n_x) \in \mathcal{C}$; also we can assume that for each $p \in \widehat{X}$ there exists a $V_p \in p$ with $V_p(p) \in \mathcal{C}$. Now, $\{U_x : x \in X\} \cup \{V_p \cup \{p\} : p \in \widehat{X}\}$ is an open cover of κX. Since κX is H-closed, there exist finitely many $x_1, x_2, \ldots, x_n$ and $p_1, \ldots, p_m$ such that $\kappa X = \bigcup_{i=1}^n \mathrm{cl}_{\kappa X} U_{x_i} \cup \bigcup_{i=1}^m \mathrm{cl}_{\kappa X} V_{p_i}$. Hence, $X \subseteq \bigcup_{i=1}^n \mathrm{cl}_X U_{x_i} \cup \bigcup_{i=1}^m \mathrm{cl}_X V_{p_i} \subseteq \bigcup_{i=1}^n \mathrm{cl}_Z U_{x_i}(n_{x_i}) \cup \bigcup_{i=1}^m \mathrm{cl}_Z V_{p_i}(p_i)$.

(3) Take a cover $\mathcal{U}$ of $X \cup \widehat{X}$ with open sets of Z. Now extend $\mathcal{U}$ to an open cover, $\mathcal{U}'$ of all of Z, by adding in the isolated singletons not already covered. Then there is a finite subfamily $\mathcal{V} \subset \mathcal{U}'$ with $X \subseteq \mathrm{cl}_Z \bigcup \mathcal{V}$. However, it is clear that $\mathcal{V}$ need not contain any of the isolated singletons added to extend $\mathcal{U}$. Hence, we may take $\mathcal{V} \subset \mathcal{U}$.

(4) We must show every point in Z is the intersection of a countable collection of closed neighborhoods. This is certainly true for the isolated points of $X \times \omega \times \widehat{X}$. If, on the other hand, $x \in X \subset Z$, take $\{U_n : n \in \omega\} \subseteq \tau(X)$ with $\bigcap_\omega \mathrm{cl}_X U_n = \{x\}$. Then $\bigcap_\omega \mathrm{cl}_Z U_n(n) = \bigcap_\omega (\mathrm{cl}_X U_n \cup (U_n \times [n, \omega) \times \widehat{X}) \cup o(U_n)) = \{x\}$. Finally, for points of $\widehat{X} \subseteq Z$, consider the following: let $p \in \widehat{X}$; then p is a free open ultrafilter on X and $\bigcap_{U \in p} \mathrm{cl}_X U = \varnothing$. But X is also weakly real-compact; hence, there is a countable family $\mathcal{C} \subseteq p$ with $\bigcap_{U \in \mathcal{C}} \mathrm{cl}_X U = \varnothing$. Considering the family $\mathcal{C}' = \{U(p) : U \in \mathcal{C}\}$, we have $\bigcap_{\mathcal{C}'} \mathrm{cl}_Z U(p) = \bigcap_{\mathcal{C}} \mathrm{cl}_X U \cup (\bigcap_{\mathcal{C}} U \times \omega \times \{p\}) \cup \{p\} = \{p\}$. Hence, $\{p\}$ is the intersection of a countable collection of closed neighborhoods and Z has countable closed-pseudocharacter. $\square$

Now we construct a space, again modifying a construction of Bella and Yaschenko in [4], which is Urysohn, has countable closed-pseudocharacter, and has a large H-set.

Theorem 5. *There is a space Z with the following properties:*

(1) *Z is Urysohn;*
(2) *Z has countable closed-pseudocharacter;*
(3) *Z has an H-set H of cardinality greater than 2^ω.*

Construction 6. Let $\{X_n : n \in \omega\}$ be a sequence of spaces defined inductively as follows: $X_0 = \omega$ and $X_{n+1} = \widehat{X}_n = \kappa X_n \setminus X_n$. For each $n \in \omega$, let $Z_n = X_n \cup (X_n \times \omega \times \widehat{X}_n) \cup \widehat{X}_n$. Finally, let Z_ω be the quotient space formed from $\bigcup_\omega Z_n$ by identifying $\widehat{X}_{n-1}$ with X_n, and let $Z = Z_\omega \cup \{\infty\}$. A basic neighborhood of ∞ in Z will be $\{\infty\} \cup \bigcup_{i \in \omega \setminus n} \{X_i \times \omega \times \widehat{X}_i : n \in \omega\}$.

Fact 7. (1) *The space Z is Urysohn.*

(2) *Z has countable closed-pseudocharacter.*
(3) *The set $H = \bigcup_\omega X_n \cup \{\infty\}$ is an H-set of Z.*
(4) *The cardinality of H is larger than $2^{\bar{\psi}(Z)} = \mathfrak{c}$.*

Proof: (1) That the points which are isolated can be separated from the other points of Z with closed neighborhoods is clear. Now for $x, y \in H$, if $x, y \in Z_n$ for some $n \in \omega$, then, by Fact 4(1), the two points can be separated by closed neighborhoods. If, on the other hand, $x \in Z_n$ and $y \in Z_m$ where $n \neq m$, then we may assume further that $x \in X_n$, $y \in X_m$, and $n < n+2 \leq m$. Now let U_x be a basic open set of x and let U_y be a basic open set of y. Then

$$\mathrm{cl}_Z U_x \subseteq X_{n-1} \cup (X_{n-1} \times \omega \times X_n) \cup X_n \cup (X_n \times \omega \times X_{n+1})$$

while

$$\mathrm{cl}_Z U_y \subseteq X_{m-1} \cup (X_{m-1} \times \omega \times X_m) \cup X_m \cup (X_m \times \omega \times X_{m+1}).$$

Hence, $\mathrm{cl}_Z U_x \cap \mathrm{cl}_Z U_y = \varnothing$.

Finally, if $x = \infty$ and $y \in Z_n$ for some n, we may simply take the basic open neighborhood $\{\infty\} \cup \bigcup_{i=n+2}^{\infty} (X_i \times \omega \times \widehat{X}_i)$ for $x = \infty$ and a typical neighborhood for y.

(2) This is clear for each point in Z_n for some n. For ∞, we let $\mathcal{U}$ be the neighborhood base $\{\{\infty\} \cup \bigcup_{i \in \omega \setminus n} (X_i \times \omega \times \widehat{X}_i) : n \in \omega\}$. Then $\bigcap_{U \in \mathcal{U}} \mathrm{cl}_Z U = \{\infty\}$.

(3) Let $\mathcal{U}$ be a cover of H with basic open sets of Z. There is some $U \in \mathcal{U}$ for which $\infty \in U$. Then for some $m \in \omega$, $\mathrm{cl}_Z U$ contains X_i for all $i \geq m$. The cover $\mathcal{U}$ contains, for each $n < m$, a subfamily $\mathcal{U}_n$ which covers $X_n \cup X_{n+1}$. Now, as in Fact 4(3) above, we have $H(X_n; X_n \cup X_{n+1}; Z_n)$. Hence, we obtain a finite subfamily $\mathcal{F}_n$ of $\mathcal{U}_n$, therefore of $\mathcal{U}$, for which $X_n \subseteq \bigcup_{U \in \mathcal{F}_n} \mathrm{cl}_Z U$. The collection $\{U\} \cup \bigcup\{\mathcal{F}_n : n < m\}$ is a finite family whose closure contains H.

(4) This is clear since $|X_1| = |\kappa\omega| = 2^{\mathfrak{c}} \geq \mathfrak{c}$. □

Theorem 8. *Let X be a space, $\kappa = \bar{\psi}(X)$, A an H-set of X, K a compact Hausdorff space, and $f : K \to X$ a θ-continuous function with $f[K] = A$; then $|A| \leq 2^\kappa$.*

Proof: The proof follows the outline of the comments after Theorem 5 in [4]. Let $x \in A$ and take a family of open neighborhoods of x, $\{U_\alpha : \alpha \leq \kappa\}$, such that $\bigcap_\kappa \mathrm{cl}_X U_\alpha = \{x\}$. For every $p \in f^{\leftarrow}(x)$, fix an open neighborhood $W_{\alpha,p}$ satisfying $f[\mathrm{cl}_K W_{\alpha,p}] \subseteq \mathrm{cl}_X U_\alpha$ and let $W_\alpha = \bigcup\{W_{\alpha,p} : p \in f^{\leftarrow}(x)\}$. Then $f[W_\alpha] \subseteq \mathrm{cl}_X U_\alpha$ and hence, $f^{\leftarrow}(x) \subseteq \bigcap_\kappa W_\alpha \subseteq f^{\leftarrow}[\bigcap_\kappa \mathrm{cl}_X U_\alpha] = f^{\leftarrow}(x)$. Now Theorem 4 of [1] implies the family $\{f^{\leftarrow}(x) : x \in A\}$ must have cardinality not more than 2^κ, and hence $|A| \leq 2^\kappa$. □

Corollary 9. *There is no compact Hausdorff space K and θ-continuous function $f : K \to Z$ with $f[K] = H$ for the space Z and embedded H-set H in Construction 6 above.*

What is most notable about the above corollary is that the space Z is a Urysohn counterexample to Vermeer's conjecture. If the base space X_0 in the construction above is discrete, a few other properties of the space Z are noteworthy. In this case, Z is a simple Urysohn extension of the discrete space $\bigcup_\omega (X_i \times \omega \times \widehat{X}_i) = Z \setminus H$. One may also say that Z is the disjoint union of discrete spaces, namely $Z = H \cup (Z \setminus H)$.

Theorem 10. *If A is an H-set of a Urysohn space X, then $|A| \leq 2^{\chi(X_s)}$.*

Proof: Let A be an H-set in a Urysohn space X. In [2], Bella shows $|A| \leq 2^{\chi(X)}$. Now $A \subset X_s$ is also an H-set of X_s, and X_s is Urysohn. Hence, $|A| \leq 2^{\chi(X_s)}$. □

Acknowledgments. The author wishes to express thanks to Dr. Jack Porter for his encouragement and guidance during the preparation of this paper.

The useful suggestions of the referee are appreciated.

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