TOPOLOGY PROCEEDINGS

Volume 36, 2010

Pages 141–143

http://topology.auburn.edu/tp/

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Electronically published on February 19, 2010

Topology Proceedings

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E-mail: topolog@auburn.edu
ISSN: 0146-4124

http://topology.auburn.edu/tp/
TOPOLOGY PROCEEDINGS
Volume 36 (2010)
Pages 141-143
E-Published on February 19, 2010

A NOTE ON COMPLETELY METRIZABLE SPACES OF CONTINUOUS INJECTIONS

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ABSTRACT. In this paper, we present a stronger version of R. A. McCoy's theorem given in *Completely metrizable spaces of embeddings* [Proc. Amer. Math. Soc. **84** (1982), no. 3, 437–442] on complete metrizability of the space $I(X,Y)$, of continuous injections from a Tychonoff space X to a complete metric space Y equipped with the compact-open topology.

The space $C_k(X,Y)$ of all continuous functions from a Tychonoff space X to a complete metric space (Y,d) equipped with the compact-open topology is completely metrizable if X is a hemicompact $k_{\mathbb{R}}$-space. For each $f \in C(X,Y)$, compact set A in X, and $\epsilon > 0$, define

$$\langle f, A, \epsilon \rangle = \{g \in C(X,Y) : d(f(x), g(x)) < \epsilon \text{ for all } x \in A\}.$$

Then for each $f \in C_k(X,Y)$, the collection $\{\langle f, A, \epsilon \rangle : A$ is a compact set in $X, \epsilon > 0\}$ forms a neighborhood base at f in $C_k(X,Y)$. Since complete metrizability is not a hereditary property, some natural subspaces of $C_k(X,Y)$ may not be completely metrizable even if $C_k(X,Y)$ is completely metrizable. In the theorem given in [2], a sufficient condition has been given on X for the subspace $I(X,Y)$ of all injections in $C_k(X,Y)$ to be completely metrizable. This theorem says that if X is a hemicompact metric space and Y is a complete metric space, then $I(X,Y)$ is completely metrizable. In

2010 *Mathematics Subject Classification.* Primary 54C35; Secondary 54E50.
Key words and phrases. completely metrizable space, function space, $k_{\mathbb{R}}$-space, submetrizable.

this paper, we show that the condition of metrizability on X in R. A. McCoy's theorem can be removed. The idea of the proof lies in the fact that, if $I(X,Y)$ is non-empty, then X is submetrizable. *A space X is said to be submetrizable if it has a weaker metrizable topology.* If $I(X,Y)$ is nonempty, then there exists a continuous injection from X to the complete metric space Y. This existence shows that X is submetrizable. Also it can be easily shown that every submetrizable space is a $k_{\mathbb{R}}$-space.

Theorem. Suppose X is a hemicompact space and Y is completely metrizable, then $I(X,Y)$ is completely metrizable.

Proof: Without loss of generality, we may assume that $I(X,Y) \neq \emptyset$. So there exists a one-to-one continuous function $f : X \longrightarrow Y$. So X is submetrizable. Consequently, X is a hemicompact $k_{\mathbb{R}}$-space. Hence, $C_k(X,Y)$ is completely metrizable.

Since X is hemicompact, then there exists a sequence $\{A_n : n \in \mathbb{N}\}$ of compact subsets of X such that $X = \cup A_n$ and whenever A is compact in X, $A \subseteq A_n$ for some n.

Since each A_n is compact submetrizable, each A_n is separable. Hence, X is a countable union of separable spaces. Consequently, X is also separable.

Let $f \in I(X,Y)$. Then $f : X \longrightarrow f(X)$ is a continuous bijection. Since X is separable, $f(X)$ is separable. So X has a separable metrizable compression. Hence, $C_k(X)$ is separable. (Here $C_k(X) = C_k(X,\mathbb{R})$, where $\mathbb{R}$ is the real line with the usual topology.) Let $D = \{\phi_n : n \in \mathbb{N}\}$ be a dense subset of $C_k(X)$. Then D separates the points of X.

For each compact set A in X, $\phi \in C_k(X)$, and $\epsilon > 0$, define

$$[\phi, A, \epsilon] = \{f \in C(X,Y) : |\phi(x) - \psi f(x)| < \epsilon \ \forall\ x \in A, \text{ for some continuous function } \psi : Y \longrightarrow \mathbb{R}\}$$

Claim 1. Each $[\phi, A, \epsilon]$ is open in $C_k(X,Y)$.

Let $f \in [\phi, A, \epsilon]$. Then there exists $\psi \in C(Y)$ such that $|\phi(x) - \psi f(x)| < \epsilon$ for all $x \in A$. Since A is compact and $\phi - \psi f$ is continuous, there exists $\epsilon_1 > 0$ such that $|\phi(x) - \psi f(x)| < \epsilon_1 < \epsilon\ \forall\ x \in A$. Choose $\epsilon_2 > 0$ such that $\epsilon_1 + \epsilon_2 < \epsilon$. Since $f(A)$ is compact, ψ is uniformly continuous on $f(A)$. So for $\epsilon_2 > 0$, there exists $\delta > 0$ such that for every $y_1, y_2 \in f(A)$, $|\psi(y_1) - \psi(y_2)| < \epsilon_2$ whenever $d(y_1, y_2) < \delta$.

Now we show that $\langle f, A, \delta\rangle \subseteq [\phi, A, \epsilon]$. Let $g \in \langle f, A, \delta\rangle$, then $d(f(x), g(x)) < \delta$ for all $x \in A$. So $|\psi f(x) - \psi g(x)| < \epsilon_2$ for all $x \in A$. Let $x \in A$ be arbitrary. Then $|\phi(x) - \psi g(x)| = |\phi(x) - \psi f(x) + \psi f(x) - \psi g(x)| \leq |\phi(x) - \psi f(x)| + |\psi f(x) - \psi g(x)| < \epsilon_1 + \epsilon_2 < \epsilon$. So $g \in [\phi, A, \epsilon]$.

CLAIM 2. $I(X,Y) = \cap\{[\phi_n, A_m, \frac{1}{p}] : m, n, p \in \mathbb{N}\}$.

Suppose $f \in \cap\{[\phi_n, A_m, \frac{1}{p}] : m, n, p \in \mathbb{N}\}$. Let $x, y \in X$ such that $x \neq y$. Since D separates the points of X, there exists $n \in \mathbb{N}$ such that $\phi_n(x) \neq \phi_n(y)$. Choose $p \in \mathbb{N}$ such that $\frac{2}{p} < |\phi_n(x) - \phi_n(y)|$. Choose $m \in \mathbb{N}$ such that $\{x, y\} \subseteq A_m$. Since $f \in [\phi_n, A_m, \frac{1}{p}]$, there exists a continuous function $\psi : Y \longrightarrow \mathbb{R}$ such that $|\phi_n(x) - \psi f(x)| < \frac{1}{p}\ \forall\ x \in A_m$. If possible, let $f(x) = f(y)$. Then $\frac{2}{p} < |\phi_n(x) - \phi_n(y)| = |\phi_n(x) - \psi f(x) + \psi f(y) - \phi_n(y)| \leq |\phi_n(x) - \psi f(x)| + |\psi f(y) - \phi_n(y)| < \frac{1}{p} + \frac{1}{p} = \frac{2}{p}$. Consequently, $f(x) \neq f(y)$. Hence, $f \in I(X,Y)$.

Conversely, let $f \in I(X,Y)$. Then $f : A_m \longrightarrow f(A_m)$ is a homeomorphism. Thus, $\phi_n f^{-1} : f(A_m) \longrightarrow \mathbb{R}$ is a continuous real-valued function on $f(A_m)$. So there exists a continuous extension $\psi : Y \longrightarrow \mathbb{R}$ of $\phi_n f^{-1}$. Then $|\phi_n(x) - \psi f(x)| = 0 < \frac{1}{p}$ for all $x \in A_m$. So $f \in [\phi_n, A_m, \frac{1}{p}]$ for all $m, n, p \in \mathbb{N}$. Consequently, $I(X,Y) = \cap\{[\phi_n, A_m, \frac{1}{p}] : m, n, p \in \mathbb{N}\}$.

Since $I(X,Y)$ is a G_δ-subset of the completely metrizable space $C_k(X,Y)$, $I(X,Y)$ is completely metrizable. □

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