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ON TOPOLOGICAL HOMOTOPY GROUPS OF n-HAWAIIAN LIKE SPACES

F. H. GHANE, Z. HAMED, B. MASHAYEKHY, AND H. MIREBRAHIMI

Abstract. By an n-Hawaiian like space X, we mean the natural inverse limit, $\varprojlim(Y_i^{(n)}, y_i^*)$, where

$$(Y_i^{(n)}, y_i^*) = \bigvee_{j \le i} (X_j^{(n)}, x_j^*)$$

is the wedge of $X_j^{(n)}$'s in which $X_j^{(n)}$'s are $(n-1)$-connected, locally $(n-1)$-connected, n-semilocally simply connected, and compact CW spaces. First, we show that the natural homomorphism $\beta_n : \pi_n(X, *) \to \varprojlim \pi_n(Y_i^{(n)}, y_i^*)$ is a bijection. Second, using this fact, we prove that the topological n-homotopy group of an n-Hawaiian like space $\pi_n^{top}(X, x^*)$ is a topological group for all $n \ge 2$ which is a partial answer to the open question whether $\pi_n^{top}(X, x^*)$ is a topological group for any space X and $n \ge 1$. Moreover, we show that $\pi_n^{top}(X, x^*)$ is metrizable.

1. Introduction

In 2002, a work of Daniel K. Biss [1] initiated the development of a theory in which the familiar fundamental group $\pi_1(X, x^*)$ of a topological space X becomes a topological space denoted by $\pi_1^{top}(X, x^*)$ by endowing it with the quotient topology inherited from the path components of based loops in X with the compact-open topology. An important feature of the theory is that if X and

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Y have the same homotopy type, then $\pi_1^{top}(X,x^*)$ and $\pi_1^{top}(Y,y^*)$ are homeomorphic. Among other things, Biss claimed that $\pi_1^{top}(X,x^*)$ is a topological group and π_1^{top} is a functor from the category of based spaces to the category of topological groups. However, there is a gap in the proof of Proposition 3.1 in [1]. For more details, see [5] and [2].

In [7], the authors extended the above theory to higher homotopy groups by introducing a topology on the n-th homotopy group of a pointed space (X,x^*) as a quotient of the n-loop space $\Omega^n(X,x^*)$ equipped with the compact-open topology. Call this space the topological homotopy group and denote it by $\pi_n^{top}(X,x^*)$. The misstep in the proof is repeated by the authors to prove that $\pi_n(X,x^*)$ is a topological group [7, Theorem 2.1]; see also [2]. Hence, there is a question whether or not $\pi_n^{top}(X,x^*)$, $n\geq 1$, is a topological group.

Note that if X is locally contractible, then $\pi_1^{top}(X,x^*)$ inherits the discrete topology [5], and thus there is no information other than algebraic data. The same thing happens in the case of higher homotopy groups when X is a locally n-connected metric space; see [7, Theorem 3.6]. So spaces that are not locally n-connected, $n\geq 1$, are interesting. One of the simplest nonlocally n-connected spaces is the n-dimensional Hawaiian earring $\mathcal{H}_n$, $n\geq 1$.

John W. Morgan and Ian Morrison [9], presenting a van Kampen theorem for 1-Hawaiian like spaces, proved that the natural homomorphism

$$\beta:\pi_1(\mathcal{H}_1,*)\to\varprojlim\pi_1(Y_i^{(1)},y_i^*)$$

is injective, where $Y_i^{(1)}=\bigvee_{j\leq i}S_j^1$ is the wedge of 1-spheres S_j^1 of radius $\frac{1}{j}$ and center $(\frac{1}{j},0)$ and hence $\mathcal{H}_1=\varprojlim Y_i^{(1)}$ with respect to the natural inverse system for the $Y_i^{(1)}$.

Now consider the n-dimensional Hawaiian earring $\mathcal{H}_n$, $n\geq 2$, which is the union of a sequence of n-spheres S_j^n of radius $\frac{1}{j}$ identified at a common point $*$ as a subspace of $\mathbb{R}^{n+1}$. This paper aims to explore $\mathcal{H}_n$ in the context of inverse limit space, i.e., $\mathcal{H}_n=\varprojlim Y_i^{(n)}$, where $\mathcal{H}_n$ can be approximated by factors $Y_i^{(n)}=\bigvee_{j\leq i}S_j^n$.

As in [9], we consider a natural homomorphism

$$\beta_n:\pi_n(\mathcal{H}_n,*)\to\varprojlim\pi_n(Y_i^{(n)},y_i^*)$$

as follows: Let $R_j : \mathcal{H}_n \to Y_j^{(n)}$ denote the projection fixing $Y_j^{(n)}$ pointwise and collapsing $\bigcup_{i=j+1}^{\infty} Y_i^{(n)}$ to the based point $*$. The formula $\beta_n([f]) = ([R_1(f)], [R_2(f)], ...)$ determines the induced homomorphism β_n into the inverse limit. We intend to show that β_n is a bijection, for all $n \geq 2$. Then we present two natural ways of imparting a topology on $\pi_n(\mathcal{H}_n, *)$, for $n \geq 2$:

(1) Since β_n is an isomorphism, one can pull back via β_n to create the prodiscrete metric space $\pi_n^{lim}(\mathcal{H}_n, *)$. Indeed, as mentioned before, $Y_i^{(n)}$ is locally n-connected and thus by [7, Theorem 3.6], $\pi_n^{top}(Y_i^{(n)}, y_i^*)$ is discrete, which implies that $\varprojlim \pi_n^{top}(Y_i^{(n)}, y_i^*)$ is a prodiscrete metric space.
(2) We can endow the quotient topology on $\pi_n(\mathcal{H}_n, *)$ inherited by the compact-open topology of n-loop space $\Omega^n(X, x^*)$, denoted by $\pi_n^{top}(\mathcal{H}_n, *)$; see [7].

Additionally, we will show that these two topologies agree. Therefore, $\pi_n^{top}(\mathcal{H}_n, *)$ is metrizable, for $n \geq 2$. However, a result of Paul Fabel [4] shows that the topological fundamental group $\pi_1^{top}(\mathcal{H}_n, *)$ fails to be metrizable. Moreover, we assert that $\pi_n^{top}(\mathcal{H}_n, *)$ is a topological group, for $n \geq 2$. This statement answers the question whether $\pi_n^{top}(X, x^*)$ is a topological group for n-Hawaiian like spaces. In fact, the main results of the paper are formulated as follows.

Theorem 1.1. *Suppose that for each j, $X_j^{(n)}$ is an $(n-1)$-connected, locally $(n-1)$-connected, n-semilocally simply connected, compact CW space and X is approximated by the factors $Y_i^{(n)}$, that is, $(X, *) = \varprojlim (Y_i^{(n)}, y_i^*)$, where $(Y_i^{(n)}, y_i^*) = \bigvee_{j \leq i} (X_j^{(n)}, x_j^*)$ is the wedge of $X_j^{(n)}$'s. Then the homomorphism*

$$\beta_n : \pi_n(X, *) \to \varprojlim \pi_n(Y_i^{(n)}, y_i^*)$$

is a bijection.

We call a space X that satisfies the assumptions of Theorem 1.1 an *n-Hawaiian like space.*

Theorem 1.2. *If X is an n-Hawaiian like space, then $\pi_n^{top}(X, x^*)$ is a topological group, for $n \geq 2$. Moreover, it is a prodiscrete metric space.*

2. Proof of Theorem 1.1

First, we extend the van Kampen theorem [9] to higher homotopy groups of n-Hawaiian like spaces. As a suitable model, one can consider the n-dimensional Hawaiian earring $\mathcal{H}_n$. Katsuya Eda and Kazuhiro Kawamura [3] determined the nth homotopy group of $\mathcal{H}_n$ by showing that $\pi_n(\mathcal{H}_n, *)$ is isomorphic to $\mathbb{Z}^\omega$. But this section aims to determine $\pi_n(\mathcal{H}_n, *)$ by a form of van Kampen theorem applicable to Hawaiian like spaces which is needed to prove the further results.

We begin by fixing some notation. Let $X_j^{(n)}$, $j \in \mathbb{N}$, be a based compact CW-complex which is also $(n-1)$-connected, locally $(n-1)$-connected, and n-semilocally simply connected. Take the wedge $Y_i^{(n)} = \bigvee_{j\leq i} X_j^{(n)}$ with collapsing maps $r_k^i : Y_i^{(n)} \to Y_k^{(n)}$, where $k \leq i$, which are the identity on $Y_k^{(n)}$ and collapse $X_j^{(n)}$ to the base point x_k^* if $k < j \leq i$. Let $G_i = \bigoplus_{j=1}^i \pi_n(X_j^{(n)}, x_j^*)$ with projections $\pi_k^i : G_i \to G_k$ defined by $\pi_k^i(\eta_1, \cdots, \eta_i) = (\eta_1, \cdots, \eta_k)$, $k \leq i$. Then $\{Y_i^{(n)}, r_k^i\}$ and $\{G_i, \pi_k^i\}$ are inverse systems of topological spaces and groups whose limits we denote by $\mathcal{H}_n$ and $\mathcal{G}$, respectively.

Example 2.1. Consider $\mathcal{H}_n$ to be a finite family $\{\mathcal{H}_n^k\}_{k=1}^m$ of n-Hawaiian earring spaces, which are joint to m points of an n-sphere S^n at their based points. One can see that $\mathcal{H}_n$ is an n-Hawaiian like space. Indeed, if $\mathcal{H}_n^k$ is approximated by $Y_{i,k}^{(n)}$'s, then $\mathcal{H}_n = \varprojlim Y_i^{(n)}$, where $Y_i^{(n)} = (\cup_{k=1}^m Y_{i,k}^{(n)}) \cup S^n$.

We start with a lemma from [6].

Lemma 2.2. *Suppose that X is an $(n-1)$-connected, locally $(n-1)$-connected, compact metric space and $\pi_n(X)$ is not finitely generated. Then there exists $x \in X$ such that for each positive integer m, there exists an n-loop f_m at x with diameter less than 2^{-m} which is not nullhomotopic. In particular, X is not n-semilocally simply connected at x.*

The next assertion follows immediately.

Corollary 2.3. *Let $X_j^{(n)}$, $j \in \mathbb{N}$, be as above. Then $\pi_n(X_j^{(n)}, x_j^*)$ is finitely generated.*

For each $j \in \mathbb{N}$, we denote the generators of $\pi_n(X_j^{(n)}, x_j^*)$ by $\alpha_{j,1}, \cdots, \alpha_{j,k_j}$.

Now we recall a result of [12, Proposition 6.36].

Proposition 2.4. *If X is an n-connected CW-complex and Y is an m-connected CW-complex, then the maps $i_X : (X, x^*) \to (X \vee Y, *)$ and $i_Y : (Y, y^*) \to (X \vee Y, *)$ given by $i_X(x) = (x, y^*)$ and $i_Y(y) = (x^*, y)$ induce an isomorphism $(i_{X_*}, i_{Y_*}) : \pi_k(X, x^*) \bigoplus \pi_k(Y, y^*) \to \pi_k(X \vee Y, *)$ for $2 \leq k \leq n + m$, provided X or Y is locally finite.*

So, one can determine $\pi_n(Y_i^{(n)}, y_i^*)$ as follows (see [12]).

Corollary 2.5. *With the previous notation, let $Y_i^{(n)}$ be the wedge $\bigvee_{j \leq i} X_j^{(n)}$. Then*

$$\pi_n(Y_i^{(n)}, y_i^*) \cong \bigoplus_{j=1}^{i} \pi_n(X_j^{(n)}, x_j^*) \qquad (\textit{for all } n \geq 2).$$

Remark 2.6. By corollaries 2.3 and 2.5, $\pi_n(Y_i^{(n)}, y_i^*)$ is finitely generated. Since $r_k^i(Y_i^{(n)}) = Y_k^{(n)}$, then clearly, $\alpha_{1,1}, \cdots, \alpha_{1,k_1}$, $\alpha_{2,1}, \cdots, \alpha_{2,k_2}, \cdots, \alpha_{i,1}, \cdots, \alpha_{i,k_i}$ are the generators of $\pi_n(Y_i^{(n)}, y_i^*)$. This means that if $\gamma \in \pi_n(Y_i^{(n)}, y_i^*)$, then $\gamma = (\alpha_{1,1}^{l_{1,1}} \cdots \alpha_{1,k_1}^{l_{1,k_1}}, \alpha_{2,1}^{l_{2,1}} \cdots \alpha_{2,k_2}^{l_{2,k_2}}, \cdots, \alpha_{i,1}^{l_{i,1}} \cdots \alpha_{i,k_i}^{l_{i,k_i}})$ for some integers $l_{1,1}, \cdots, l_{1,k_1}, \cdots, l_{i,1}, \cdots, l_{i,k_i}$, where $\alpha^l = \alpha \cdots \alpha$ is the concatenation l-times of the homotopy class of α with itself. We embed the generators $\alpha_{j1}, \cdots, \alpha_{jk_j}$, $(j \leq i)$, in $\pi_n(Y_i^{(n)}, y_i^*)$ by a map induced by inclusion $X_j^{(n)} \to Y_i^{(n)} = \bigvee_{k=1}^{i} X_k^{(n)}$. For simplicity, one can denote the embedded classes by the same notations $\alpha_{j,1} \cdots \alpha_{j,k_j}$. It is easy to see that the map

$$\alpha_{1,1}^{l_{1,1}} \cdots \alpha_{1,k_1}^{l_{1,k_1}} \cdots \alpha_{i,1}^{l_{i,1}} \cdots \alpha_{i,k_i}^{l_{i,k_i}} \mapsto (\alpha_{1,1}^{l_{1,1}} \cdots \alpha_{1,k_1}^{l_{1,k_1}}, \cdots, \alpha_{i,1}^{l_{i,1}} \cdots \alpha_{i,k_i}^{l_{i,k_i}})$$

induces an isomorphism between $\pi_n(Y_i^{(n)}, y_i^*)$ and $\bigoplus_{j=1}^{i} \pi_n(X_j^{(n)}, x_j^*)$.

Let $f : I^n \to \mathcal{H}_n$ be an n-loop with $f(\partial I^n) = \{*\}$; f is said to be standard if $f(J_i^n) \subseteq X_i^{(n)}$, where $J_i^n = [a_i, b_i] \times I^{(n-1)}$ with $a_i = 1 - \frac{1}{2^{i-1}}$ and $b_i = 1 - \frac{1}{2^i}$, for $i \in \mathbb{N}$.

Now we recall a definition from [11]. Suppose that (X, x) is a pointed space. Given an n-loop f based at x in X, then any other n-loop g based at x in X, with $H : g \simeq f(rel\partial I^n)$ and $g(I^n \backslash I_1^n) = \{x\}$, is called a concentration of f on subcube I_1^n.

We will need the following lemma which is a key step in the proof of Theorem 1.1.

Lemma 2.7. *Each n-homotopy class in $\pi_n(\mathcal{H}_n, *)$ is represented by a standard n-loop.*

Proof: Let f be any n-loop in $\mathcal{H}_n$ based at $*$. Then f determines a sequence of n-loops f_i in $Y_i^{(n)}$ defined by $f_i = R_i \circ f$, where $R_i : \mathcal{H}_n \to Y_i^{(n)}$ denotes the retraction fixing $Y_i^{(n)}$ pointwise and collapsing $\bigcup_{j=n+1}^{\infty} Y_j^{(n)}$ to the point $*$. By Corollary 2.5, the n-homotopy class of f_i is contained in $\bigoplus_{j=1}^{i} \pi_n(X_j^{(n)}, x_j^*)$. Therefore, $[f_i] = (\alpha_{1,1}^{l_{1,1}} \cdots \alpha_{1,k_1}^{l_{1,k_1}}, \cdots, \alpha_{i,1}^{l_{i,1}} \cdots \alpha_{i,k_i}^{l_{i,k_i}})$ for some integers $l_{1,1}, \cdots, l_{1,k_1}, \cdots, l_{i,1}, \cdots, l_{i,k_i}$. Let $g_1, \cdots, g_i$ be n-loops in $X_1^{(n)}$, $\cdots, X_i^{(n)}$ representing the n-homotopy classes $\gamma_1, \cdots, \gamma_i$, where $\gamma_j = \alpha_{j,1}^{l_{j,1}} \cdots \alpha_{j,k_j}^{l_{j,k_j}}$. By Remark 2.6, $f_i \simeq g_1 * \cdots * g_i$, where $*$ denotes the product of n-loops in $\Omega^n(Y_i^{(n)}, y_i^*)$. Note that each n-loop in $X_j^{(n)}$ $(j < i)$ can be embedded in $Y_i^{(n)}$ or in $\mathcal{H}_n$, if it is necessary, by maps induced by the inclusions $X_j^{(n)} \to Y_i^{(n)}$ and $Y_i^{(n)} \to \mathcal{H}_n$, respectively.

Let h_j be a concentration of g_j on subcube J_j^n, for $j \in \mathbb{N}$. By [11, Lemma 2.5.2], such concentration exists. Then $f_i \simeq h_1 * \cdots * h_i$ $(rel\partial I^n)$ by a homotopy H_i.

We proceed by induction, constructing homotopies $H_i : I^n \times [0,1] \to Y_i^{(n)}$ satisfying

(1) $H_i(x, 0) = f_i(x)$;
(2) $H_i(x, 1) = h_1 * \cdots * h_i(x)$;
(3) $r_k^i \circ H_i = H_k$, $(k \leq i)$.

Such homotopies give, in the limit, a homotopy H of n-loop f to a standard n-loop h (the homotopies H_i's are endowed with the uniform metric, so the limit H exists and it is continuous; see [10, Theorem 46.8 and Corollary 46.6]). □

Now we prove the main result of this section.

Proof of Theorem 1.1: Suppose f is a standard n-loop based at $*$ in $\mathcal{H}_n$, $n \geq 2$, with corresponding sequence of n-loops $f_i = R_i \circ f$ in $Y_i^{(n)}$. The homomorphism β_n is well defined, since $r_k^i(f_i) = f_k$, and by construction of standard map in Lemma 2.7 and Remark 2.6, we have $\pi_k^i[f_i] = [f_k]$.

To show the injectivity of β_n, we must prove that given a standard n-loop f in $\mathcal{H}_n$ with $\beta_n([f]) = e_{\mathcal{G}}$, there is a based homotopy between f and the constant n-loop at $*$, where $e_{\mathcal{G}}$ is the identity of $\mathcal{G} = \varprojlim \pi_n(Y_i^{(n)}, y_i^*)$. Let $f_i = R_i \circ f$. Clearly, $[f_i] = (e_1, \cdots, e_i)$, where e_j is the identity of $\pi_n(X_j^{(n)}, x_j^*)$, for $j = 1, \cdots, i$. Then there are based homotopies K_i between f_i and the constant n-loop at y_i^*. Now the limit of K_i's is a homotopy between f and constant n-loop at $*$, denoted by K.

Now, it is sufficient to show that β_n is surjective. Let $\mathfrak{g} = (g_i) \in \mathcal{G}$. Then $g_i = (\eta_1, \cdots, \eta_i) \in \bigoplus_{j=1}^{i} \pi_n(X_j^{(n)}, x_j^*) \cong \pi_n(Y_i^{(n)}, y_i^*)$. Suppose f_i represents n-homotopy class g_i. So $r_j^i(f_i) = f_j$, since $\pi_j^i(\eta_1, \cdots, \eta_i) = (\eta_1, \cdots, \eta_j)$ for $j \leq i$. Also, there are integers $l_{1,1}, \cdots, l_{1,k_1}, \cdots, l_{i,1}, \cdots, l_{i,k_i}$ such that $[f_i] = \alpha_{1,1}^{l_{1,1}} \cdots \alpha_{1,k_1}^{l_{1,k_1}} \cdots \alpha_{i,1}^{l_{i,1}} \cdots \alpha_{i,k_i}^{l_{i,k_i}}$. Let the n-loop t_j represent $\alpha_{j,1}^{l_{j,1}} \cdots \alpha_{j,k_j}^{l_{j,k_j}}$, let s_j be a concentration of t_j on J_j^n, and let $h_i = s_1 \cdots s_i$. Clearly, the limit of h_i's is an n-loop denoted by h and $[h] = g$ (the homotopies K_i's and the n-loops h_i's are endowed with the uniform metric so their limits K and h exist and they are also continuous; see [10, Theorem 46.8 and Corollary 46.6]). This completes the proof. □

Corollary 2.8. *Let $\mathcal{H}_n$ be the n-dimensional Hawaiian earring with based point $*$. Then $\pi_n(\mathcal{H}_n, *) \cong \mathcal{G} = \varprojlim G_i$, where G_i is the direct sum of i copies of integers $\mathbb{Z}$.*

3. Proof of Theorem 1.2

In conclusion, we assert that $\pi_n^{top}(\mathcal{H}_n, *)$ is a topological group homeomorphic to $\pi_n^{lim}(\mathcal{H}_n, *)$ which implies that $\pi_n^{top}(\mathcal{H}_n, *)$ is metrizable.

As mentioned in the introduction, it is an open question whether or not in general $\pi_n^{top}(X, *)$ is a topological group. If X is a locally

n-connected metric space, then $\pi_n^{top}(X,x^*)$, and hence, $\pi_n^{top}(X,x^*)\times \pi_n^{top}(X,x^*)$ is discrete (see [7]), and therefore multiplication is continuous. In general, the continuity of multiplication remains an unsettled question.

The following lemma shows that if (X,x^*) is a pointed topological space, then left and right translations by a fixed element in $\pi_n^{top}(X,x^*)$ are homeomorphisms.

Lemma 3.1. *Let (X,x^*) be a pointed topological space. If $[f]\in \pi_n^{top}(X,x^*)$, then left and right translations by $[f]$ are homeomorphisms of $\pi_n^{top}(X,x^*)$.*

Proof: First, we show that the multiplication

$$\Omega^n(X,x^*)\times\Omega^n(X,x^*)\xrightarrow{\tilde{m}}\Omega^n(X,x^*)$$

is continuous, where $\tilde{m}$ is concatenation of n-loops and $\Omega^n(X,x^*)$ is equipped with compact-open topology.

Let $\langle K,U\rangle$ be a subbasis element in $\Omega^n(X,x^*)$. Define

$$K_1=\{(t_1,\ldots,t_n);(t_1,\ldots,t_{n-1},\frac{t_n}{2})\in K\}$$

and

$$K_2=\{(t_1,\ldots,t_n);(t_1,\ldots,t_{n-1},\frac{t_n+1}{2})\in K\}.$$

Then

$$\tilde{m}^{-1}(\langle K,U\rangle)=\{(f_1,f_2);(f_1*f_2)(K)\subseteq U\}=\langle K_1,U\rangle\times\langle K_2,U\rangle$$

is open in $\Omega^n(X,x^*)\times\Omega^n(X,x^*)$ and so $\tilde{m}$ is continuous.

Now, fix $[f]\in\pi_n^{top}(X,x^*)$ and consider left translation by $[f]$ on $\pi_n^{top}(X,x^*)$

$$\pi_n^{top}(X,x^*)\to\pi_n^{top}(X,x^*)$$
$$[g]\mapsto[f]\cdot[g].$$

Clearly, the following diagram is commutative.

$$\begin{array}{ccc}
\Omega^n(X,x^*) & \xrightarrow{\tilde{m}_f} & \Omega^n(X,x^*)\\
\downarrow & & \downarrow\\
\pi_n^{top}(X,x^*) & \xrightarrow{m_{[f]}} & \pi_n^{top}(X,x^*),
\end{array}$$

where $\tilde{m}_f$ is defined by $g \mapsto f * g$. By the universal property of quotient maps [10, Theorem 11.1], $m_{[f]}$ is continuous. Since $m_{[f]}^{-1} = m_{[f^{-1}]}$ is also continuous, so $m_{[f]}$ is a homeomorphism, as desired. A similar argument implies that right translation is also a homeomorphism. □

Note that $\pi_n^{top}(X, x^*)$ acts on itself by left and right translations as a group of homeomorphisms. It is easy to see that these actions are both transitive. So, we have the following result.

Proposition 3.2. *If (X, x^*) is a pointed topological space, then $\pi_n^{top}(X, x^*)$ is a homogeneous space.*

Now, let U be an open neighborhood of x^* in X and $\tilde{U}$ be the set of all n-loops based at x^* lying inside U. Also, let $\hat{U}$ be its quotient under homotopy, that is

$$\hat{U} = \{[f]; f \in \Omega^n(X, x^*) \text{ and } im f \subseteq U\}.$$

Suppose $W \subset \pi_n^{top}(X, x^*)$ is an open neighborhood containing the identity element $[e_{x_*}]$. Then

$$e_{x_*} \in q^{-1}(W) = \bigcup_{\alpha \in J} \Big(\bigcap_{i=1}^{k_\alpha} < K_i^\alpha, V_i^\alpha >\Big),$$

where e_{x_*} is the constant n-loop based at x^*. Therefore, there exists an index $\alpha \in J$ such that $e_{x_*} \in \bigcap_{i=1}^{k_\alpha} < K_i^\alpha, V_i^\alpha >$, which implies that

$$x^* \in \bigcup_{i=1}^{k_\alpha} V_i^\alpha = V^\alpha \ , \quad \tilde{V}^\alpha \subset \bigcap_{i=1}^{k_\alpha} < K_i^\alpha, V_i^\alpha > \subset \ q^{-1}(W)$$

and then $[e_{x_*}] \in \hat{V}^\alpha \subset W$.

Since left translation is continuous in $\pi_n^{top}(X, x^*)$,

$$[f]W = \{[f] \cdot [g]; \ [g] \in W\}$$

runs through a basis at $[f]$ for $\pi_n^{top}(X, x^*)$, as W runs through a basis at $[e_{x_*}]$ in its topology.

Now we use a classical theorem in the theory of topological groups [8] which asserts that for a given group G with a filter base $\{U\}$, satisfying the following conditions

- each U is symmetric, i.e., $U^{-1} = U$,

- for each U in $\{U\}$, there exists a V in $\{U\}$ such that $V^2 \subset U$, where $V^2 = \{xy; x, y \in V\}$,
- for each U in $\{U\}$ and $a \in G$, there exists a V in $\{U\}$ such that $V \subset a^{-1}Ua$ or $aVa^{-1} \subset U$,

then $\{U\}$ forms a fundamental system of neighborhoods of e. In particular, G with the topology induced by this fundamental system becomes a topological group.

Since $\pi_n(X, x^*)$ is an abelian group, for $n \geq 2$, it is easy to see that the filter base $\{\hat{U}\}$ forms a fundamental system of neighborhoods of the identity element e, and hence $\pi_n(X, x^*)$ with this topology becomes a topological group, denoted by $\pi_n^{lim}(X, x^*)$. By the above statements, this topology, denoted by τ^{lim}, is coarser than quotient topology τ^{top} on $\pi_n(X, x^*)$ inherited from $\Omega^n(X, x^*)$ with the compact-open topology.

If $X = \mathcal{H}_n$ and $W_i^{(n)}$ is an n-connected neighborhood of x_i^* in $X_i^{(n)}$, then the following sets provide a basis for $\mathcal{H}_n$ at $*$

$$\mathcal{U}_k = (\bigcup_{i \leq k} W_i^{(n)}) \cup (\bigcup_{i > k} X_i^{(n)}).$$

Also, a basis of neighborhoods of the identity $e_{\mathcal{G}}$ in $\mathcal{G}$ is given by the subgroups

$$\mathcal{G}_k = \{\mathfrak{g} = (g_i) \in \mathcal{G};\ g_k = (e_1, \dots, e_k)\}, \quad for\ k \in \mathbb{N}.$$

Theorem 3.3. *The map $\beta_n : \pi_n^{lim}(\mathcal{H}_n, *) \to \mathcal{G}$ is a homeomorphism and therefore an isomorphism of topological groups.*

Proof: Since $\{\hat{\mathcal{U}}_k\}$ and $\{\mathcal{G}_k\}$ form bases at $*$ and $e_{\mathcal{G}}$ for topologies on $\pi_n^{lim}(\mathcal{H}_n, *)$ and $\mathcal{G}$ (see also the statement before Theorem 3.3), it is sufficient to show that $\beta_n(\hat{\mathcal{U}}_k) = \mathcal{G}_k$. For, let $[f] \in \hat{\mathcal{U}}_k$ and h be a standard n-loop representing $[f]$. Since $W_i^{(n)}$ is n-connected for $i \leq k$, we have that $h_k = R_k \circ h$ is nullhomotopic. Therefore, $\beta_n([f]) \in \mathcal{G}_k$.

Conversely, if $\mathfrak{g} = (g_i) \in \mathcal{G}_k$, then $g_k = (e_1, \dots, e_k)$. The homomorphism β_n is a bijection, so there is a standard n-loop f such that $[f] \in \pi_n(\mathcal{H}_n, *)$ and $\beta_n([f]) = g$. But $\beta_n([f]) = ([R_1(f)], [R_2(f)], \dots)$. This implies that $[R_k(f)] = (e_1, \dots, e_k)$ and therefore, $R_k(f) = f_k$ is nullhomotopic. Take h a standard n-loop such that $h \simeq f$ $(rel\ \partial I^n)$ and $R_k \circ h = y_k^*$. Then $\beta_n\ [h] = \beta_n\ [f] \in \hat{\mathcal{U}}_k$. $\square$

The homomorphism β_n gives a compatible sequence of homomorphisms $\beta_{n,i} : \pi_n(\mathcal{H}_n, *) \to \pi_n(Y_i^{(n)}, y_i^*)$. By [7], the homomorphisms $\beta_{n,i} : \pi_n^{top}(\mathcal{H}_n, *) \to \pi_n^{top}(Y_i^{(n)}, y_i^*)$ are continuous when we dealing with topological homotopy groups, which implies that $\beta_n : \pi_n^{top}(\mathcal{H}_n, *) \to \mathcal{G} = \varprojlim \pi_n^{top}(Y_i^{(n)}, y_i^*)$ is also continuous.

So that there is no ambiguity in notation, we denote

$$\beta_n : \pi_n^{lim}(\mathcal{H}_n, *) \to \mathcal{G} \qquad \text{and} \qquad \beta_n : \pi_n^{top}(\mathcal{H}_n, *) \to \mathcal{G}$$

by β_n^{lim} and β_n^{top}, respectively.

Now consider the following commutative diagram.

$$\begin{array}{ccc} \pi_n^{top}(\mathcal{H}_n, *) & \xrightarrow{\beta_n^{top}} & \mathcal{G} \\ {\scriptstyle id}\downarrow & \nearrow {\scriptstyle \beta_n^{lim}} & \\ \pi_n^{lim}(\mathcal{H}_n, *) & & \end{array}$$

Since β_n^{lim} is a homeomorphism, the identity map id is continuous. This fact shows that quotient topology τ^{top} inherited from compact-open topology of n-loop space is coarser than τ^{lim}. But we have already seen that $\tau^{top} \subset \tau^{lim}$. Therefore, these two topologies on $\pi_n(\mathcal{H}_n, *)$ are equivalent. This means that $\pi_n^{top}(\mathcal{H}_n, *)$ is a topological group and also a prodiscrete metric space.

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References

[1] Daniel K. Biss, *The topological fundamental group and generalized covering spaces*, Topology Appl. **124** (2002), no. 3, 355–371.

[2] Jack S. Calcut and John D. McCarthy, *Discreteness and homogeneity of the topological fundamental group*, Topology Proc. **34** (2009), 339–349.

[3] Katsuya Eda and Kazuhiro Kawamura, *Homotopy and homology groups of the n-dimensional Hawaiian earring*, Fund. Math. **165** (2000), no. 1, 17–28.

[4] Paul Fabel, *The Hawaiian earring group and metrizability*. Available at arXiv:math/0603252v1 [math/GT].

[5] ———, *Topological fundamental groups can distinguish spaces with isomorphic homotopy groups*, Topology Proc. **30** (2006), no. 1, 187–195.

[6] H. Ghane and Z. Hamed, *On nondiscreteness of a higher topological homotopy group and its cardinality*, Bull. Belg. Math. Soc. Simon Stevin **16** (2009), no. 1, 179–183.

[7] H. Ghane, Z. Hamed, B. Mashayekhy, and H. Mirebrahimi, *Topological homotopy groups*, Bull. Belg. Math. Soc. Simon Stevin **15** (2008), no. 3, 455–464.

[8] Taqdir Husain, *Introduction to Topological Groups*. Philadelphia, Pa.-London: W. B. Saunders Co., 1966.

[9] John W. Morgan and Ian Morrison, *A van Kampen theorem for weak joins*, Proc. London Math. Soc. (3) **53** (1986), no. 3, 562–576.

[10] James R. Munkres, *Topology: A First Course*. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1975.

[11] Gregory L. Naber, *Topology, Geometry, and Gauge Fields: Foundations*. Texts in Applied Mathematics, 25. New York: Springer-Verlag, 1997.

[12] Robert M. Switzer, *Algebraic Topology—Homotopy and Homology*. Reprint of the 1975 original [New York: Springer]. Classics in Mathematics. Berlin: Springer-Verlag, 2002.

(Ghane, Hamed, Mashayekhy, Mirebrahimi) Department of Pure Mathematics; Center of Excellence in Analysis on Algebraic Structures; Ferdowsi University of Mashhad; P.O.Box 1159-91775; Mashhad, Iran.