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by

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## ON SOME POSITIONAL DIMENSION-LIKE FUNCTIONS

D. N. GEORGIU, S. D. ILIADIS, AND A. C. MEGARITIS

**ABSTRACT.** In *The universality property for some dimension-like functions* [Questions Answers Gen. Topology **27** (2009), no. 2, 141–156], the authors define some positional dimension-like functions. These functions were studied only with respect to the property of universality. Here, we first compare and then study these functions with respect to other standard properties of dimension theory (subspace, product, and sum theorems).

### 1. INTRODUCTION AND PRELIMINARIES

All spaces are considered to be  $T_0$ -spaces of weight  $\leq \tau$ , where  $\tau$  is a fixed infinite cardinal. The least cardinal greater than  $\tau$  is denoted by  $\tau^+$ . The class of all ordinals is denoted by  $\mathcal{O}$  and the first infinite cardinal is denoted by  $\omega$ . In the class  $\mathcal{O}$  we denote by  $(+)$  the natural sum of Hessenberg (see, for example, [7]). We note the following properties of the natural sum:

- (1)  $\alpha(+)\beta = \beta(+)\alpha$ ,
- (2) if  $\alpha_1 < \alpha_2$ , then  $\alpha_1(+)\beta < \alpha_2(+)\beta$ , and
- (3)  $\alpha(+n) = \alpha + n$  for  $n < \omega$ .

We also consider two symbols “ $-1$ ” and “ $\infty$ .” It is assumed that  $-1 < \alpha < \infty$  for every  $\alpha \in \mathcal{O}$  and  $-1(+)\alpha = \alpha(+)(-1) = \alpha$ ,  $\infty(+)\alpha = \alpha(+)\infty = \infty$  for every  $\alpha \in \mathcal{O} \cup \{-1, \infty\}$ .

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Let  $Q$  be a subset of a space  $X$ . We denote by  $\text{Int}_X(Q)$ ,  $\text{Cl}_X(Q)$ , and  $\text{Bd}_X(Q)$  the interior, the closure, and the boundary of  $Q$  in  $X$ , respectively.

We recall (see [6]) that a family  $B$  of open subsets of  $X$  (containing  $X$  and the empty set) is said to be a *p-base for  $Q$  in  $X$*  if the set  $\{Q \cap U : U \in B\}$  is a base for the subspace  $Q$ . A p-base  $B$  for  $Q$  in  $X$  is said to be a *pos-base* if for every  $x \in Q$  and an open neighborhood  $U$  of  $x$  in  $X$  there exists an element  $V$  of  $B$  such that  $x \in V \subseteq U$ . A p-base  $B$  for  $Q$  in  $X$  is said to be a *ps-base* if  $B$  is a base for the space  $X$ . By a *class of p-bases*, we mean a class  $\mathbb{ID}$  consisting of triads  $(Q, B, X)$ , where  $B$  is a p-base for a subset  $Q$  in a space  $X$  such that  $|B| \leq \tau$  and  $\emptyset, X \in B$ . We say that a p-base  $B$  for  $Q$  in  $X$  is a  *$\mathbb{ID}$ -p-base*, where  $\mathbb{ID}$  is a class of p-bases if  $(Q, B, X) \in \mathbb{ID}$ . By a *class of subsets*, we mean a class consisting of pairs  $(Q, X)$ , where  $Q$  is a subset of a space  $X$ .

We recall (see, for example, [2]) that a subset  $L$  of a space  $X$  *separates* two disjoint subsets  $A$  and  $B$  of  $X$  if there exist two open subsets  $U$  and  $W$  of  $X$  such that (a)  $A \subseteq U$  and  $B \subseteq W$ , (b)  $U \cap W = \emptyset$ , and (c)  $X \setminus L = U \cup W$ .

In [8], two dimensions  $dm$  and  $Dm$  are introduced and studied. In [1], transfinite extensions of these dimensions denoted by  $trdm$  and  $trDm$  are given. Below, we give the definition of these transfinite extensions (using the original notations  $dm$  and  $Dm$ ).

**Definition 1.1** (See [8] and [1]). We denote by  $dm$  and  $Dm$  the *dimension-like functions* (or *dimensions*) with as domain the class of all spaces and as range the class  $\mathcal{O} \cup \{-1, \infty\}$  satisfying the following conditions:

- (1)  $dm(X) = Dm(X) = -1$  if and only if  $X = \emptyset$ .
- (2)  $Dm(X) \leq \alpha$ , where  $\alpha \in \mathcal{O}$ , if and only if for any pair of distinct points  $x$  and  $y$  of  $X$  there exists a subset  $L$  of  $X$  which separates the singletons  $\{x\}$  and  $\{y\}$  such that  $dm(L) < \alpha$ .
- (3)  $dm(X) \leq \alpha$ , where  $\alpha \in \mathcal{O}$ , if and only if  $X = \cup\{Q_i^X : i \in \omega\}$  such that the subset  $Q_i^X$  of  $X$  is closed and  $Dm(Q_i^X) \leq \alpha$ ,  $i \in \omega$ .

In [3] and [4], modifications of dimensions  $dm$  and  $Dm$  are given and studied. These new dimension-like functions are denoted by

$dm_{\mathbb{E},\nu}^{\mathbb{K},\mathbb{B}}$  and  $Dm_{\mathbb{E},\nu}^{\mathbb{K},\mathbb{B}}$ , where  $\mathbb{E}$  is a class of spaces,  $\mathbb{K}$  is a class of subsets,  $\mathbb{B}$  is a class of bases, and  $\nu \leq \tau$  is an infinite cardinal. In [5], reliant on dimension-like functions defined in [3] and [4], some positional dimension-like functions modulo a class  $\mathbb{F}$  of subsets are defined using a given class  $\mathbb{K}$  of subsets and a class  $\mathbb{ID}$  of p-bases. These functions were studied only with respect to the property of universality.

In what follows we denote by  $\nu$  a fixed infinite cardinal such that  $\omega \leq \nu \leq \tau$ .

**Definition 1.2** (See [5]). Let  $\mathbb{ID}$  be a class of p-bases. A class  $\mathbb{F}$  of subsets is said to be  $\mathbb{ID}$ -p<sub>0</sub>-hereditary separated (respectively,  $\mathbb{ID}$ -p<sub>1</sub>-hereditary separated) if for every  $(Q, X) \in \mathbb{F}$  there exists a  $\mathbb{ID}$ -p-base  $B = \{U_\delta : \delta \in \tau\}$  for  $Q$  in  $X$  such that for every two elements  $U_{\delta_1}$  and  $U_{\delta_2}$  of  $B$  with  $\text{Cl}(U_{\delta_1}) \cap \text{Cl}(U_{\delta_2}) = \emptyset$  there exists a subspace  $L$  of  $X$  separating the sets  $\text{Cl}(U_{\delta_1})$  and  $\text{Cl}(U_{\delta_2})$  with  $(Q \cap L, L) \in \mathbb{F}$  (respectively,  $(Q \cap L, X) \in \mathbb{F}$ ).

**Definition 1.3** (See [5]). Let  $\mathbb{ID}$  be a class of p-bases,  $\mathbb{F}$  a  $\mathbb{ID}$ -p<sub>0</sub>-hereditary separated class of subsets, and  $\mathbb{K}$  a class of subsets with  $(X, X) \in \mathbb{K}$  for every space  $X$ . We denote by  $p_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  and  $p_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  the dimension-like functions with as domain the class of all subsets and as range the set  $\mathcal{O} \cup \{-1, \infty\}$  satisfying the following conditions:

- (1)  $p_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}(Q, X) = p_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}(Q, X) = -1$  if and only if  $(Q, X) \in \mathbb{F}$ .
- (2)  $p_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}(Q, X) \leq \alpha$ , where  $\alpha \in \mathcal{O}$ , if and only if there exists a  $\mathbb{ID}$ -p-base  $B = \{U_\delta : \delta \in \tau\}$  for  $Q$  in  $X$  such that for every two elements  $U_{\delta_1}, U_{\delta_2}$  of  $B$  with  $\text{Cl}_X(U_{\delta_1}) \cap \text{Cl}_X(U_{\delta_2}) = \emptyset$  there exists a subset  $L$  of  $X$  separating  $\text{Cl}_X(U_{\delta_1})$  and  $\text{Cl}_X(U_{\delta_2})$  with  $p_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}(Q \cap L, L) < \alpha$ .
- (3)  $p_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}(Q, X) \leq \alpha$ , where  $\alpha \in \mathcal{O}$ , if and only if  $X = \cup\{S_i : i \in \nu\}$  such that (a) the subset  $S_i$  of  $X$  is closed, (b)  $(S_i, X) \in \mathbb{K}$ , and (c)  $p_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}(Q \cap S_i, S_i) \leq \alpha$ .

**Notation.** (1) In the case where  $\mathbb{F} = \{(\emptyset, \emptyset)\}$ , the dimension-like functions  $p_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  and  $p_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  are denoted by  $p_0\text{-}dm_{\nu}^{\mathbb{K},\mathbb{ID}}$  and  $p_0\text{-}Dm_{\nu}^{\mathbb{K},\mathbb{ID}}$ .

(2) If in Definition 1.3 instead of the p-base  $B$  we consider a pos-base (respectively, a ps-base), then the dimension-like functions  $p_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  and  $p_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  will be denoted by  $\text{pos}_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  and  $\text{pos}_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  (respectively, by  $\text{ps}_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  and  $\text{ps}_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$ ).

**Definition 1.4** (See [5]). Let  $\mathbb{ID}$  be a class of p-bases,  $\mathbb{F}$  a  $\mathbb{ID}$ -p<sub>1</sub>-hereditary separated class of subsets, and  $\mathbb{K}$  a class of subsets with  $(X, X) \in \mathbb{K}$  for every space  $X$ . We denote by  $p_1\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  and  $p_1\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  the dimension-like functions with as domain the class of all subsets and as range the set  $\mathcal{O} \cup \{-1, \infty\}$  satisfying the following conditions:

- (1)  $p_1\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}(Q, X) = p_1\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}(Q, X) = -1$  if and only if  $(Q, X) \in \mathbb{F}$ .
- (2)  $p_1\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}(Q, X) \leq \alpha$ , where  $\alpha \in \mathcal{O}$ , if and only if there exists a  $\mathbb{ID}$ -p-base  $B = \{U_{\delta} : \delta \in \tau\}$  for  $Q$  in  $X$  such that for every two elements  $U_{\delta_1}, U_{\delta_2}$  of  $B$  with  $\text{Cl}_X(U_{\delta_1}) \cap \text{Cl}_X(U_{\delta_2}) = \emptyset$  there exists a subset  $L$  of  $X$  separating  $\text{Cl}_X(U_{\delta_1})$  and  $\text{Cl}_X(U_{\delta_2})$  with  $p_1\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}(Q \cap L, X) < \alpha$ .
- (3)  $p_1\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}(Q, X) \leq \alpha$ , where  $\alpha \in \mathcal{O}$ , if and only if  $X = \cup\{S_i : i \in \nu\}$  such that (a) the subset  $S_i$  of  $X$  is closed, (b)  $(S_i, X) \in \mathbb{K}$ , and (c)  $p_1\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}(Q \cap S_i, X) \leq \alpha$ .

**Notation.** (1) In the case where  $\mathbb{F} = \{(\emptyset, X)\}$ , the dimension-like functions  $p_1\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  and  $p_1\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  are denoted by  $p_1\text{-}dm_{\nu}^{\mathbb{K},\mathbb{ID}}$  and  $p_1\text{-}Dm_{\nu}^{\mathbb{K},\mathbb{ID}}$ .

(2) If in Definition 1.4 instead of the p-base  $B$  we consider a pos-base (respectively, a ps-base), then the dimension-like functions  $p_1\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  and  $p_1\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  will be denoted by  $\text{pos}_1\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  and  $\text{pos}_1\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  (respectively, by  $\text{ps}_1\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  and  $\text{ps}_1\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$ ).

In this paper, we first compare the functions  $p_i\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$  and  $p_i\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{ID}}$ ,  $i \in \{0, 1\}$  with the functions defined in [3] and [4] and then

study these dimension-like functions with respect to other standard properties of dimension theory (subspace, product, and sum theorems).

## 2. PROPERTIES AND COMPARISON OF DIMENSION-LIKE FUNCTIONS

**Proposition 2.1.** *Let  $i \in \{0, 1\}$ . For every subset  $Q$  of a space  $X$ , we have*

- (a)  $p_i\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \leq p_i\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X)$ ,
- (b)  $\text{pos}_i\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \leq \text{pos}_i\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X)$ , and
- (c)  $\text{ps}_i\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \leq \text{ps}_i\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X)$ .

*Proof:* We prove the inequality

$$p_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \leq p_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X). \quad (1)$$

The proofs of all other inequalities are similar. Let

$$p_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) = \alpha \in \{-1, \infty\} \cup \mathcal{O}.$$

Inequality (1) is clear if  $\alpha = -1$  or  $\alpha = \infty$ . Suppose that  $\alpha \in \mathcal{O}$ . We have  $X = \cup\{S_i : i \in \nu\}$ , where  $S_i = X$ . Since  $(S_i, X) = (X, X) \in \mathbb{K}$  and  $p_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q \cap S_i, S_i) = Dm_{\mathbb{E},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \leq \alpha$ , Definition 1.3(3) implies that  $p_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \leq \alpha$ .  $\square$

**Proposition 2.2.** *Let  $i \in \{0, 1\}$ . For every subset  $Q$  of a space  $X$  we have*

$$p_i\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \leq \text{pos}_i\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \leq \text{ps}_i\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X)$$

and

$$p_i\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \leq \text{pos}_i\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \leq \text{ps}_i\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X).$$

*Proof:* We prove by induction only the inequalities

$$p_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \leq \text{pos}_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \quad (2)$$

and

$$p_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \leq \text{pos}_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X). \quad (3)$$

The proofs of all other inequalities are similar. If

$$\text{pos}_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) = \text{pos}_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) = -1,$$

then  $(Q, X) \in \mathbb{F}$  and

$$p_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) = p_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) = -1,$$

proving the proposition. Suppose that (2) is true for every pair  $(Q, X)$  with  $\text{pos}_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) < \alpha$  and that (3) is true for every pair  $(Q, X)$  with  $\text{pos}_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) < \alpha$ , where  $\alpha \in \mathcal{O}$ . First, we suppose that  $(Q, X)$  is a pair with  $\text{pos}_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) = \alpha$ . We need to prove  $p_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \leq \alpha$ . There exists a  $\mathbb{D}$ -pos-base  $B = \{U_\delta : \delta \in \tau\}$  for  $Q$  in  $X$  such that for every two elements  $U_{\delta_1}, U_{\delta_2}$  of  $B$  with  $\text{Cl}_X(U_{\delta_1}) \cap \text{Cl}_X(U_{\delta_2}) = \emptyset$  there exists a subset  $L$  of  $X$  separating  $\text{Cl}_X(U_{\delta_1})$  and  $\text{Cl}_X(U_{\delta_2})$  with  $\text{pos}_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q \cap L, L) < \alpha$ . By induction,

$$p_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q \cap L, L) \leq \text{pos}_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q \cap L, L),$$

and since  $B$  is also a  $p$ -base for  $Q$  in  $X$ ,  $p_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \leq \alpha$ . Now, we suppose that  $(Q, X)$  is a pair with  $\text{pos}_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) = \alpha$ . We need to prove that  $p_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \leq \alpha$ . We have

$$X = \cup\{S_i : i \in \nu\},$$

where

- (a)  $S_i$  is closed subset of  $X$ ,
- (b)  $(S_i, X) \in \mathbb{K}$ , and
- (c)  $p_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q \cap S_i, S_i) \leq \alpha$ .

By the preceding,

$$p_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q \cap S_i, S_i) \leq \text{pos}_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q \cap S_i, S_i), \quad i \in \omega.$$

Therefore,  $p_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \leq \alpha$ . The proposition is trivial if

$$\text{pos}_0\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) = \text{pos}_0\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) = \infty. \quad \square$$

**Proposition 2.3.** *For every subset  $Q$  of a space  $X$ , we have*

$$\text{ps}_i\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \in \{-1, \infty\} \cup \tau^+, \quad i \in \{0, 1\}$$

and, therefore,

$$\text{ps}_i\text{-}dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X) \in \{-1, \infty\} \cup \tau^+, \quad i \in \{0, 1\}.$$

*Proof:* We prove that

$$\text{ps}_0\text{-Dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}(Q, X) \in \{-1, \infty\} \cup \tau^+. \quad (4)$$

The case  $i = 1$  is similar. Suppose that relation (4) is not true. Let  $\alpha$  be the minimal element of  $\mathcal{O} \setminus \tau^+$  such that there exists a space  $X$  and a subset  $Q$  of  $X$  with  $\text{ps}_0\text{-Dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}(Q, X) = \alpha$ . Let  $B = \{U_\delta : \delta \in \tau\}$  be the  $\text{ID}$ -ps-base for  $Q$  in  $X$  mentioned in Definition 1.3(2). Denote by  $P$  the set of all pairs  $(\delta_1, \delta_2) \in \tau \times \tau$  with

$$\text{Cl}_X(U_{\delta_1}) \cap \text{Cl}_X(U_{\delta_2}) = \emptyset.$$

For every  $(\delta_1, \delta_2) \in P$ , let  $L(\delta_1, \delta_2)$  be a subset of  $X$  separating the sets  $\text{Cl}_X(U_{\delta_1})$  and  $\text{Cl}_X(U_{\delta_2})$  with

$$\text{ps}_0\text{-dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}(Q \cap L(\delta_1, \delta_2), L(\delta_1, \delta_2)) = \beta(\delta_1, \delta_2) < \alpha.$$

First, we suppose that  $\beta(\delta_1, \delta_2) < \tau^+$  for every  $(\delta_1, \delta_2) \in P$ . Since  $|P| \leq \tau$ , there exists an ordinal  $\beta \in \tau^+$  such that  $\beta(\delta_1, \delta_2) < \beta$  for every  $(\delta_1, \delta_2) \in P$ . Then

$$\text{ps}_0\text{-dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}(Q \cap L(\delta_1, \delta_2), L(\delta_1, \delta_2)) < \beta$$

and, by Definition 1.3(2),  $\text{ps}_0\text{-Dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}(Q, X) \leq \beta$ , which is a contradiction. Now, we suppose that there exists  $(\delta_1, \delta_2) \in P$  such that  $\tau^+ \leq \beta(\delta_1, \delta_2)$ . Since

$$\text{ps}_0\text{-dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}(Q \cap L(\delta_1, \delta_2), L(\delta_1, \delta_2)) = \beta(\delta_1, \delta_2),$$

there exist closed subsets  $L_i$  of  $L(\delta_1, \delta_2)$ ,  $i \in \nu$ , such that

- (a)  $L(\delta_1, \delta_2) = \cup\{L_i : i \in \nu\}$ ,
- (b)  $(L_i, L(\delta_1, \delta_2)) \in \mathbb{K}$ , and
- (c)  $\text{ps}_0\text{-Dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}(Q \cap L_i, L_i) = \beta_i \leq \beta(\delta_1, \delta_2) < \alpha$ .

If  $\beta_i < \tau^+$  for all  $i \in \nu$ , then there exists an ordinal  $\beta \in \tau^+$  such that  $\beta_i \leq \beta$ , which means that  $\text{ps}_0\text{-Dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}(Q \cap L_i, L_i) \leq \beta$ . Therefore,

$$\text{ps}_0\text{-dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}(Q \cap L(\delta_1, \delta_2), L(\delta_1, \delta_2)) \leq \beta < \tau^+ \leq \beta(\delta_1, \delta_2),$$

which is a contradiction. Thus, there exists  $i \in \nu$  such that

$$\tau^+ \leq \text{ps}_0\text{-Dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}(Q \cap L_i, L_i) < \alpha.$$

The last relation contradicts the choice of the ordinal  $\alpha$ , completing the proof of the proposition.  $\square$



**Notation.** Let  $\mathbb{B}$  be a class of bases and  $\mathbb{E}$  a  $\mathbb{B}$ -hereditary-separated class of spaces (see [4, Definition 2.1]). We set

$$\mathbb{D}_{\mathbb{B}} = \{(Q, B, X) : Q \subseteq X \text{ and } (B, X) \in \mathbb{B}\}$$

and

$$\mathbb{F}_{\mathbb{E}} = \{(Q, X) : Q \subseteq X \text{ and } X \in \mathbb{E}\}.$$

It is easy to see that the class  $\mathbb{F}_{\mathbb{E}}$  of subsets is  $\mathbb{D}_{\mathbb{B}}$ - $p_i$ -hereditary separated,  $i \in \{0, 1\}$ .

By induction, the following proposition can be proved.

**Proposition 2.4.** *Let  $\mathbb{D} \supseteq \mathbb{D}_{\mathbb{B}}$ . For every subset  $Q$  of a space  $X$ , we have*

$$\text{ps}_0\text{-}dm_{\mathbb{F}_{\mathbb{E}}, \nu}^{\mathbb{K}, \mathbb{D}}(Q, X) \leq dm_{\mathbb{E}, \nu}^{\mathbb{K}, \mathbb{B}}(X)$$

and

$$\text{ps}_0\text{-}Dm_{\mathbb{F}_{\mathbb{E}}, \nu}^{\mathbb{K}, \mathbb{D}}(Q, X) \leq Dm_{\mathbb{E}, \nu}^{\mathbb{K}, \mathbb{B}}(X).$$

In particular,

$$\text{ps}_0\text{-}dm_{\nu}^{\mathbb{K}, \mathbb{D}}(Q, X) \leq dm_{\nu}^{\mathbb{K}, \mathbb{B}}(X)$$

and

$$\text{ps}_0\text{-}Dm_{\nu}^{\mathbb{K}, \mathbb{D}}(Q, X) \leq Dm_{\nu}^{\mathbb{K}, \mathbb{B}}(X).$$

**Definition 2.5.** It is said that a class  $\mathbb{K}$  of subsets is *closed with respect to the subspaces* if  $(Q \cap Y, Y) \in \mathbb{K}$  for every  $(Q, X) \in \mathbb{K}$  and  $Y \subseteq X$ .

**Proposition 2.6.** *Let  $\mathbb{D}$  be a class of  $p$ -bases and  $\mathbb{K}$  be a class of subsets closed with respect to the subspaces. For every subset  $Q$  of a regular space  $X$ , we have*

$$dm(Q) \leq \text{pos}_i\text{-}dm_{\omega}^{\mathbb{K}, \mathbb{D}}(Q, X)$$

and

$$Dm(Q) \leq \text{pos}_i\text{-}Dm_{\omega}^{\mathbb{K}, \mathbb{D}}(Q, X), \quad i \in \{0, 1\}.$$

*Proof:* We prove by induction that

$$dm(Q) \leq \text{pos}_0\text{-}dm_{\omega}^{\mathbb{K}, \mathbb{D}}(Q, X) \tag{5}$$

and

$$Dm(Q) \leq \text{pos}_0\text{-}Dm_{\omega}^{\mathbb{K}, \mathbb{D}}(Q, X). \tag{6}$$

The case  $i = 1$  is similar. If

$$\text{pos}_0\text{-}dm_{\omega}^{\mathbb{K}, \mathbb{D}}(Q, X) = \text{pos}_0\text{-}Dm_{\omega}^{\mathbb{K}, \mathbb{D}}(Q, X) = -1,$$

then  $Q = \emptyset$  and  $dm(Q) = Dm(Q) = -1$ , proving the proposition. Suppose that relation (5) is true for every pair  $(Q, X)$  with  $\text{pos}_0\text{-}dm_\omega^{\mathbb{K}, \mathbb{D}}(Q, X) < \alpha$  and that relation (6) is true for every pair  $(Q, X)$  with  $\text{pos}_0\text{-}Dm_\omega^{\mathbb{K}, \mathbb{D}}(Q, X) < \alpha$ , where  $\alpha \in \mathcal{O}$ . First we suppose that  $(Q, X)$  is a pair with  $\text{pos}_0\text{-}Dm_\omega^{\mathbb{K}, \mathbb{D}}(Q, X) = \alpha$ . We need to prove  $Dm(Q) \leq \alpha$ . Let  $x$  and  $y$  be two distinct points of  $Q$ . Since  $\text{pos}_0\text{-}Dm_\omega^{\mathbb{K}, \mathbb{D}}(Q, X) = \alpha$ , there exists a  $\mathbb{D}$ -pos-base  $B = \{U_\delta : \delta \in \tau\}$  for  $Q$  in  $X$  such that for every two elements  $U_{\delta_1}, U_{\delta_2}$  of  $B$  with  $\text{Cl}_X(U_{\delta_1}) \cap \text{Cl}_X(U_{\delta_2}) = \emptyset$  there exists a subset  $L$  of  $X$  separating  $\text{Cl}_X(U_{\delta_1})$  and  $\text{Cl}_X(U_{\delta_2})$  with  $\text{pos}_0\text{-}dm_\omega^{\mathbb{K}, \mathbb{D}}(Q \cap L, L) < \alpha$ . Since  $X$  is regular, there exist  $U_{\delta_1}, U_{\delta_2} \in B$  such that

$$x \in \text{Cl}_X(U_{\delta_1}), y \in \text{Cl}_X(U_{\delta_2}), \text{ and } \text{Cl}_X(U_{\delta_1}) \cap \text{Cl}_X(U_{\delta_2}) = \emptyset.$$

Thus, there exists a subset  $L$  of  $X$  separating the sets  $\text{Cl}_X(U_{\delta_1})$  and  $\text{Cl}_X(U_{\delta_2})$  with  $\text{pos}_0\text{-}dm_\omega^{\mathbb{K}, \mathbb{D}}(Q \cap L, L) < \alpha$ . Obviously,  $Q \cap L$  separates the singletons  $\{x\}$  and  $\{y\}$  in  $Q$ . Moreover, by inductive assumption,

$$dm(Q \cap L) \leq \text{pos}_0\text{-}dm_\omega^{\mathbb{K}, \mathbb{D}}(Q \cap L, L) < \alpha.$$

Therefore,  $Dm(Q) \leq \alpha$ . Now, we suppose that  $(Q, X)$  is a pair with  $\text{pos}_0\text{-}dm_\omega^{\mathbb{K}, \mathbb{D}}(Q, X) = \alpha$ . We need to prove that  $dm(Q) \leq \alpha$ . We have  $X = \cup\{S_i : i \in \omega\}$ , where

- (a)  $S_i$  is closed subset of  $X$ ,
- (b)  $(S_i, X) \in \mathbb{K}$ , and
- (c)  $\text{pos}_0\text{-}Dm_\omega^{\mathbb{K}, \mathbb{D}}(Q \cap S_i, S_i) \leq \alpha$ .

For every  $i \in \omega$ , we set  $Q_i = Q \cap S_i$ . Obviously,  $Q_i$  is closed subset of  $Q$ . Since the class  $\mathbb{K}$  is closed with respect to the subspaces,  $(Q_i, Q) \in \mathbb{K}$ . By the preceding,  $Dm(Q_i) \leq \alpha, i \in \omega$ . Therefore,  $dm(Q) \leq \alpha$ . The proposition is trivial if

$$\text{pos}_0\text{-}dm_\omega^{\mathbb{K}, \mathbb{D}}(Q, X) = \text{pos}_0\text{-}Dm_\omega^{\mathbb{K}, \mathbb{D}}(Q, X) = \infty. \quad \square$$

### 3. THE SUBSPACE, PRODUCT, AND SUM THEOREMS

**Proposition 3.1.** *Let  $i \in \{0, 1\}$  and  $Q$  and  $K$  be two subsets of a space  $X$  with  $K \subseteq Q$ . Then*

- (a)  $p_i\text{-}dm_\nu^{\mathbb{K}, \mathbb{D}}(K, X) \leq p_i\text{-}dm_\nu^{\mathbb{K}, \mathbb{D}}(Q, X)$   
 $p_i\text{-}Dm_\nu^{\mathbb{K}, \mathbb{D}}(K, X) \leq p_i\text{-}Dm_\nu^{\mathbb{K}, \mathbb{D}}(Q, X),$
- (b)  $\text{pos}_i\text{-}dm_\nu^{\mathbb{K}, \mathbb{D}}(K, X) \leq \text{pos}_i\text{-}dm_\nu^{\mathbb{K}, \mathbb{D}}(Q, X)$   
 $\text{pos}_i\text{-}Dm_\nu^{\mathbb{K}, \mathbb{D}}(K, X) \leq \text{pos}_i\text{-}Dm_\nu^{\mathbb{K}, \mathbb{D}}(Q, X),$  and

$$(c) \text{ps}_i\text{-}dm_\nu^{\mathbb{K},\mathbb{D}}(K, X) \leq \text{ps}_i\text{-}dm_\nu^{\mathbb{K},\mathbb{D}}(Q, X) \\ \text{ps}_i\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(K, X) \leq \text{ps}_i\text{-}Dm_{\mathbb{F},\nu}^{\mathbb{K},\mathbb{D}}(Q, X).$$

*Proof:* We prove the inequalities

$$p_0\text{-}dm_\nu^{\mathbb{K},\mathbb{D}}(K, X) \leq p_0\text{-}dm_\nu^{\mathbb{K},\mathbb{D}}(Q, X) \quad (7)$$

and

$$p_0\text{-}Dm_\nu^{\mathbb{K},\mathbb{D}}(K, X) \leq p_0\text{-}Dm_\nu^{\mathbb{K},\mathbb{D}}(Q, X). \quad (8)$$

The proofs of all other inequalities are similar. If

$$p_0\text{-}dm_\nu^{\mathbb{K},\mathbb{D}}(Q, X) = p_0\text{-}Dm_\nu^{\mathbb{K},\mathbb{D}}(Q, X) = -1,$$

then  $Q = X = \emptyset$  and, therefore,  $K = X = \emptyset$ . Hence,

$$p_0\text{-}dm_\nu^{\mathbb{K},\mathbb{D}}(K, X) = p_0\text{-}Dm_\nu^{\mathbb{K},\mathbb{D}}(K, X) = -1.$$

Suppose that relation (7) is true for any  $K \subseteq Q \subseteq X$  with  $p_0\text{-}dm_\nu^{\mathbb{K},\mathbb{D}}(Q, X) < \alpha$  and that relation (8) is true for any  $K \subseteq Q \subseteq X$  with  $p_0\text{-}Dm_\nu^{\mathbb{K},\mathbb{D}}(Q, X) < \alpha$ , where  $\alpha \in \mathcal{O}$ . First, we suppose that  $p_0\text{-}Dm_\nu^{\mathbb{K},\mathbb{D}}(Q, X) = \alpha$  for some  $K \subseteq Q \subseteq X$ . We need to prove  $p_0\text{-}Dm_\nu^{\mathbb{K},\mathbb{D}}(K, X) \leq \alpha$ . Since  $p_0\text{-}Dm_\nu^{\mathbb{K},\mathbb{D}}(Q, X) = \alpha$ , there exists a  $\mathbb{D}$ -p-base  $B = \{U_\delta : \delta \in \tau\}$  for  $Q$  in  $X$  such that for every two elements  $U_{\delta_1}, U_{\delta_2}$  of  $B$  with  $\text{Cl}_X(U_{\delta_1}) \cap \text{Cl}_X(U_{\delta_2}) = \emptyset$  there exists a subset  $L$  of  $X$  separating  $\text{Cl}_X(U_{\delta_1})$  and  $\text{Cl}_X(U_{\delta_2})$  with  $p_0\text{-}dm_\nu^{\mathbb{K},\mathbb{D}}(Q \cap L, L) < \alpha$ . By inductive assumption,

$$p_0\text{-}dm_\nu^{\mathbb{K},\mathbb{D}}(K \cap L, L) \leq p_0\text{-}dm_\nu^{\mathbb{K},\mathbb{D}}(Q \cap L, L) < \alpha,$$

and since  $B$  is also a  $\mathbb{D}$ -p-base for  $K$  in  $X$ ,  $p_0\text{-}Dm_\nu^{\mathbb{K},\mathbb{D}}(K, X) \leq \alpha$ . Now, we suppose that  $p_0\text{-}dm_\nu^{\mathbb{K},\mathbb{D}}(Q, X) = \alpha$  for some  $K \subseteq Q \subseteq X$ . We need to prove that  $p_0\text{-}dm_\nu^{\mathbb{K},\mathbb{D}}(K, X) \leq \alpha$ . We have

$$X = \cup\{S_i : i \in \nu\},$$

where

- (a)  $S_i$  is closed subset of  $X$ ,
- (b)  $(S_i, X) \in \mathbb{K}$ , and
- (c)  $p_0\text{-}Dm_\nu^{\mathbb{K},\mathbb{D}}(Q \cap S_i, S_i) \leq \alpha$ .

By the preceding,

$$p_0\text{-}Dm_\nu^{\mathbb{K},\mathbb{D}}(K \cap S_i, S_i) \leq p_0\text{-}Dm_\nu^{\mathbb{K},\mathbb{D}}(Q \cap S_i, S_i) \leq \alpha, \quad i \in \nu.$$

Therefore,  $p_0\text{-}dm_\nu^{\mathbb{K},\mathbb{D}}(K, X) \leq \alpha$ . The proposition is trivial if

$$p_0\text{-}dm_\nu^{\mathbb{K},\mathbb{D}}(Q, X) = p_0\text{-}Dm_\nu^{\mathbb{K},\mathbb{D}}(Q, X) = \infty. \quad \square$$

**Definition 3.2.** It is said that

- (a) a class  $\mathbb{K}$  of subsets and
- (b) a class  $\mathbb{ID}$  of p-bases

are *closed with respect to the products* if we have, respectively,

- (a)  $(Q^X \times Q^Y, X \times Y) \in \mathbb{K}$  for every two elements  $(Q^X, X)$  and  $(Q^Y, Y)$  of  $\mathbb{K}$  and
- (b)  $(Q^X \times Q^Y, B^{X \times Y}, X \times Y) \in \mathbb{ID}$  for every two elements  $(Q^X, B^X, X)$  and  $(Q^Y, B^Y, Y)$  of  $\mathbb{ID}$ , where  $B^{X \times Y} = \{U^X \times U^Y : U^X \in B^X, U^Y \in B^Y\}$ .

**Proposition 3.3.** *Let  $i \in \{0, 1\}$ ,  $Q^X$  be a subset of a space  $X$ , and  $Q^Y$  be a subset of a space  $Y$ . Then,*

$$p_i\text{-dm}_\nu^{\mathbb{K}, \mathbb{ID}}(Q^X \times Q^Y, X \times Y) \leq p_i\text{-dm}_\nu^{\mathbb{K}, \mathbb{ID}}(Q^X, X)(+)p_i\text{-dm}_\nu^{\mathbb{K}, \mathbb{ID}}(Q^Y, Y) \quad (9)$$

and

$$p_i\text{-Dm}_\nu^{\mathbb{K}, \mathbb{ID}}(Q^X \times Q^Y, X \times Y) \leq p_i\text{-Dm}_\nu^{\mathbb{K}, \mathbb{ID}}(Q^X, X)(+)p_i\text{-Dm}_\nu^{\mathbb{K}, \mathbb{ID}}(Q^Y, Y), \quad (10)$$

where  $\mathbb{K}$  and  $\mathbb{ID}$  are classes closed with respect to the products.

*Proof:* We prove the proposition by induction and only for the case  $i = 1$ . The case  $i = 0$  is similar. If

$$p_1\text{-dm}_\nu^{\mathbb{K}, \mathbb{ID}}(Q^X, X)(+)p_1\text{-dm}_\nu^{\mathbb{K}, \mathbb{ID}}(Q^Y, Y) = -1$$

or

$$p_1\text{-Dm}_\nu^{\mathbb{K}, \mathbb{ID}}(Q^X, X)(+)p_1\text{-Dm}_\nu^{\mathbb{K}, \mathbb{ID}}(Q^Y, Y) = -1,$$

then  $Q^X$  and  $Q^Y$  are empty and so

$$p_1\text{-dm}_\nu^{\mathbb{K}, \mathbb{ID}}(Q^X \times Q^Y, X \times Y) = p_1\text{-Dm}_\nu^{\mathbb{K}, \mathbb{ID}}(Q^X \times Q^Y, X \times Y) = -1.$$

Suppose that inequality (9) is true for any  $Q^X \subseteq X$  and  $Q^Y \subseteq Y$  with

$$p_1\text{-dm}_\nu^{\mathbb{K}, \mathbb{ID}}(Q^X, X)(+)p_1\text{-dm}_\nu^{\mathbb{K}, \mathbb{ID}}(Q^Y, Y) < \alpha$$

and inequality (10) is true for any  $Q^X \subseteq X$  and  $Q^Y \subseteq Y$  with

$$p_1\text{-Dm}_\nu^{\mathbb{K}, \mathbb{ID}}(Q^X, X)(+)p_1\text{-Dm}_\nu^{\mathbb{K}, \mathbb{ID}}(Q^Y, Y) < \alpha,$$

where  $\alpha$  is a fixed ordinal. First, we suppose that

$$p_1\text{-Dm}_\nu^{\mathbb{K}, \mathbb{ID}}(Q^X, X)(+)p_1\text{-Dm}_\nu^{\mathbb{K}, \mathbb{ID}}(Q^Y, Y) = \alpha$$

for some  $Q^X \subseteq X$  and  $Q^Y \subseteq Y$ . We need to prove that

$$\text{p}_1\text{-Dm}_\nu^{\mathbb{K}, \mathbb{D}}(Q^X \times Q^Y, X \times Y) \leq \alpha.$$

If  $\text{p}_1\text{-Dm}_\nu^{\mathbb{K}, \mathbb{D}}(Q^X, X) = -1$  or  $\text{p}_1\text{-Dm}_\nu^{\mathbb{K}, \mathbb{D}}(Q^Y, Y) = -1$ , then  $Q^X \times Q^Y = \emptyset$  and, therefore,

$$\text{p}_1\text{-Dm}_\nu^{\mathbb{K}, \mathbb{D}}(Q^X \times Q^Y, X \times Y) = -1 < \alpha.$$

Let  $\text{p}_1\text{-Dm}_\nu^{\mathbb{K}, \mathbb{D}}(Q^X, X) = \beta$  and  $\text{p}_1\text{-Dm}_\nu^{\mathbb{K}, \mathbb{D}}(Q^Y, Y) = \gamma$ , where  $\beta, \gamma \in \mathcal{O}$ . Then there exist two  $\mathbb{D}$ -p-bases  $B^X = \{U_\delta : \delta \in \tau\}$  and  $B^Y = \{V_\delta : \delta \in \tau\}$  for  $Q^X$  in  $X$  and  $Q^Y$  in  $Y$ , respectively, such that Definition 1.4(2) is satisfied. Since the class  $\mathbb{D}$  is closed with respect to the products, the set

$$B^{X \times Y} = \{U_\delta \times V_{\delta'} : \delta, \delta' \in \tau\}$$

is a  $\mathbb{D}$ -p-base for  $Q^X \times Q^Y$  in  $X \times Y$ . Suppose that  $U_{\delta_1} \times V_{\delta'_1}, U_{\delta_2} \times V_{\delta'_2} \in B^{X \times Y}$ , and

$$\text{Cl}_{X \times Y}(U_{\delta_1} \times V_{\delta'_1}) \cap \text{Cl}_{X \times Y}(U_{\delta_2} \times V_{\delta'_2}) = \emptyset.$$

We have

$$\begin{aligned} & \text{Cl}_{X \times Y}(U_{\delta_1} \times V_{\delta'_1}) \cap \text{Cl}_{X \times Y}(U_{\delta_2} \times V_{\delta'_2}) = \\ & (\text{Cl}_X(U_{\delta_1}) \times \text{Cl}_Y(V_{\delta'_1})) \cap (\text{Cl}_X(U_{\delta_2}) \times \text{Cl}_Y(V_{\delta'_2})) = \\ & (\text{Cl}_X(U_{\delta_1}) \cap \text{Cl}_X(U_{\delta_2})) \times (\text{Cl}_Y(V_{\delta'_1}) \cap \text{Cl}_Y(V_{\delta'_2})) = \emptyset. \end{aligned}$$

If  $\text{Cl}_X(U_{\delta_1}) \cap \text{Cl}_X(U_{\delta_2}) = \emptyset$ , then there exists a subset  $L^X$  of  $X$  separating  $\text{Cl}_X(U_{\delta_1})$  and  $\text{Cl}_X(U_{\delta_2})$  with  $\text{p}_1\text{-dm}_\nu^{\mathbb{K}, \mathbb{D}}(Q^X \cap L^X, X) < \beta$ . Therefore, there exist two open subsets  $W_{\delta_1}$  and  $H_{\delta_2}$  of  $X$  such that

- (a)  $\text{Cl}_X(U_{\delta_1}) \subseteq W_{\delta_1}, \text{Cl}_X(U_{\delta_2}) \subseteq H_{\delta_2}$ ,
- (b)  $W_{\delta_1} \cap H_{\delta_2} = \emptyset$ , and
- (c)  $X \setminus L^X = W_{\delta_1} \cup H_{\delta_2}$ .

Let  $W = W_{\delta_1} \times Y, H = H_{\delta_2} \times Y$ , and  $P = L^X \times Y$ . Then we have

$$\text{Cl}_{X \times Y}(U_{\delta_1} \times V_{\delta'_1}) = \text{Cl}_X(U_{\delta_1}) \times \text{Cl}_Y(V_{\delta'_1}) \subseteq W,$$

$$\text{Cl}_{X \times Y}(U_{\delta_2} \times V_{\delta'_2}) = \text{Cl}_X(U_{\delta_2}) \times \text{Cl}_Y(V_{\delta'_2}) \subseteq H,$$

$$W \cap H = \emptyset,$$

and

$$(X \times Y) \setminus P = W \cup H,$$

which means that  $P$  separates the sets

$$\text{Cl}_{X \times Y}(U_{\delta_1} \times V_{\delta'_1}) \text{ and } \text{Cl}_{X \times Y}(U_{\delta_2} \times V_{\delta'_2})$$

in the space  $X \times Y$ . By Proposition 2.1(a),

$$\text{p}_1\text{-dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^Y, Y) \leq \text{p}_1\text{-Dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^Y, Y).$$

Therefore,

$$\begin{aligned} & \text{p}_1\text{-dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^X \cap L^X, X)(+) \text{p}_1\text{-dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^Y, Y) \leq \\ & \text{p}_1\text{-dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^X \cap L^X, X)(+) \text{p}_1\text{-Dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^Y, Y) < \beta(+) \gamma = \alpha. \end{aligned}$$

By inductive assumption,

$$\begin{aligned} & \text{p}_1\text{-dm}_{\nu}^{\mathbb{K}, \mathbb{D}}((Q^X \times Q^Y) \cap P, X \times Y) = \\ & \text{p}_1\text{-dm}_{\nu}^{\mathbb{K}, \mathbb{D}}((Q^X \cap L^X) \times Q^Y, X \times Y) \leq \\ & \text{p}_1\text{-dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^X \cap L^X, X)(+) \text{p}_1\text{-Dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^Y, Y) < \alpha. \end{aligned}$$

Similar to the above, if  $\text{Cl}_Y(V_{\delta'_1}) \cap \text{Cl}_Y(V_{\delta'_2}) = \emptyset$ , then in the space  $X \times Y$ , we can construct a subset  $P'$  which separates the subsets  $\text{Cl}_{X \times Y}(U_{\delta_1} \times V_{\delta'_1})$  and  $\text{Cl}_{X \times Y}(U_{\delta_2} \times V_{\delta'_2})$  of  $X \times Y$  such that

$$\text{p}_1\text{-dm}_{\nu}^{\mathbb{K}, \mathbb{D}}((Q^X \times Q^Y) \cap P', X \times Y) < \alpha.$$

Thus,

$$\text{p}_1\text{-Dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^X \times Q^Y, X \times Y) \leq \alpha. \quad (11)$$

Now, we suppose that

$$\text{p}_1\text{-dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^X, X)(+) \text{p}_1\text{-dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^Y, Y) = \alpha$$

for some  $Q^X \subseteq X$  and  $Q^Y \subseteq Y$ . We need to prove that

$$\text{p}_1\text{-dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^X \times Q^Y, X \times Y) \leq \alpha.$$

If  $\text{p}_1\text{-dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^X, X) = -1$  or  $\text{p}_1\text{-dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^Y, Y) = -1$ , then the set  $Q^X \times Q^Y$  is empty and, therefore,

$$\text{p}_1\text{-dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^X \times Q^Y, X \times Y) = -1 < \alpha.$$

Suppose that  $\text{p}_1\text{-dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^X, X) = \beta$  and  $\text{p}_1\text{-dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^Y, Y) = \gamma$ , where  $\beta, \gamma \in \mathcal{O}$ . Then, by Definition 1.4(3), we have

- (a)  $X = \cup\{S_i^X : i \in \nu\}$ , where  $S_i^X$  is closed,  $(S_i^X, X) \in \mathbb{K}$ , and  $\text{p}_1\text{-Dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^X \cap S_i^X, X) \leq \beta$  and
- (b)  $Y = \cup\{S_i^Y : i \in \nu\}$ , where  $S_i^Y$  is closed,  $(S_i^Y, Y) \in \mathbb{K}$ , and  $\text{p}_1\text{-Dm}_{\nu}^{\mathbb{K}, \mathbb{D}}(Q^Y \cap S_i^Y, Y) \leq \gamma$ .

We observe that

$$X \times Y = \cup\{S_i^X \times S_j^Y : i, j \in \nu\},$$

the subset  $S_i^X \times S_j^Y$  of  $X \times Y$  is closed, and

$$\begin{aligned} & \text{p1-}Dm_\nu^{\mathbb{K}, \mathbb{D}}(Q^X \cap S_i^X, X)(+) \\ & \text{p1-}Dm_\nu^{\mathbb{K}, \mathbb{D}}(Q^Y \cap S_j^Y, Y) \leq \beta(+)\gamma = \alpha, i, j \in \nu. \end{aligned}$$

Since the class  $\mathbb{K}$  is closed with respect to the products, we have

$$(S_i^X \times S_j^Y, X \times Y) \in \mathbb{K}.$$

Setting in relation (11)  $Q^X = Q^X \cap S_i^X$  and  $Q^Y = Q^Y \cap S_j^Y$ , we have

$$\begin{aligned} & \text{p1-}Dm_\nu^{\mathbb{K}, \mathbb{D}}((Q^X \times Q^Y) \cap (S_i^X \times S_j^Y), X \times Y) = \\ & \text{p1-}Dm_\nu^{\mathbb{K}, \mathbb{D}}((Q^X \cap S_i^X) \times (Q^Y \cap S_j^Y), X \times Y) \leq \alpha, i, j \in \nu. \end{aligned}$$

Thus, by Definition 1.4(3),

$$\text{p1-}dm_\nu^{\mathbb{K}, \mathbb{D}}(Q^X \times Q^Y, X \times Y) \leq \alpha.$$

Obviously, the proposition is true if

$$\begin{aligned} & \text{p1-}dm_\nu^{\mathbb{K}, \mathbb{D}}(Q^X, X)(+)\text{p1-}dm_\nu^{\mathbb{K}, \mathbb{D}}(Q^Y, Y) = \\ & \text{p1-}Dm_\nu^{\mathbb{K}, \mathbb{D}}(Q^X, X)(+)\text{p1-}Dm_\nu^{\mathbb{K}, \mathbb{D}}(Q^Y, Y) = \infty. \end{aligned}$$

The proof of the proposition is complete.  $\square$

**Remark 3.4.** Proposition 3.3 is also true if we replace the positional dimension-like functions  $\text{p}_i\text{-}dm_\nu^{\mathbb{K}, \mathbb{D}}$  and  $\text{p}_i\text{-}Dm_\nu^{\mathbb{K}, \mathbb{D}}$  by the positional dimension-like functions  $\text{pos}_i\text{-}dm_\nu^{\mathbb{K}, \mathbb{D}}$  and  $\text{pos}_i\text{-}Dm_\nu^{\mathbb{K}, \mathbb{D}}$ , respectively, or by the positional dimension-like functions  $\text{ps}_i\text{-}dm_\nu^{\mathbb{K}, \mathbb{D}}$  and  $\text{ps}_i\text{-}Dm_\nu^{\mathbb{K}, \mathbb{D}}$ , respectively.

**Proposition 3.5.** *Let  $Q$  be a subset of a space  $X$  and  $\mathbb{K}$  be a class of subsets such that  $(K, Y) \in \mathbb{K}$  for every space  $Y$  and for every closed subset  $K$  of  $Y$ . If  $X$  is the union of closed subsets  $F_i$ ,  $i \in \nu$ , such that  $\text{p}_0\text{-}dm_{\mathbb{F}, \nu}^{\mathbb{K}, \mathbb{D}}(Q \cap F_i, F_i) \leq \alpha \in \mathcal{O}$ , then  $\text{p}_0\text{-}dm_{\mathbb{F}, \nu}^{\mathbb{K}, \mathbb{D}}(Q, X) \leq \alpha$ .*

*Proof:* Since  $\text{p}_0\text{-}dm_{\mathbb{F}, \nu}^{\mathbb{K}, \mathbb{D}}(Q \cap F_i, F_i) \leq \alpha$ , Definition 1.3(3) implies that  $F_i = \cup\{S_j^i : j \in \nu\}$  such that for every  $j \in \nu$ , we have

- (a) the subset  $S_j^i$  of  $F_i$  is closed,
- (b)  $(S_j^i, F_i) \in \mathbb{K}$ , and

$$(c) \text{p}_0\text{-dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}(Q \cap S_j^i, S_j^i) \leq \alpha.$$

Since the subset  $S_j^i$  of  $X$  is closed in  $X$ , we have  $(S_j^i, X) \in \mathbb{K}$ . Also  $X = \cup\{S_j^i : i, j \in \nu\}$ . The above conditions mean that  $\text{p}_0\text{-dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}(Q, X) \leq \alpha$ .  $\square$

**Remark 3.6.** Proposition 3.5 is also true if we replace the positional dimension-like function  $\text{p}_0\text{-dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}$  by the positional dimension-like function  $\text{pos}_0\text{-dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}$  or by the positional dimension-like function  $\text{ps}_0\text{-dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}$ .

The proof of the following proposition is similar to the proof of Proposition 3.5.

**Proposition 3.7.** *Let  $Q$  be a subset of a space  $X$  and  $\mathbb{K}$  be a class of subsets such that  $(K, Y) \in \mathbb{K}$  for every space  $Y$  and for every closed subset  $K$  of  $Y$ . If  $X$  is the union of closed subsets  $F_i$ ,  $i \in \nu$ , such that  $\text{p}_1\text{-dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}(Q \cap F_i, X) \leq \alpha \in \mathcal{O}$ , then  $\text{p}_1\text{-dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}(Q, X) \leq \alpha$ .*

**Remark 3.8.** Proposition 3.7 is also true if the positional dimension-like function  $\text{p}_1\text{-dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}$  is replaced by the positional dimension-like function  $\text{pos}_1\text{-dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}$  or by the positional dimension-like function  $\text{ps}_1\text{-dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}$ .

**Question.** Does a sum theorem hold for the positional dimension-like functions  $\text{p}_i\text{-Dm}_{\mathbb{F},\nu}^{\mathbb{K},\text{ID}}$ ,  $i \in \{0, 1\}$ ?

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