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COMPLETENESS TYPE PROPERTIES OF SEMITOPOLOGICAL GROUPS, AND THE THEOREMS OF MONTGOMERY AND ELLIS

ALEXANDER V. ARHANGEL'SKII AND MITROFAN M. CHOBAN

Abstract. In this paper the class of fan-complete spaces is introduced. Every G_δ-subspace of a pseudocompact space is fan-complete. We prove that a paratopological group is a topological group provided it contains a dense fan-complete subspace. Moreover, if a semitopological group contains a dense Čech-complete subspace, then it is a Čech-complete topological group. This improves some results of A. Bouziad, P. Kenderov, I.S. Kortezov, and W.B. Moors. Some other new results are obtained (see, in particular, Theorems 5.1 and 5.3, and Corollaries 5.4, 6.5 and 6.6).

1. Introduction

By a space we understand a regular topological T_1-space. We use the terminology from [7], [17]. Let ω be the first infinite ordinal and the first infinite cardinal, $\omega = \{0, 1, 2, ...\}$. By $cl_X H$ we denote the closure of a set H in a space X. A *paratopological group* is a group with a topology such that the multiplication is jointly continuous, and a *semitopological group* is a group with a topology such that the multiplication is separately continuous. Every paratopological group is a semitopological group. A semitopological group in which the inverse operation is continuous is called a *quasitopological group.*

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The space S of reals with the topology generated by the base consisting of the sets $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$, where $a, b \in \mathbb{R}$ and $a < b$, is called *Sorgenfrey line*. Sorgenfrey line has the following properties (see [7]):

- S is an Abelian paratopological group with the Baire property;
- S is a hereditarily Lindelöf first-countable hereditarily separable non-metrizable space;
- Each metrizable subspace of S is countable.
- Each compact subspace of S is countable.
- S does not admit a structure of a topological group.

In 1936 D. Motgomery [28] has proved the following two theorems:

Theorem 1M. *Every separable semitopological group metrizable by a complete metric is a topological group.*

Theorem 2M. *Every metrizable by a complete metric semitopological group is a paratopological group.*

In 1957 R. Ellis [16] showed that every locally compact semitopological group is a topological group. Further interesting results on semitopological and paratopological groups were established by Z. Zelazko [38], N. Brand [11], L. G. Brown [12], P. Kenderov, I. S. Kortezov, and W. B. Moors [23], E. A. Reznichenko [34], and other authors (see [7], [20], [25], [33]). Some important recent advances in this direction were made by A. Bouziad (see [9], [10]). In [10] A. Bouziad has proved that every Čech-complete (every Čech-analytic with the Baire property) semitopological group is a topological group. A. V. Arhangel'skii and E. A. Reznichenko (see [7], [8]) have proved that a paratopological group G is a topological group provided it is a G_δ-subspace of some pseudocompact space. In [23] P. Kenderov, I. S. Kortezov and W. B. Moors introduced the class of strongly Baire (strongly β-unfavorable) spaces and proved that a strongly Baire semitopological group is a topological group. Some remarkable relations between separate continuity and joint continuity were established in [25], [30], [31], [37], [19], [34]. The monograph [7] contains many references of this kind.

In this paper we develop the methods from [7] and extend the theorems of D. Montgomery and R. Ellis over the very wide class of fan-complete spaces (see Theorems 5.1 and 5.2). Fan-complete spaces have the following properties:

- All compact spaces, all countably compact spaces, and all pseudocompact spaces are fan-complete.
- Every dense G_δ-subspace of a fan-complete space is fan-complete.
- Any image of a fan-complete space under an open continuous mapping is fan-complete.
- Any locally fan-complete space is fan-complete.

Besides, the definition of fan-complete spaces is very transparent and easy to work with.

2. Various types of completeness of topological spaces

For a sequence $\{H_n : n \in \omega\}$ of subsets of a space X, $Lim\{H_n : n \in \omega\}$ is the set of all accumulation points of $\{H_n : n \in \omega\}$. If $H_{n+1} \subseteq H_n$ for any $n \in \omega$, then $Lim\{H_n : n \in \omega\} = \cap\{cl_X H_n : n \in \omega\}$.

A sequence $\{U_n : n \in \omega\}$ of open subsets of a space X is called a *stable sequence* if it satisfies the following conditions:

(S1) $\emptyset \neq U_{n+1} \subseteq U_n$ for any $n \in \omega$;

(S2) Every sequence $\{V_n : n \in \omega\}$ of open non-empty sets in X such that $V_n \subseteq U_n$ for each $n \in \omega$, has an accumulation point in X, i.e. $Lim\{V_n : n \in \omega\} \neq \emptyset$.

A subset L of a space X is *bounded* if for every locally finite family γ of open subsets in X the set $\{U \in \gamma : U \cap L \neq \emptyset\}$ is finite.

A space X is *feebly compact* if every locally finite family of open subsets in X is finite, i.e. X is bounded in X. For Tychonoff spaces, feeble compactness is equivalent to pseudocompactness. Every countably compact space is feebly compact. A subset L of a Tychonoff space X is bounded if and only if every continuous function on X is bounded on L (see [7], [17]).

From conditions (S1) and (S2) it follows that $H = \cap\{cl_X U_n : n \in \omega\} = Lim\{U_n : n \in \omega\}$ is a bounded non-empty subset of X.

A space X is called *μ-complete* if the closure of each bounded subset is compact.

Let Y be a dense subspace of a space X, $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \omega\}$ be a sequence of families of open subsets of X, and let $\pi = \{\pi_n : A_{n+1} \to A_n : n \in \omega\}$ be a sequence of mappings. A sequence $\alpha = \{\alpha_n : n \in \omega\}$ is called a *c-sequence* if $\alpha_n \in A_n$ and $\pi_n(\alpha_{n+1}) = \alpha_n$ for every n. Let $H(\alpha) = \cap\{U_{\alpha_n} ; n \in \omega\}$. Consider the following conditions:

(SC1) $\cup\{U_\beta : \beta \in A_n\}$ is a dense subset of X for each $n \in \omega$.

(SC2) $\cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\}$ is a dense subset of the set U_α for all $\alpha \in A_n$ and $n \in \omega$.

(SC3) $U_\alpha = \cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\}$ for all $\alpha \in A_n$ and $n \in \omega$.

(SC4) $\cup\{cl_X U_\beta : \beta \in \pi_n^{-1}(\alpha)\} \subseteq U_\alpha$ for all $\alpha \in A_n$ and $n \in \omega$.

(C1) For any c-sequence $\alpha = \{\alpha_n \in A_n : n \in \omega\}$, the sequence $\{U_{\alpha_n}; n \in \omega\}$ is stable.

(C2) For any c-sequence $\alpha = \{\alpha_n \in A_n : n \in \omega\}$, each sequence $\{y_n \in Y \cap U_{\alpha_n}; n \in \omega\}$ has an accumulation point in X.

(C3) For any c-sequence $\alpha = \{\alpha_n \in A_n : n \in \omega\}$, the sequence $\{U_{\alpha_n}; n \in \omega\}$ is a base of open neighbourhoods of the set $H(\alpha)$ in X.

(C4) For any c-sequence $\alpha = \{\alpha_n \in A_n : n \in \omega\}$, the set $H(\alpha)$ is a non-empty compact subset of X.

(C5) For any c-sequence $\alpha = \{\alpha_n \in A_n : n \in \omega\}$, the set $H(\alpha)$ is a non-empty countably compact subset of X.

Sequence γ and π are called a wA-sieve if they have the Property (SC3) and each γ_n is a cover of X;

They are an A-sieve if they have the Properties (SC3), (SC4) and each γ_n is a cover of X;

They are called a dense wA-sieve if they have the Properties (SC1), (SC2), and a dense A-sieve if they have the Properties (SC1), (SC2), (SC4).

A space X is called *densely sieve-complete* if there exist a dense subspace Y and a dense A-sieve with the Properties (C2) and (C4). A space X is called *sieve-complete* if there exists an A-sieve with the Properties (C2) and (C4) for $Y = X$.

A space X is called *densely q-complete* if there exist a dense subspace Y and a dense A-sieve with the Property (C2). A space X is called *q-complete* if there exists an A-sieve with the Propertiies (C2) and (C5) for $Y = X$.

A space X is called *densely fan-complete* if there exists a dense A-sieve on X with the Property (C1). A space X is called *fan-complete* if there exists an A-sieve on X with the Property (C1).

The sieve-complete and q-complete spaces were examined in [13], [14], [35], [36], [32].

Following E. Michael [27] and [23], a point $x \in X$ is called a *q_D-point* if there exist a dense subspace Y of X and a sequence of neighbourhoods $\{U_n : n \in \omega\}$ of the point x in X such that if $x_n \in Y \cap U_n$, then the sequence $\{x_n : n \in \omega\}$ has a cluster point in X.

If $Y = X$, then x is a q-point. A space X is called a *densely q-space* if each $x \in X$ is a q_D-point. A space X is called a q-space if each $x \in X$ is a q-point. Any q-complete space is a q-space.

Let $\{\gamma_n = \{U_\alpha : \alpha \in A_n\}, \pi_n : A_{n+1} \to A_n : n \in \omega\}$ be a densely wA-sieve with the Property (C2) and $X_1 = \cup\{H(\alpha) : \alpha$ *is a c-sequence*$\}$. Then every point $x \in X_1$ is a q_D-point. The set X_1 is dense in X.

A point $x \in X$ is called a *point of countable type* if there exists a compact subset F with a countable base of open neighbourhoods $\{U_n : n \in \omega\}$ in X such that $x \in F$. A space X is called a space of *pointwise countable type* if each $x \in X$ is a point of countable type [3], [17].

Remark 2.1. By Proposition 2.10 from [14], for every wA-sieve $\mathcal{U}$ $= \{\gamma_n = \{U_\alpha : \alpha \in A_n\}, \pi_n : A_{n+1} \to A_n : n \in \omega\}$ there exist an A-sieve $\mathcal{V} = \{\xi_n = \{V_\beta : \beta \in B_n\}, q_n : B_{n+1} \to B_n : n \in \omega\}$ and the mappings $\{h_n : B_n \to A_n : n \in \omega\}$ such that:

1. $cl_X V_\beta \subseteq U_{q_n(\beta)}$ and $h_n \circ q_n = \pi_n \circ h_{n+1}$ for each $\beta \in B_n$ and for any $n \in \omega$.
2. If $\mathcal{U}$ has Property (C1), then $\mathcal{V}$ has Property (C1) too.
3. If $\mathcal{U}$ has Properties (C2) and (C3), then $\mathcal{V}$ has Properties (C2) and (C3) too.
4. If $\mathcal{U}$ has Properties (C2) and (C4), then $\mathcal{V}$ has Properties (C2) and (C4) too.

Moreover, if Y is a dense subspace of X and $\mathcal{U}$ is a dense wA-sieve, then $\mathcal{V}$ is a dense A-sieve with properties:

- the family ξ_n is disjoint for any $n \in \omega$;
- $q_n^{-1}(\beta) = \{\mu \in B_{n+1} : V_\mu \subseteq V_\beta\}$ for each $\beta \in B_n$ and for any $n \in \omega$;
- if k $\in \{1, 2, 3, 4, 5\}$ and $\mathcal{U}$ has Property (Ck), then $\mathcal{V}$ has Property (Ck), too.

Remark 2.2. Every feebly compact space is fan-complete. Since every Tychonoff space is a closed subspace of some pseudocompact space, a closed subspace of a fan-complete space needn't be fan-complete. The following statements are obvious:

- Any closed subspace of a sieve-complete space is sieve-complete;
- Any closed subspace of a q-complete space is q-complete;
- Any q-complete space is densely q-complete;

- Every sieve-complete space is q-complete, and every q-complete space is fan-complete;
- Every fan-complete μ-complete space is sieve-complete;
- A space X is densely fan-complete if and only if it contains a dense fan-complete subspace.

Proposition 2.3. *Every G_δ-subspace of a fan-complete space is fan-complete. In particular, every Čech-complete space is sieve-complete.*

Proof. Let $\mathcal{U} = \{\gamma_n = \{U_\alpha : \alpha \in A_n\}, \pi_n : A_{n+1} \to A_n : n \in \omega\}$ be an A-sieve with the Property (C1) on a space X, and let $Y = \cap\{U_n : n \in \omega\}$, where $\{U_n : n \in \omega\}$ is a sequence of open subsets of X. Then there exist a sequence $\{\xi_n = \{V_\beta : \beta \in B_n\} : n \in \omega\}$ of families of open subsets of X, sequences of mappings $\{q_n : B_{n+1} \to B_n : n \in \omega\}$ and $\{h_n : B_n \to A_n : n \in \omega\}$ such that:
- $cl_X V_\beta \subseteq U_n \cap U_{q_n(\beta)}$ and $h_n \circ q_n = \pi_n \circ h_{n+1}$ for each $\beta \in B_n$ and for any $n \in \omega$;
- $V_\mu \cap Y = \cup\{V_\beta \cap Y : \beta \in q_n^{-1}(\mu)\} = \cup\{Y \cap cl_X U_\beta : \beta \in \pi_n^{-1}(\alpha)\}$ for all $\mu \in B_n$ and $n \in \omega$.

Then $\mathcal{W} = \{W_\beta = Y \cap V_\beta : \beta \in B_n\}, q_n : B_{n+1} \to B_n : n \in \omega\}$ is an A-sieve with the Property (C1) on Y. □

The next statement can be proved in a standard way.

Proposition 2.4. *If a space X contains a dense fan-complete subspace, then X has the Baire property.*

A space X is said to be a *complete M-space* (see [29]) if there exists a continuous closed mapping $f : X \to Y$ onto a complete metrizable space Y such that the fibers $f^{-1}(y)$ are countably compact. In this case we say that f is a *quasiperfect* mapping. Every complete M-space is q-complete.

For a metric space X the following theorem was proved in [15].

Theorem 2.5. *Let $f : X \to Y$ be an open continuous mapping of a space X onto a space Y, and X_1 be a dense fan-complete subspace of X. Then there exists a dense fan-complete subspace Y_1 of Y such that $f(X_1) \subseteq Y_1$. Moreover, if X, Y are Tychonoff spaces, and $\beta f : \beta X \to \beta Y$ is the continuous extension of f, then there exists a paracompact G_δ-subspace Z of βX such that:*
- *$S = \beta f(Z)$ is a paracompact dense subspace of βY;*

- $g = \beta f|Z : Z \to S$ is a perfect mapping, $S_1 = Y \cap S$ is a dense G_δ-subspace of Y_1 and $Z \cap f^{-1}(y) = X \cap g^{-1}(y) \subseteq g^{-1}(y) = cl_{\beta X}(Z \cap f^{-1}(y))$ for each $y \in S_1$;

- if X_1 is a q-complete space, $Z_1 = Z \cap X$ and $S_1 = f(Z_1)$, then $h = f|Z_1 : Z_1 \to S_1$ is a quasiperfect mapping, and Z_1 is a complete M-space;

- if X_1 is a sieve-complete space, then $Z \subseteq X$;

- if X or X_1 is μ-complete, then $Z \subseteq X$.

Proof. There exist a sequence $\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \omega\}$ of families of open subsets of X and a sequence $\{\pi_n : A_{n+1} \to A_n : n \in \omega\}$ of mappings such that:

(RC1) $X_1 \subseteq \cup\{U_\alpha : \alpha \in A_0\}$ and $U_\alpha \cap X_1 \subseteq \cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\} \subseteq U_\alpha$ for all $\alpha \in A_n$ and $n \in \omega$;

(RC2) for any c-sequence $\alpha = \{\alpha_n \in A_n : n \in \omega\}$, the sequence $\{X_1 \cap U_{\alpha_n}; n \in \omega\}$ is stable in X_1.

Let $\eta_n = \{V_\alpha = f(U_\alpha) : \alpha \in A_n\}$ for any $n \in \omega$, $X(\alpha) = \cap\{U_{\alpha_n} : n \in \omega\}$ and $Y(\alpha) = \cap\{V_{\alpha_n} : n \in \omega\}$ for any c-sequence $\alpha = \{\alpha_n : n \in \omega\}$. We put $X_2 = \cup\{X(\alpha) : \alpha \textit{ is a c-sequence}\}$ and $Y_1 = \cup\{Y(\alpha) : \alpha \textit{ is a c-sequence}\}$. By the construction, $f(X(\alpha)) \subseteq Y(\alpha)$, $X_1 \subseteq X_2$ and $f(X_2) \subseteq Y_1$.

It is obvious that $Y_1 \subseteq \cup\{V_\alpha : \alpha \in A_0\}$ and $V_\alpha \cap Y_1 \subseteq \cup\{V_\beta : \beta \in \pi_n^{-1}(\alpha)\} \subseteq V_\alpha$ for all $\alpha \in A_n$ and $n \in \omega$. Let $\alpha = \{\alpha_n : n \in \omega\}$ be a c-sequence and $\{V_n : n \in \omega\}$ be a sequence of open non-empty sets in Y such that $V_n \subseteq V_{\alpha_n}$ for any $n \in \omega$. We put $U_n = U_{\alpha_n} \cap f^{-1}(V_n)$. Then $Lim\{U_n : n \in \omega\} \neq \emptyset$ and $f(Lim\{U_n : n \in \omega\}) \subseteq Lim\{V_n : n \in \omega\}$. Thus, Y_1 is a dense fan-complete subspace of Y.

Assume that X and Y are Tychonoff spaces. Then there are disjoint families $\{\xi_n = \{H_\beta : \beta \in B_n\} : n \in \omega\}$ of open subsets of βX, disjoint families $\{\xi_n = \{W_\beta : \beta \in B_n\} : n \in \omega\}$ of open subsets of βY, mappings $\{q_n : B_n \to A_n : n \in \omega\}$ and mappings $\{p_n : B_{n+1} \to B_n : n \in \omega\}$ such that:

- $W_n = \{W_\beta : \beta \in B_n\}$ is an open dense subset of βY for each $n \in \omega\}$;
- $\cup\{cl_{\beta X} H_\mu : \mu \in p_n^{-1}(\beta)\} \subseteq H_\beta$ and $\cup\{cl_{\beta Y} W_\mu : \mu \in p_n^{-1}(\beta)\} \subseteq W_\beta$ for all $\beta \in B_n$ and $n \in \omega\}$;
- $X \cap H_\beta \subseteq U_{p_n(\beta)}$ and $\beta f(H_\beta) = W_\beta$ for all $\beta \in B_n$ and $n \in \omega\}$;
- $\pi_n \circ q_{n+1} = q_n \circ p_{n+1}$ for each $n \in \omega\}$.

We put $H_n = \{H_\beta : \beta \in B_n\}$, $Z = \cap\{H_n : n \in \omega\}$ and $S = \cap\{W_n : n \in \omega\}$. The subspaces Z, S and the mapping $g = \beta f|Z : Z \to S$ are what we want. □

Corollary 2.6. *If a μ-complete space X contains a dense fan-complete subspace, then X contains a dense Čech-complete paracompact subspace.*

Theorem 2.7. *Every paracompact fan-complete space is Čech-complete.*

Proof. If X is a paracompact fan-complete space, then there exist a sequence $\{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \omega\}$ of open locally finite covers and a sequence $\{\pi_n : A_{n+1} \to A_n : n \in \omega\}$ of mappings with the properties (C1), (C2) such that U_α is a F_σ-set in X for all $\alpha \in A_n$ and $n \in \omega$. Fix continuous non-negative functions $\{g_\alpha : \alpha \in A_n, n \in \omega\}$ such that $\Sigma\{g_\beta : \beta \in A_n\} = 2^{-n}$ and $g_\alpha^{-1}(0) = X \setminus U_\alpha$ for all $\alpha \in A_n$ and $n \in \omega$. Then $\rho(x,y) = \Sigma\{g_\alpha : \alpha \in A_n, n \in \omega\}$ is a continuous pseudometric on X. Thus, there exist a metric space (Y,d) and a continuous mapping $f : X \to Y$ onto Y such that $\rho(x,y) = d(f(x), f(y))$ for all $x, y \in X$. Fix $b \in X$ and a c-sequence $\alpha = \{\alpha_n : n \in \omega\}$ such that $b \in \cap\{U_{\alpha_n} : n \in \omega\}$. There exists a sequence $\{\varepsilon(n) : n \in \omega\}$ such that $\varepsilon(n) > \varepsilon(n+1) > 0$ and $V_n = \{y \in X : \rho(x,y) < \varepsilon(n)\} \subseteq U_{\alpha_n}$ for any $n \in \omega$. Since $f^{-1}(f(b)) \subseteq H = \cap\{U_{\alpha_n} : n \in \omega\}$ and H is a bounded subset of the paracompact space X, the sets $f^{-1}(f(b))$ and $cl_X H$ are compact, and f is a perfect mapping. Now it is easy to verify that (Y,d) is a complete metric space. □

The following facts are easy to establish:

Proposition 2.8. *Let X be a space such that for every non-empty subset U there exist an open non-empty subset V and a fan-complete subspace Z of X such that $Z \subseteq V \subseteq U \subseteq cl_X Z$. Then X contains a dense fan-complete subspace.*

Proposition 2.9. *Any locally fan-complete space is fan-complete.*

Proposition 2.10. *Let G be a semitopological group or a homogeneous space, V be an open subset of G, Z be a fan-complete subspace, and let $Z \subseteq V \subseteq cl_X Z$. Then:*

1. *G contains some dense fan-complete subspace.*
2. *If $Z = V$, then G is fan-complete.*

A set Z is *normally* in a space X if for any closed subset F of the subspace Z and each open subset U of X that contains F there exists an open subset V of X such that $F \subseteq V \subseteq cl_X V \subseteq U$. Obviously, that Z is a normal closed subspace of X.

Proposition 2.11. *Let Y be a dense subset of X and $\{U_n : n \in \omega\}$ be a sequence of open non-empty subsets of X with the properties:*

- $F = \cap\{U_n : n \in \omega\}$ is normally in X subset;

- $cl_X U_{n+1} \subseteq U_n$ for each $n \in \omega$;

- each sequence $\{y_n \in Y \cap U_n : n \in \omega\}$ has an accumulation point in X.

Then:

1. F is a closed normal subspace of the space X.

2. The subspace F is countably compact.

3. $\{U_n : n \in \omega\}$ is a base of the set F in X.

Proof. Assertion 1 is obvious. Let $L = \{x_n \in F : n \in \omega\}$ be a discrete sequence of the subspace F. Then there exists a disjoint family $\{V_n : n \in \omega\}$ of open subsets of X such that $x_n \in V_n$ for any $n \in \omega$. There exists an open subset W of X such that $L \subseteq W \subseteq cl_X W \subseteq \cup\{V_n : n \in \omega\}$. We put $W_n = W \cap U_n \cap V_n$. Then $\{W_n : n \in \omega\}$ is a discrete family of open subsets of X, a contradiction. The assertion 2 is proved.

Let U and V be two open subsets of X and $F \subseteq V \subseteq cl_X V \subseteq U$. Then $\{H_n = U_n \setminus cl_X V : n \in \omega\}$ is a discrete family of open subsets of X. Thus $H_m = \emptyset$ and $U_n \subseteq U$ for some $m \in \omega$. The assertion 3 is proved. □

Corollary 2.12. *Let X be a normal space. The following assertions are equivalent:*

1. The space X is densely q-complete.

2. The space X contains a dense q-complete subspace.

3. The space X is densely fan-complete.

4. The space X contains a dense fan-complete subspace.

Corollary 2.13. *A space X is densely sieve-complete if and only if it contains a dense Čech-complete subspace.*

Example 2.14. Let W_0 be the set of all countable ordinal numbers and ω_1 be the first uncountable ordinal number. In the set $V_0 = W_0 \times [0, 1)$ consider the linear order: $(\alpha, u) < (\beta, v)$ whenever $\alpha < \beta$ or $\alpha = \beta$ and $u < v$. The space V_0 with the topology induced by the linear order is called the long line ([17], Problem 3.12.18). The space $V = V_0 \cup \{\omega_1\}$, where $x < \omega_1$ for any $x \in V_0$, with the topology induced by the linear order is the Stone-Čech compactification of the space V_0. Let Y be the set of all points $(\alpha, 0) \in V_0$, where α is a non-limit ordinal number. Obviously, the subspace $X = V_0 \backslash Y$ of V_0 is locally metrizable and locally Čech-complete. Hence X is a sieve-complete space. We affirm that X is not a G_δ-subspace of some pseudocompact space. Really, assume that X is a subspace of a pseudocompact space Z and $X = Z \cup (\cup\{F_n : n \in \omega\})$, where $\{F_n : n \in \omega\}$ is a sequence of closed subsets of Z. Consider the continuous mappings $\varphi : \beta X \to \beta Z$ and $\psi : \beta X \to V$, where $\varphi(x) = \psi(x) = x$ for any $x \in X$. By construction, $\Phi_n = \varphi(\psi^{-1}(\psi(\varphi^{-1}(cl_{\beta Z} F_n)))))$ is a compact subset of $\beta Z \backslash X$. We can suppose that $F_n = Z \cap \Phi_n$. The space $Z_1 = \beta X \backslash \psi^{-1}(\omega_1)$ is countably compact and locally compact. If $(\alpha, 0) \in Y$ and $H(\alpha) = \varphi(\psi^{-1}(\alpha, 0))$, then $H(\alpha) \subseteq \beta Z \backslash X$. Since Z is pseudocompact $X \cap H(\alpha) \neq \emptyset$. Thus $Z \cap H(\alpha) \neq \emptyset$ and $H(\alpha) \subseteq \Phi_{n(\alpha)}$ for some $n(\alpha) \in \omega$. Therefore $\varphi(Z_1 \setminus X) \subseteq \cup\{\Phi_n : n \in \omega\}$. Consequently, $X = \cap\{Z_1 \setminus \varphi^{-1}(\Phi_n) : n \in \omega\}$ is a G_δ-subset of the space Z_1 and X is Čech-complete, a contradiction (see [2], [17]).

There exists a pseudocompact space X_1 such that the space of rationals $\mathbb{Q}$ is a closed subspace. Let $X_2 = X \times X_1$. By construction, X_2 is a fan-complete space, which is not locally Čech-complete and is not a G_δ-subspace of some pseudocompact space.

3. On quasicontinuous mappings

A mapping $f : X \times Y \to Z$ of a product space $X \times Y$ into a space Z is called *right strongly quasicontinuous at a point* $(a, b) \in X \times Y$ if for each open neighbourhood W of $f(a, b)$ in Z and every open neighbourhood U of a in X, there exist a non-empty open set U_1 in X and an open set V in Y such that $U_1 \subset U$, $b \in V$, and $f(U_1 \times V) \subset W$. In a similar way one can define a *left strongly quasicontinuous at a point* $(a, b) \in X \times Y$ mapping. A mapping f is *strongly quasicontinuous at a point* $(a, b) \in X \times Y$ if it is left and right strongly quasicontinuous at $(a, b) \in X \times Y$.

A mapping $f : X \to Y$ of a space X into a space Y is called *quasicontinuous at a point* $b \in X$ if for each open neighbourhood V of $f(b)$ in Y and every open neighbourhood U of b in X there exists a non-empty open set W in X such that $W \subset U$ and $f(W) \subset V$.

If a mapping $f : X \times Y \to Z$ is right or left strongly quasicontinuous, then f is quasicontinuous (about quasicontinuity see [7], [19], [20], [22]).

Proposition 3.1. *Let X_1 be a dense subspace of a space X, Y_1 be a dense subspace of a space Y, and f be a separately continuous mapping of $X \times Y$ into a space Z, $g = f|(X_1 \times Y_1)$, $a \in X_1$, and $b \in Y_1$. Then:*

1. The mapping f is right (left) strongly quasicontinuous at a point (a, b) if and only if the mapping g is right (left) strongly quasicontinuous at (a, b).

2. The mapping f is quasicontinuous at a point (a, b) if and only if the mapping g is quasicontinuous at (a, b).

Proof. Let $c = f(a, b)$ and W be a neighbourhood of c in Z. Fix two open subsets W_1 and W_2 of Z such that $c \in W_2 \subseteq cl_Z W_2 \subseteq W_1 \subseteq cl_Z W_1 \subseteq W$. Suppose that U is an open subset of X and V is an open subset of Y such that $g(U_1 \times V_1) \subseteq W_2$, where $U_1 = U \cap X_1$ and $V_1 = V \cap Y_1$. For every $x \in U_1$, we have $f(\{x\} \times cl_Y V_1) \subseteq cl_Z W_2 \subseteq W_1$, by the separate continuity of f. Since $V \subseteq cl_Y V_1$, we have $f(U_1 \times V) \subseteq cl_Z W_2 \subseteq\in W_1$. For every $y \in V$, we have $f(cl_X U_1 \times \{y\}) \subseteq cl_Z W_1 \subseteq W$, by the separate continuity of f. Since $U \subseteq cl_X U_1$, we have $f(U \times V) \subseteq cl_Z W_1 \subseteq W$. □

The following statement can be found in ([7], Lemma 2.3.7), for the Čech-complete spaces, and in [23], for strongly β-unfavorable spaces.

Proposition 3.2. *Suppose that X is a densely q-complete space, $a \in X$, b is a q_D-point of a space Y and f is a separately continuous mapping of $X \times Y$ into a space Z. Then:*

- f is right strongly quasicontinuous at (a, b);

- If X and Y are homogeneous spaces, then f is right strongly quasicontinuous at every point of $X \times Y$.

Proof. Fix a dense subspace X_1 of X and a dense wA-sieve $\mathcal{U}= \{\gamma_n = \{U_\alpha : \alpha \in A_n\}, \pi_n : A_{n+1} \to A_n : n \in \omega\}$ with the Property (C2) on the space X. We can assume that $a \in X_1$. Now we fix a dense subspace Y_1 of Y and a sequence of neighbourhoods $\{H_n : n \in \omega\}$ of the point b in Y such that if $y_n \in Y_1 \cap H_n$, then the sequence $\{y_n : n \in \omega\}$ has a cluster point in Y. We can assume that $b \in Y_1$ (we take $Y_1 \cup \{b\}$ in the place Y_1, in the contrary). Let $g = f|(X_1 \times Y_1)$.

Put $c = f(a, b)$. Suppose that the mapping f is not right strongly quasicontinuous at (a, b). By virtue of the Proposition 3.1, the mapping g is not right strongly quasicontinuous at (a, b). Then there exist a neighbourhood W_1 of c and a neighbourhood U of a such that for each non-empty open in X subset U' of U and each neighbourhood V of b in Y we have $f((U' \times V) \cap (X_1 \times Y_1)) \setminus W_1 \neq \emptyset$. Since Z is regular, we can find open neighbourhoods W_0 and W of c such that $cl_Z W_0 \subset W \subset cl_Z W \subset W_1$.

We will construct by induction certain decreasing sequences of open sets $\{U_n : n \in \omega\}$ and $\{V_n : n \in \omega\}$ in X and Y, respectively, a c-sequence $\alpha = \{\alpha_n : n \in \omega\}$ and sequences $\{x_n : n \in N\} \subset X_1$ and $\{y_n : n \in \omega\} \subset Y_1$ with the following properties:

(i) $x_n \in U_n \subseteq cl_X U_n \subseteq U_{\alpha_n} \cap U$, $y_n \in V_n \subseteq cl_Y V_n \subseteq H_n$, $cl_X U_{n+1} \subseteq U_n$, $cl_Y V_{n+1} \subseteq V_n$, and $f(x_n, y_n) \in f(U_n \times V_n) \setminus W_1 \subseteq Z \setminus cl_Z W$ for each $n \in \omega$;

(ii) If $M_0 = \{x \in X : f(x, b) \in W_0\}$, then $U_0 \subseteq M_0$;

(iii) If $n \geq 1$, $M_n = \{x \in U_{n-1} : f(x, y_{n-1}) \in Z \setminus cl_Z W\}$ and $L_n = \{y \in V_{n-1} : f(x_{n-1}, y) \in W_0\}$, then $cl_X U_n \subseteq M_n$, and $cl_Y V_n \subseteq L_n$.

We proceed as follows.

Step 0. Put $M_0 = \{x \in X : f(x, b) \in W_0\}$. Then M_0 is open, by the separate continuity of f, and $a \in M_0$, since $f(a, b) \in W_0$. Fix $\alpha_0 \in A_0$ and an open subset U_0 of X such that $a \in U_0 \subseteq cl_X U_0 \subseteq M_0 \cap U_{\alpha_0} \cap U$. Let V_0 be an open subset of Y such that $b \in V_0 \subseteq H_0$. By the assumption, the set $f(U_0 \times V_0) \setminus W_1$ is not empty. Therefore, we can choose $x_0 \in U_0 \cap X_1$ and $y_0 \in V_0 \cap Y_1$ such that $f(x_0, y_0) \in Z \setminus W_1$.

Step 1. Put $M_1 = \{x \in U_0 : f(x, y_0) \in Z \setminus cl_Z W\}$. Then $x_0 \in M_1$, and M_1 is open in X, by the separate continuity of f. Let $\alpha_1 \in \pi_0^{-1}(\alpha_0)$ and U_1 be an open neighbourhood of x_0 in X such that $cl_X U_1 \subset U_0 \cap M_1 \cap U_{\alpha_1}$. Put $L_1 = \{y \in V_0 : f(x_0, y) \in W_0\}$. It follows from $x \in U_1 \subset M_0$ that $b \in L_1$, and that L_1 is open in Y, by the separate continuity of f. Let V_1 be an open neighbourhood of b in Y such that $cl_Y V_1 \subset V_0 \cap L_1 \cap H_1$. By the assumption, the set $f(U_1 \times V_1) \setminus W_1$ is not empty. Then we can choose $x_1 \in U_1 \cap X_1$ and $y_1 \in V_1 \cap Y_1$ such that $f(x_1, y_1) \in Z \setminus W_1$.

Step n + 1. Assume that we have already defined the open sets U_n and V_n in X and Y, respectively, an element $\alpha_n \in A_n$ and the points $x_n \in U_n$ and $y_n \in V_n$ such that $U_n \subset M_0 \cap U_{\alpha_n}$, $b \in V_n$, and $f(x_n, y_n) \in Z \setminus cl_Z W$. Then we put $M_{n+1} = \{x \in U_n : f(x, y_n) \in Z \setminus cl_Z W\}$. Clearly, the set M_{n+1}, is open, and $x_n \in M_{n+1}$. Now we let $\alpha_{n+1} \in \pi_n^{-1}(\alpha_n)$, U_{n+1} be any open neighbourhood of x_n in X such that $cl_X U_{n+1} \subseteq M_{n+1} \cap U_{\alpha_{n+1}} \cap U_n$. Put $L_{n+1} = \{y \in V_n : f(x_n, y) \in W_0\}$. Clearly, this set is open and contains the point b, since $x_n \in U_n \subseteq M_0$. Now let V_{n+1} be any open neighbourhood of b in Y such that $cl_Z V_{n+1} \subseteq V_n \cap H_{n+1} \cap L_{n+1}$.

Again, by the assumption, the set $f(U_{n+1} \times V_{n+1}) \setminus W_1$ is not empty, and we can choose points $x_{n+1} \in U_{n+1} \cap X_1$ and $y_{n+1} \in V_{n+1} \cap Y_1$ such that $f(x_{n+1}, y_{n+1}) \in Z \setminus W_1$. Step $n+1$ is complete.

Let $P = \cap\{cl_X U_n : n \in \omega\}$ and $H = \cap\{cl_Y V_n : n \in \omega\}$. By construction, there exist an accumulation point $x^* \in P$ for the sequence $\{ x_n : n \in \omega\}$ in X, and an accumulation point $y^* \in H$ for the sequence $\{ y_n : n \in \omega\}$ in Y. Since $cl_X U_{n+1} \subset U_n$, the point x^* belongs to each U_n. Now, from $x^* \in U_{n+1} \subset M_{n+1}$ it follows that $f(x^*, y_n) \in Z \setminus cl_Z W$, for each $n \in \omega$. Since f is separately continuous, we conclude that $f(x^*, y^*) \in Z \setminus W$.

Fix $n \in \omega$. Then $y_{n+1} \in V_{n+1} \subset \{y \in Y : f(x_n, y) \in W_0\}$. Therefore, $f(x_n, y_m) \in W_0$ for each $m > n$. By the separate continuity of f, it follows that $f(x_n, y^*) \in cl_Z W_0$. Then again, by the separate continuity of f, $f(x^*, y^*) \in cl_Z W_0 \subset W$, a contradiction. The proof is complete. □

Corollary 3.3. *Suppose that X and Y are densely q-complete spaces, and that f is a separately continuous mapping of $X \times Y$ into a space Z. Then there exist a dense subspace X_1 of the space X and a dense subspace Y_1 of the space Y such that the mapping f is strongly quasicontinuous at every point of $X_1 \times Y_1$. In particular, the mapping f is quasicontinuous.*

4. On semitopological groups with a quasicontinuous multiplication

We need the following simple lemma.

Lemma 4.1. *Let U be an open neighbourhood of the neutral element e in a semitopological group G with a right strongly quasicontinuous multiplication $m : G \times G \to G$. Then:*

- $e \in cl_X U^*$, where $U^* = \{x \in G : (x.e) \in Int(m^{-1}(U))\}$;

- for each point $a \in U^*$ there exists an open neighbourhood V of the neutral element such that $aV^2 \subseteq U$.

Proof. By the definition, $U^* = \cup\{W \subseteq U : U$ *is open in* G *and* $W \cdot V \subseteq U$ *for some neighbourhood* V *of the neutral element* $e\}$. Thus the statements follows from right strongly quasicontinuity of the multiplication. □

The following lemma was proved in ([23], Lemma 2).

Lemma 4.2. *Let* G *be a semitopological group with a right strongly quasicontinuous multiplication,* X *be a dense subspace of* G, $\{H'_n : n \in \omega\}$ *and* $\{H''_n : n \in \omega\}$ *be two sequences of open subsets of* G, $\cap\{H'_n : n \in \omega\} \neq \emptyset$ *and* $\cap\{H''_n : n \in \omega\} \neq \emptyset$. *Suppose that every sequence* $\{x_n \in X \cap (H'_n \cdot H''_n) : n \in \omega\}$ *has an accumulation point in* G. *Then* G *is a paratopological group.*

Proof. We can assume that $e \in H_n = H'_n = H''_n$ and $cl_G H_{n+1} \subseteq H_n$ for each $n \in \omega$.

Assume that the multiplication is not jointly continuous. Then there exists two neighbourhoods W and V of the neutral element such that $cl_G V \subseteq W$ and $U^2 \setminus cl_G W \neq \emptyset$ for each neighbourhood U of the neutral element. We put $H = Int(m^{-1}(V))$ and $V^* = \{x \in G : (x.e) \in H\}$.

We will inductively define two sequences $\{b_n \in X : n \in \omega\}$ and $\{c_n \in X : n \in \omega\}$ of points, and two sequences $\{V_n : n \in \omega\}$ and $\{W_n : n \in \omega\}$ of open neighbourhoods of the neutral element such that:

- $c_i \in V^* \cap V_{i-1} \cap X$ and $c_i \cdot W_i \cdot W_i \subseteq V$ for each $n \in \omega$;
- $b_i \in W_i^2 \setminus cl_G W$ and $V_i \cdot b_i \subseteq G \setminus cl_G W$ for each $n \in \omega$;
- $W_{i+1} \subseteq W_i \cap H_{i+1}$ and $V_{i+1} \subseteq V_i \cap W_{i+1}$ for each $n \in \omega$.

Step 0. Fix $c_0 \in X \cap V^*$. Since $(c_0, e) \in H$, then there exists a neighbourhood W_0 of e such that $W_0 \subseteq H_0$ and $c_0 \cdot W_0^2 \subseteq V$. Now choose a point $b_0 \in X \cap (W_0^2 \setminus cl_G W)$ and a neighbourhood V_0 of e such that $V_0 \subseteq W_0$ and $V_0 \cdot b_0 \subseteq G \setminus clW$. Finally fix a point $c_1 \in V_0 \cap X$.

Step n+1. Since $c_{n+1} \in V^* \cap V_n$, then there exists a neighbourhood W_{n+1} of e such that $W_{n+1} \subseteq H_{n+1} \cap V_n \cap W_n$ and $c_{n+1} \cdot W_{n+1}^2 \subseteq V$. Now choose a point $b_{n+1} \in X \cap (W_{n+1}^2 \setminus cl_G W)$ and a neighbourhood V_{n+1} of e such that $V_{n+1} \subseteq W_{n+1}$ and $V_{n+1} \cdot b_{n+1} \subseteq G \setminus clW$. Finally, fix a point $c_{n+2} \in V_{n+1} \cap X$. This completes the induction.

Let b be an accumulation point of the sequence $\{b_n\}$, and c be an accumulation point of the sequence $\{c_n\}$. Assume that $n < k$.

Then $a_{nk} = c_n \cdot b_n \in c_n \cdot W_n \cdot W_k \subseteq c_n \cdot W_n^2 \subseteq V$. Therefore, $a_n = c_n \cdot b \in cl_G V$, and $a = c \cdot b \in cl_G V \subseteq W$. Since $c_k \cdot b_n \in V_k \cdot b_n \subseteq G \setminus cl_G W$, we have $a \in G \setminus W$, a contradiction. □

5. Main results

The next three theorems improve Montgomery Theorem [28] and Ellis Theorem [16]; they are closely related to some results in [7], [8], [9], [10], [11], [12], [20], [23], [38].

Theorem 5.1. *Suppose that a paratopological group G contains a dense fan-complete subspace. Then G is a topological group.*

Proof. Assume that G is not a topological group. Then (see [7], Lemma 2.3.19, or [8], Lemma 1.2) there exists an open subset U of G such that $e \in U$ and the set $U \cap U^{-1}$ is nowhere dense in G.

Let X be a dense fan-complete subspace of G. There exist a sequence $\{\gamma_n = \{V_\alpha : \alpha \in A_n\} : n \in \omega\}$ of families of open subsets of the space G and a sequence $\{\pi_n : A_{n+1} \to A_n : n \in \omega\}$ of mappings such that:

(RC1) $X \subseteq \cup\{V_\alpha : \alpha \in A_0\}$ and $V_\alpha \cap X \subseteq \cup\{V_\beta : \beta \in \pi_n^{-1}(\alpha)\} \subseteq V_\alpha$ for all $\alpha \in A_n$ and $n \in \omega$;

(RC2) for any c-sequence $\alpha = \{\alpha_n \in A_n : n \in \omega\}$, the sequence $\{X \cap V_{\alpha_n}; n \in \omega\}$ is stable in X.

Fix $\beta_0 \in A_0$ and an open non-empty subset W of G such that $e \in W$ and $cl_G W \cdot W \subseteq U \cap V_{\beta_0}$. Put $O = W \setminus cl_G(U \cap U^{-1})$. Then $O \subseteq W \subseteq cl_G(X \cap O)$ and $U \cap O^{-1} = \emptyset$.

We are going to define a sequence $\{U_n : n \in \omega\}$ of open subsets of G, a sequence $\{x_n \in X : n \in \omega\}$ of points, and a c-sequence $\{\alpha_n \in A_n : n \in \omega\}$ such that:

- $x_n \in X \cap U_n$ for any $n \in \omega$;
- $x_{n+1} \in U_n \cap x_n O$ for any $n \in \omega$;
- $cl_G U_{n+1} \subseteq U_n \cap V_{\alpha_{n+1}} \cap O$ for any $n \in \omega$;
- $\alpha_0 = \beta_0$ and $\pi_n(\alpha_{n+1}) = \alpha_n$ for any $n \in \omega$.

Let $\alpha_0 = \beta_0$, $U_0 = O$ and $x_0 \in X \cap O$.

Assume that $n \in \omega$ and α_n, U_n and x_n are already constructed. Since $e \in W \subseteq cl_G O$, we have that $x_n \in cl_G x_n O = cl_G(X \cap x_n O)$. Thus, $U_n \cap x_n O \cap X \neq \emptyset$. Take $x_{n+1} \in X \cap U_n \cap x_n O$. Since $x_{n+1} \in X \cap V_{\alpha_n} \subseteq \cup\{V_\beta : \beta \in \pi_n^{-1}(\alpha_n)\}$, there exist $\alpha_{n+1} \in \pi_n^{-1}(\alpha_n)$ and an open subset U_{n+1} of G such that $x_{n+1} \in U_{n+1} \subseteq cl_G U_{n+1} \subseteq U_n \cap V_{\alpha_{n+1}} \cap x_n O$.

Put $F = \cap\{cl_G U_n : n \in \omega\}$. Obviously, $F = \cap\{U_n : n \in \omega\} = Lim\{U_n : n \in \omega\}$ and $F \neq \emptyset$. The set $V = F \cdot W$ is open in G, and $F \subseteq V$. Let $P = cl_G V$ and $H = cl_G(X \setminus P) = cl_G(G \setminus P)$. By the construction, P and H are regular closed subsets of G (a *regular closed* set is the closure of an open set).

Since $F \subseteq V$ and $H \cap V = \emptyset$, we have: $F \cap H = \emptyset$.

We claim that $H \cap cl_G U_k = \emptyset$ for some $k \in \omega$. Indeed, assume the contrary. Then $U_n \cap H \neq \emptyset$ for any $n \in \omega$. It follows that $W_n = U_n \cap (G \setminus P) \neq \emptyset$ and $W_n \subseteq V_{\alpha_n}$, for any $n \in \omega$. Since the sequence $\{V_{\alpha_n} : n \in \omega\}$ is stable in G, we have $\Phi = Lim\{W_n; n \in \omega\} \neq \emptyset$. Hence, $\Phi \subseteq F \cap H$ and $F \cap H \neq \emptyset$, a contradiction. Therefore, $H \cap cl_G U_k = \emptyset$ for some $k \in \omega$.

Thus, $U_n \subseteq cl_G U_n \subseteq V \subseteq P$, for all $n \geq k$. Hence $x_n \in V$ for all $k \geq n$. However, $F \subseteq U_{n+2} \subseteq x_{n+1}O \subseteq x_{n+1}W$ for all $k \geq n$. Hence, $x_k \in cl_G V \subseteq x_{k+1} cl_G WW$. Taking into account that $x_{k+1} \in x_k O$, we obtain $x_k \in x_k O \cdot cl_G WW$. Hence, $e \in O \cdot cl_G WW \subseteq O \cdot U$. Therefore, $O \cap U^{-1} \neq \emptyset$, a contradiction. □

The next theorem could be derived as a corollary from [23], Theorem 1.

Theorem 5.2. *Suppose that a semitopological group G is densely q-complete. Then G is a topological group.*

Proof. Fix a dense subspace X of G and a dense A-sieve $\mathcal{U}= \{\gamma_n = \{U_\alpha : \alpha \in A_n\}, \pi_n : A_{n+1} \to A_n : n \in \omega\}$ with the Property (C2) on the space G. We can assume that $e \in X$ and there exists a c-sequence $\alpha = \{\alpha_n : n \in \omega\}$ such that $e \in \cap\{U_{\alpha_n} : n \in \omega\}$.

By virtue of Proposition 3.2, the multiplication $m : G \times G \to G$ is right strongly quasicontinuous at every point of $G \times G$.

We will inductively define a c-sequence $\beta = \{\beta_n : n \in \omega\}$, a sequence of points $\{b_n \in X : n \in \omega\}$, a sequence $\{H_n : n \in \omega\}$ of open sets, and a sequence $\{V_n : n \in \omega\}$ of open neighbourhoods of the neutral element such that:

- $b_0 \in b_0 \cdot V_0 \subseteq H_0 \subseteq U_{\beta_0}$ and $H_0 \cdot V_0 \subseteq U_{\alpha_0} \cap U_{\beta_0}$;
- $b_n \cdot V_n \subseteq H_n$ for each $n \in \omega$;
- $V_{n+1} \subseteq cl_G V_{n+1} \subseteq V_n \cap U_{\alpha_{n+1}}$ for each $n \in \omega$;
- $H_{n+1} \subseteq cl_G H_{n+1} \subseteq H_n \cap U_{\beta_{n+1}}$ for each $n \in \omega$;
- $H_{n+1} \cdot V_{n+1} \subseteq U_{\beta_{n+1}} \cap H_n$ for each $n \in \omega$.

Step 0. Let $\beta_0 = \alpha_0$. Since the multiplication is right strongly quasicontinuous at (e, e), there exists a neighbourhood W_0 of e and an open non-empty subset H_0 of U_{β_0} such that $H_0 \cdot W_0 \subseteq U_{\alpha_0}$.

Fix a point $b_0 \in H_0$ and a neighbourhood V_0 of e so that $cl_G V_0 \subseteq W_0$ and $cl_G(b_0 \cdot V_0 \subseteq H_0$.

Step n+1. Fix $\beta_{n+1} \in \pi^{-1}(\beta_n)$ such that $b_n \in U_{\beta_{n+1}}$. Since the multiplication is right strongly quasicontinuous at (b_n, e), there exists a neighbourhood W_{n+1} of e and an open non-empty subset H_{n+1} of $U_{\beta_{n+1}} \cap b_n V_n$ such that $cl_G(H_{n+1} \cdot W_{n+1}) \subseteq U_{\beta_{n+1}} \cap b_n V_n$. Fix a point $b_{n+1} \in H_{n+1}$ and an open neighbourhood V_{n+1} of e such that $cl_G V_{n+1} \subseteq W_{n+1}$ and $cl_G(b_{n+1} \cdot V_{n+1}) \subseteq H_{n+1}$. This completes the induction.

By the construction, $cl_G(H_n \cdot V_n) \subseteq U_{\beta_n}$ and any sequence $\{x_n \in X \cap (H_n \cdot V_n) : n \in \omega\}$ has an accumulation point in G. Lemma 4.2 implies that G is a paratopological group. Since every densely q-complete space is densely fan-complete, by Theorem 5.1, it follows that G is a topological group. □

Theorem 5.3. *Lef G be a semitopological group of pointwise countable type. If G is densely q-complete, then G is a Čech-complete topological group.*

Proof. It follows from Theorem 5.2 that G is a topological group. Every topological group of pointwise countable type is paracompact [7]. Corollary 2.6 implies that G has a dense Čech-complete subspace X.

Denote by ρG the Raikov completion of the topological group G (see [7]). There exists a sequence $\{U_n : n \in \omega\}$ of open subsets of ρG such that $X = \cap\{U_n : n \in \omega\}$. By the construction, if $Y \subseteq \rho G \setminus G$ is a dense in ρG subspace, then Y is of the first category. Suppose that $c \in \rho G \setminus G$. Then cX is a dense Čech-complete subspace of ρG, and $cX \cap X = \emptyset$, a contradiction. Thus, $G = \rho G$, that is, G is a Raikov complete group. By Theorem 4.3.15 from [7], G is Čech-complete. □

Corollary 5.4. *For any semitopological group G, the following conditions are equivalent:*

1. G is a Čech-complete topological group.

2. G has a dense Čech-complete subspace.

3. G is a Tychonoff space, and there exists a compactification bG of G such that the remainder $bG \setminus G$ is a set of the first category in bG.

4. G is a Tychonoff space, and for any compactification bG of G the remainder $bG \setminus G$ is a set of the first category in bG.

5. G is a densely q-complete space of pointwise countable type.

6. G is a paratopological group of pointwise countable type, and G contains a dense fan-complete subspace.

7. G is a μ-complete space, and G contains a dense fan-complete subspace.

Corollary 5.5. *Suppose that G is a paratopological group, and that G is a dense subspace of a feebly compact (or pseudocompact) space Z. Suppose further that $Z \setminus G$ is a set of the first category in Z. Then G is a topological group.*

The next statement means that every Čech-complete topological group is absolutely closed in the class of quasitopological groups.

Corollary 5.6. *Let H be a Čech-complete subgroup of a quasitopological group G. Then H is closed in G.*

Proof. Let $P = cl_G H$. By Proposition 1.4.13 [7], P is a subgroup of G. Theorem 5.3 implies that P is a Čech-complete topological group. Since H is Raikov complete ([7], Theorem 4.3.15), we have $H = P$. The proof is complete. □

Theorem 5.7. *Lef X be a normal dense subspace of a semitopological group G. If X is densely fan-complete, then G is a topological group.*

Proof. By virtue of Corollary 2.12, the spaces X and G are densely q-complete. By Theorem 5.2 it follows that G is a topological group. □

The last statement does not generalize to metrizable paratopological groups.

Example 5.8. By Example 1.4.17 ([7], p. 30), there exists a non-discrete first-countable paratopological group G and a discrete countable subgroup H such that H is not closed in G. Fix a point $b \in cl_G H \setminus H$. Denote by B the subgroup of G generated by $H \cup \{b\}$. Then B is a countable metrizable paratopological group, and H is a non-closed discrete subgroup. Obviously, B is not a topological group.

Remark 5.9. A. V. Korovin [24] (see also [7], Example 2.4.16) constructed a commutative quasitopological pseudocompact Boolean group which is not a topological group. Therefore, a fan-complete quasitopological group needn't be a paratopological group. In particular, Theorem 5.1 does not hold for quasitopological groups.

Problem 5.10. *Suppose that H is a Řaikov complete topological group, and that H is a subgroup and a subspace of a quasitopological group G. Is it true that H is closed in G?*

By Theorem 5.1 and Proposition 2.9, the answer to Problem 2.4.5 in [7] is positive.

6. On analytic spaces with the Baire property

Let $J = \omega^\omega$ be the space of irrational numbers, with the usual topology. If $t = (t_0, t_1, ...)$ and $n \in \omega$, then $t|n = t_0 t_1 ... t_n$. A family $\{E_{t|n} : t \in J, n \in \omega\}$ of subsets of a space X is called an *array* of subsets of X.

A subspace Y of a space X is an *A-FU-subset* of X if there exists an array $\{E_{t|n} : t \in J, n \in \omega\}$ of X such that $Y = \cup\{\cap\{E_{t|n} : n \in \omega\} : t \in J\}$ and for all $t \in J$ and $n \in \omega$ there exist a closed subset $F \subseteq X$ and an open subset U of X such that $E_{t|n} = F \cap U$.

A space Y is *Čech-analytic* if Y is an A-FU-subset of some Čech complete space (see [18], [10]).

A space Y is a *q-analytic* if Y is an A-FU-subset of some q-complete space.

A space Y is *feebly-analytic* if Y is an A-FU-subset of some fan-complete space.

Theorem 6.1. *Let Y be a dense A-FU-subset of a space X, and let Y have the Baire property. Then there exists a dense G_δ-subset Z of X such that $Z \subseteq Y$.*

Proof. Let $Y = \cup\{\cap\{E_{t|n} : n \in \omega\} : t \in J\}$ and $E_{t|n} = F_{t|n} \cap U_{t|n}$, where $U_{t|n}$ is open in X, $F_{t|n}$ is closed in X, and $F_{t|n} \subseteq cl_X U_{t|n}$. Then $X_1 = X \setminus \cap\{cl_X U_{t|n} \setminus U_{t|n} : t \in J, n \in \omega\}$ is a G_δ-subset of X, and $Y_1 = Y \cap X_1$ is dense in X. Thus, we can assume that the sets $X_1 = X$ and $E_{t|n}$ are closed in X.

We put $Y_{t_0 t_1 ... t_m} = \cup\{\cap\{E_{t|n} : n \in \omega\} : t \in J, t|m = t_0 t_1 ... t_m\}$. We can assume that $F_{t|n} = cl_X Y_{t|n}$. We have $Y_{t_0 t_1 ... t_m} = \cup\{Y_{t_0 t_1 ... t_m n} : n \in \omega\}$.

Now we will construct open subsets $V_{t|n}$ of X as follows:

1. Put $V_n = IntE_n \setminus \cup\{E_m : m < n\}$ for each $n \in \omega$. The set $W_0 = \cup\{V_n : n \in \omega\}$ is open and dense in X.

2. Assume that $n \geq 0$ and that the sets $\{V_{t|n} : t \in J\}$ have been already defined. Fix $t_0, t_1, ...t_n \in \omega$. We put $V_{t_0 t_1 ... t_n m} = V_{t_0 t_1 ... t_n} \cap (IntE_{t_0 t_1 ... t_n m} \setminus \cup\{E_{t_0 t_1 ... t_n k} : k < m\})$ for each $m \in \omega$.

The family $\{V_{t|n} : n \in \omega, t \in J\}$ is constructed. The family $\{V_{t|n} : t \in J\}$ is disjoint, and the set $W_n = \cup\{V_{t|n} : t \in J\}$ is open and dense in X, for each $n \in \omega$. By the construction, $Z = \cap\{W_n : n \in \omega\}$ is open and dense in X, and $Z \subseteq Y$. The proof is complete. □

Corollary 6.2. *Every Čech-analytic space with the Baire property contains a dense Čech complete G_δ-subspace.*

Corollary 6.3. *Every q-analytic space with the Baire property contains a dense q-complete G_δ-subspace.*

Corollary 6.4. *Every feebly-analytic space with the Baire property contains a dense fan-complete G_δ-subspace.*

Corollary 6.5. *(see* [10] *for Čech-analytic semitopological groups). Let G be a semitopological group with a dense Čech-analytic subspace with the Baire property. Then G is a Čech-complete topological group.*

Corollary 6.6. *Let X be a dense subspace with the Baire property of a semitopological group G. If X is an A-FU-subset of some densely q-complete space, then G is a topological group.*

Corollary 6.7. *Let G be a paratopological group. If G contains a dense feebly-analytic subspace with the Baire property, then G is a topological group.*

Corollary 6.8. *Let G be a semitopological group with a dense feebly-analytic subspace Y with the Baire property. Then the following conditions are equivalent:*

1. *The space G is paracompact.*
2. *The space G is μ-complete.*
3. *The space G is Čech-complete.*
4. *G is a space of pointwise countable type.*

7. Countable networks in remainders of semitopological groups

We apply some results obtained above to study the structure of semitopological groups that have a remainder with a countable network. Here is the main result:

Theorem 7.1. *If a non-locally compact semitopological group G has a compactification bG such that the remainder $X = bG \setminus G$ has a countable network, then G has a countable π-base and is first countable, and the remainder X has a dense separable metrizable subspace.*

To prove this theorem, we need a general lemma on the structure of regular spaces with a countable network.

Lemma 7.2. *Suppose that X is a regular space with a countable network $\mathcal{S}$. Then $X = Y \cup Z$, where Y is a separable metrizable space, and Z has a countable network $\mathcal{P}$ such that every element of $\mathcal{P}$ is nowhere dense in X.*

Proof. Since X is regular, we may assume that each element of $\mathcal{S}$ is closed in X. For each $M \in \mathcal{S}$ put $P_M = M \setminus Int(M)$, where $Int(M)$ is the interior of M in X. Put $\mathcal{P} = \{P_M : M \in \mathcal{S}\}$. Clearly, each element of the family $\mathcal{P}$ is closed and nowhere dense in X.

Let Z be the set of all points of X at which the countable family $\mathcal{P}$ is a network, and put $Y = X \setminus Z$. Put also $\mathcal{B} = \{Int(M) : M \in \mathcal{S}\}$.

Claim: The family $\mathcal{B}$ is a base of X at each point of Y.

Indeed, take any $y \in Y$, and let $O(y)$ be an arbitrary open neighbourhood of y in X. Therefore, we can fix an open neighbourhood V of y in X such that no element of the family $\mathcal{P}$ contains y and is contained in V. Clearly, we may assume that $V \subset O(y)$.

Since $\mathcal{S}$ is a network of X, there exists $M \in \mathcal{S}$ such that $y \in M \subset V$. Then $P_M \in \mathcal{P}$ and $P_M \subset V$. Now it follows from the choice of V that $y \notin P_M$. Since $y \in M$, the definition of P_M implies that $y \in Int(M)$. We also have $Int(M) \in \mathcal{B}$ and $Int(M) \subset V \subset O(y)$. The Claim is proved.

Hence, Y is a regular space with a countable base, that is, Y is separable and metrizable [17]. □

We are now in a position to prove Theorem 7.1.

Proof. By Lemma 7.2, $X = Y \cup Z$, where Y is a separable metrizable space, and Z has a countable network $\mathcal{P}$ such that every element of $\mathcal{P}$ is nowhere dense in X.

Observe that X is dense in bG as well, since the space G is nowhere locally compact.

Case 1. Y is dense in X. Then Y is dense in bG, since X is dense in bG. Since Y has a countable base, it follows that bG has a countable π-base. Therefore, G has a countable π-base, since G is dense in bG.

Case 2. Y is not dense in X. Denote by U the complement in bG of the closure of Y in bG. Then U is a non-empty open subspace of bG.

For an arbitrary $P \in \mathcal{P}$, denote by F_P the closure of P in bG. Since P is nowhere dense in X, it follows that F_P is nowhere dense in bG. Therefore, the set $W_P = U \setminus F_P$ is a dense open subspace of U. Since U is itself open in bG, and bG is compact, it follows that the subspace $H = \cap\{W_P : P \in \mathcal{P}\}$ of U is a Čech-complete space dense in U. Now it follows by a standard argument that G has a dense Čech-complete subspace. Therefore, G is itself a Čech-complete topological group. If a topological group G has a compactification such that the remainder has a countable network, then G is a separable metrizable space (see [5]). In this case the remainder X is a Lindelöf p-space [4] with a G_δ-diagonal and, hence, X is separable and metrizable.

Observe that G is a space of pointwise countable type, since the remainder X is Lindelöf [21]. On the other hand, every Tychonoff semitopological group with a countable π-base has a G_δ-diagonal [8]. Hence, G is first countable [17]. □

Theorem 7.3. *If a non-locally compact semitopological group G has a separable metrizable remainder, then G is also separable and metrizable.*

Proof. From Theorem 7.1 it follows that G has a countable π-base. Now using results from ([8], Corollary 2.5), we conclude that G has a G_δ-diagonal. We also see that G is a Lindelöf p-space ([4], Section 3), since G is a remainder of a separable metrizable space. It remains to recall that every Lindelöf p-space with a G_δ-diagonal is separable and metrizable (see [17], Problem 5.5.7). □

Example 7.4. Let A be the set of rational numbers $\mathbb{Q}$ as a subspace of Sorgenfrey line S. Then:
- A is a metrizable paratopological group;
- A is not a topological group;
- A has a metrizable compactification.

The paratopological group B from Example 5.8 has the same properties. The topological spaces A, B, and $\mathbb{Q}$ are homeomorphic, but they are not topologically isomorphic as paratopological groups.

Problem 7.5. *Is every separable metrizable paratopological (semi-topological) group homeomorphic to a topological group?*

Observe in this connection that *every countable metrizable semi-topological group is homeomorphic to a topological group* (to the group of rational numbers in the usual, or discrete topology).

8. On pointwise pseudocompact groups

A point $x \in X$ is called a pseudocompactness point of X if there exists a stable sequence $\{U_n : n \in \omega\}$ of open subsets of X such that $x \in \cap\{U_n : n \in \omega\}$. A space X is said to be *pointwise pseudo-compact* if each point of X is a pseudocompactness point (see [7], p. 388). Clearly, every fan-complete space is pointwise pseudocompact. Moreover, every q-space is pointwise pseudocompact.

We recall that the G_δ-closure $\omega cl_X Y$ of a set $Y \subseteq X$ in a space is the set of all points $x \in X$ such that every G_δ-set H containing x intersects Y. The set $\mu cl_X H = \cup\{cl_X A : A \subseteq H, A$ *is a bounded subset of* $X\}$ is called the *μ-closure* of H in X.

The subspace $\mu^* X = \mu cl_{\beta X} X$ of the Stone-Čech compactification βX of a Tychonoff space X is called the *μ-completion* of X. Obviously, $\mu^* X$ is a subspace of the Dieudonné completion μX of X.

Let ρG be the Raikov completion of a topological group G and $\rho_\omega G$ be the G_δ-closure of G in ρG. Clearly, $\rho_\omega G$ is a subgroup of ρG.

Lemma 8.1. *Let P be a closed bounded subset of a topological group G. Then $\omega cl_{\rho G} P = cl_{\rho G} P$.*

Proof. Let $x \in cl_{\rho G} P \setminus \omega cl_{\rho G} P$. Then there exists a sequence $\{U_n : n \in \omega\}$ of open subsets of ρG such that $x \in \cap\{U_n : n \in \omega\}$, $\cap\{P \cap U_n : n \in \omega\} = \emptyset$ and $cl_{\rho G} U_{n+1} \subseteq U_n$ for any $n \in \omega$.

Then $\{V_n = G \cap U_n : n \in \omega\}$ is an open locally finite family of subsets of G and $P \cap V_n \neq \emptyset$ for any $n \in \omega$.

A space X is called a *paracompact p-space* if it admits a perfect mapping onto a metrizable space [1]. A feathered group is a topological group whose underlying space is a paracompact p-space. A topological group is a feathered group if and only if it is a space of pointwise countable type (see [7]). □

Theorem 8.2. *For any topological group G the following conditions are equivalent:*

1. G is a fan-complete space.

2. G contains a dense fan-complete subspace.

3. $\rho_\omega G$ is a Čech-complete paracompact space.

4. G is G_δ-dense in ρG and ρG is a Čech-complete paracompact space.

5. G is a G_δ-subset of the pseudocompact space $Z = G \cup (\beta G \setminus \mu^* G)$.

Proof. The implications $1 \to 2$, $5 \to 1$ and $3 \to 4 \to 3$ are obvious.

Let G be a pointwise pseudocompact topological group. There exists a stable sequence $\{U_n : n \in \omega\}$ of open subsets of G and a sequence $\{f_n : G \to [0,1] : n \in \omega\}$ of continuous functions such that $e \in U_{n+1} = U_{n+1}^{-1} \subseteq cl_G U_{n+1}^2 \subseteq U_n$, $U_{n+1} \subseteq f^{-1}(0)$ and $G \setminus U_n \subseteq f^{-1}(1)$, for any $n \in \omega$. Then $H = \cap\{U_n : n \in \omega\}$ is a closed bounded subgroup of G. The closure Φ of H in ρG is a compact subgroup and the projection $\varphi : \rho G \to X$, where $X = \rho G/\Phi$, is a perfect and open continuous mapping onto a complete metrizable space X.

Fix a complete metric d on a space X. By Theorems 6.9.14 and 6.9.15 from [7], we have: $\rho_\omega G = \mu G = \mu^* G = \varphi^{-!}(\varphi(G)$. From Theorem 6.5.1 and Corollary 6.9.10 [7] it follows that $\rho_\omega G$ is a subgroup of the group ρG.

Assume that G is G_δ-dense in ρG. Then $\rho_\omega G = \rho G = \mu G = \mu^* G$ and $\rho G \subseteq \beta G$. Let $Z = G \cup (\beta G \setminus \rho G)$ be a subspace of βG. By the construction, G is a G_δ-subset of Z and $\beta Z = \beta G = \beta Z$. For every Tychonoff space X, the space $X \cup (\beta X \setminus \mu^* X \subseteq \beta X$ is pseudocompact. Hence, Z is a pseudocompact space. Proposition 2.3 implies that G is fan-complete. The implications $4 \to 1$ and $4 \to 5$ are proved.

Assume now that Y is a dense fan-complete subspace of G. By Theorem 2.5 and Corollary 2.6, there exists a dense Čech-complete paracompact subspace of $\rho_\omega G$. By Theorem 5.3, the topological group $\rho_\omega G$ is Čech-complete. Thus, $\rho_\omega G = \rho G$. The implication $2 \to 4$ is proved. The proof is complete. □

Suppose that a topological group G is a q-space. Then in the proof of Theorem 8.2 we can assume that H is a countably compact subset and $\{U_n : n \in \omega\}$ is a base of neighbourhoods of the set H in G. In this case the mapping $\psi = \varphi|G : G \to Y = \varphi(G)$ is quasiperfect, and G is a complete M-space. Moreover, the mapping $\beta\psi|Z : Z \to \beta X$ is quasiperfect, and Z is countably compact. Thus, we have proved the following theorem:

Theorem 8.3. *For any topological group G, the following conditions are equivalent:*

1. G is a complete M-space.

2. G is a q-space, and G has a dense fan-complete subspace.

3. G is a q-space, and $\rho_\omega G$ is a Čech-complete paracompact space.

4. G is a q-space, G is G_δ-dense in ρG, and the space ρG is Čech-complete and paracompact.

5. G is a G_δ-subset of the countably compact space $Z = G \cup (\beta G \setminus \mu^* G)$.

Let X be a normal space. If $\{U_n : n \in \omega\}$ is a stable sequence of open subsets, then each sequence $\{x_n \in U_n : n \in \omega\}$ has a $\{U_n : n \in \omega\}$ cluster point in X. In particular, each pseudocompactness point is a q-point. Therefore, Theorem 8.2 and Corollary 2.12 yield the next result:

Corollary 8.4. *Suppose that G is a semitopological group, and that the space G is normal and has a dense fan-complete subspace. Then:*

1. G is a complete M-space.

2. G is a topological group.

3. G is a G_δ-subset of the countably compact space $Z = G \cup (\beta G \setminus \mu^* G)$.

The next example shows that a metrizable topological group with the Baire property needs not be Čech-complete.

Example 8.5. Let $\mathbb{R}$ be the field of reals and $\mathbb{Q}$ be the subfield of rationals. Then there exists an additive subgroup G of $\mathbb{R}$ such that:

1. G and $S = \mathbb{R} \setminus G$ are dense subspaces with the Baire property.
2. In G and S every compact subset is countable.
3. $G \cdot \mathbb{Q} = G$.

A subspace M of $\mathbb{R}$ is a $\mathbb{Q}$-*module* if M is an additive subgroup of $\mathbb{R}$ and $M \cdot \mathbb{Q} = M$. If $L \subseteq \mathbb{R}$, then we denote by $m(L)$ the $\mathbb{Q}$-module algebraically generated by the set L.

Let $\{F_\alpha : \alpha < c = 2^\omega\}$ be the family of all non-countable closed subsets of $\mathbb{R}$. Assume that $F_0 = [1, 2]$.

Using the transfinite recursion, we construct the sequences $\{x_\alpha : \alpha < c\}$ and $\{y_\alpha : \alpha < c\}$ in the following way:

Fix $x_0 = 2$ and $y_0 \in F_0 \setminus \mathbb{Q}$. Assume that $0 < \alpha < c$, and that $L_\alpha = \{x_\beta, y_\beta : \beta < \alpha\}$ is already defined. Since $|m(L_\alpha)| < c$, we can fix $x_\alpha \in F_\alpha \setminus m(L_\alpha)$. Let $M_\alpha = L_\alpha \cup \{x_\alpha\}$, and fix $y_\alpha \in F_\alpha \setminus m(M_\alpha)$.

Now we put $X = \{x_\alpha : \alpha < c\}$, $Y = \{y_\alpha : \alpha < c\}$, and $G_\alpha = m(\{x_\beta : \beta \leq \alpha\}$, for each $\alpha < c$. Then $G = \cup\{G_\alpha : \alpha < c\}$ is the $\mathbb{Q}$-module generated by the set X. We claim that $G \cap Y = \emptyset$. Thus, G is the desired $\mathbb{Q}$-module.

Remark 8.6. There exist a separable complete metrizable linear space L and a dense linear subspace B of L such that: B and $Y = L \setminus B$ are dense subspaces of the space L; B and Y are spaces with the Baire property; B and Y are not complete metrizable spaces.

Really, let $\xi \in \beta\omega \setminus \omega$ and $X = \{\xi\} \cup \omega$. We put $L = \mathbb{R}^X$ and $B = C_p(X) \subseteq L$. D.J. Lutzer and R.A. McCoy [26] have proved that the space B is not complete metrizable and has the Baire property. By virtue of the recent result from ([6], Theorem 1.1), since the space B is not Čech-complete, the subspace Y has the Baire property.

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