
TOPOLOGY PROCEEDINGS



Volume 37, 2011

Pages 119–128

<http://topology.auburn.edu/tp/>

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Electronically published on May 18, 2010

Topology Proceedings

Web: <http://topology.auburn.edu/tp/>

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ISSN: 0146-4124

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ABSTRACT. Let M_K be the 2-fold branched cover of a knot, K , in S^3 . If $H_1(M_K) = \mathbf{Z}_3 \oplus \mathbf{Z}_{3^{2i}} \oplus G$ where 3 does not divide the order of G then K is not of order 4 in the concordance group. This obstruction detects infinite new families of knots that represent elements of order 4 in the algebraic concordance group that are not of order 4 in concordance.

1. INTRODUCTION

Levine [12, 13] defined a homomorphism ϕ from the concordance group \mathcal{C} of knots in S^3 , onto an algebraically defined group \mathcal{G} , and further proved that $\mathcal{G} \cong \oplus \mathbf{Z}^\infty \oplus \mathbf{Z}_2^\infty \oplus \mathbf{Z}_4^\infty$. It is a long standing conjecture that \mathcal{C} contains no torsion of order other than two; see for instance [4, 11]. This paper continues our investigation of the possibility of elements of order four in \mathcal{C} .

For a knot $K \subset S^3$, let M_K denote the 2-fold branched cover of S^3 branched over K , and for a prime p , let $H_1(M_K)_p$ denote the p -primary subgroup of $H_1(M_K)$; homology is with integer coefficients throughout this paper. Our earlier work on 4-torsion, [16, 17], demonstrated the following.

Theorem 1.1. *If $H_1(M_K)_p \cong \mathbf{Z}_{p^k}$ for some prime $p \equiv 3 \pmod{4}$ with k odd, then K is of infinite order in \mathcal{C} .*

2010 *Mathematics Subject Classification.* 57M25, 57N70.

Key words and phrases. Knot, Concordance.

The second author was supported in part by NSF Grant #DMS-0709625.

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This criterion is effective in ruling out the possibility of being order four for most low-crossing knots that represent four torsion in \mathcal{G} .

Since we wrote [17], several papers have appeared that apply new methods in smooth concordance theory (in particular Heegaard-Floer theory) to the study of 4-torsion. This work includes [8, 9, 14]. Given the continued interest in the structure of the concordance group, we here investigate the extension of our earlier work to the case in which $H_1(M_K)_p$ is not cyclic. Working with primes greater than three greatly complicates the algebra; our main result is restricted to the case of $p = 3$.

Theorem 1.2. *If $H_1(M_K)_3 \cong \mathbf{Z}_3 \oplus \mathbf{Z}_{3^{2i}}$ then K is not of order 4 in \mathcal{C} .*

We will also present applications of this result, describing new infinite families of knots that are of algebraic order four but do not represent 4-torsion in \mathcal{C} . A simple, easily stated application is the following, where the Alexander polynomial of a knot K is denoted $\Delta_K(t)$:

Corollary 1.3. *If $\Delta_K(t)$ is quadratic and $\Delta_K(-1)\Delta_K(1) = -27m$ where $m > 0$ and 3 does not divide m , then K is of order 4 in \mathcal{G} but not in \mathcal{C} .*

While the simplest application of our main result is to prove that particular knots that are of algebraic order four are not of order four in \mathcal{C} , we are more interested in the fact that this obstruction applies to entire S -equivalences classes of knots, and thus the calculation of the obstruction is purely algebraic, based on simple classical algorithms from knot theory.

Of further interest is that the result applies in the topological, locally flat category. The techniques we use are based on Casson-Gordon theory, which initially applied only in the smooth category (see, for example, [1, 2]), but by [5] the techniques extend to the topological locally flat category. With regards to examples taken from low-crossing prime knots, all algebraic order four knots that have been shown to be of order greater than 4 smoothly can be shown to have order greater than four topologically.

Basic results in knot theory can be found in [19] or [7]. Tables of low crossing knots and their algebraic and concordance orders can be found in [3].

2. Casson-Gordon invariants and linking forms

Let χ denote a homomorphism from $H_1(M_K)$ to \mathbf{Z}_{p^k} , for some prime p . The Casson-Gordon invariant $\sigma(K, \chi)$ is a rational invariant of the pair (K, χ) . (See [1, 2]. In the original paper, [CG1], this invariant is denoted $\sigma_1\tau(K, \chi)$, and σ is used for a closely related invariant.)

On the rational homology sphere M_K there is a nonsingular symmetric linking form, $\beta : H_1(M_K) \times H_1(M_K) \rightarrow \mathbf{Q}/\mathbf{Z}$. For a subgroup $M \leq H_1(M_K)$ we let $M^\perp = \{x \in H_1(M_K) \mid \beta(x, m) = 0 \forall m \in M\}$. The main result in [CG1] concerning Casson-Gordon invariants and slice knots that we will be using is the following:

Theorem 2.1. *If K is slice there is a subgroup $M \subset H_1(M_K)$ with $M = M^\perp$ and $\sigma(K, \chi) = 0$ for all prime power order χ vanishing on M .*

A subgroup $M \subset H_1(M_K)$ satisfying $M = M^\perp$ is called a *metabolizer*. It is useful to recall the following result.

Lemma 2.2. *For a metabolizer $M \subset H_1(M_K)$, $H_1(M_K)/M \cong M$ and in particular $|M|^2 = |H_1(M_K)|$.*

Proof. This follows quickly from the following exact sequence

$$0 \rightarrow M^\perp \rightarrow H_1(M_K) \rightarrow \text{hom}(M, \mathbf{Q}/\mathbf{Z}) \rightarrow 0,$$

the fact that $M^\perp = M$, and the observation that since M is a finite abelian group, $\text{hom}(M, \mathbf{Q}/\mathbf{Z}) \cong M$. \square

We will need Gilmer's additivity theorem [6], a vanishing result proved by Litherland [15, Corollary B2], and a simple fact that follows immediately from the definition of the Casson-Gordon invariant.

Theorem 2.3. *If χ_1 and χ_2 are defined on M_{K_1} and M_{K_2} , respectively, then we have $\sigma(K_1 \# K_2, \chi_1 \oplus \chi_2) = \sigma(K_1, \chi_1) + \sigma(K_2, \chi_2)$.*

Theorem 2.4. *If χ is the trivial character, then $\sigma(K, \chi) = 0$.*

Theorem 2.5. *For every character χ , $\sigma(K, \chi) = \sigma(K, -\chi)$.*

We will also need to use the relationship between the Casson-Gordon invariant of a knot and the linking form on its 2-fold branched cover, as developed in [16, 17].

Theorem 2.6. *If $\chi: H_1(M_K) \rightarrow \mathbf{Z}_p^r$ is a character obtained by linking with the element $x \in H$, then $\sigma(K, \chi) \equiv \beta(x, x)$ modulo \mathbf{Z} .*

This will be used later to conclude that certain Casson-Gordon invariants are nonzero.

NOTATION:

In the rest of this paper all knots will satisfy $H_1(M_K)_3 \cong \mathbf{Z}_3 \oplus \mathbf{Z}_{3^{2i}}$, $i > 0$. All characters χ will take values in $\mathbf{Z}_{3^{2i}} \subset \mathbf{Q}/\mathbf{Z}$, and such χ factor through characters defined on $\mathbf{Z}_3 \oplus \mathbf{Z}_{3^{2i}}$. Any such character is given by linking with an element of the $H_1(M_K)_3$, say $x \oplus y \in \mathbf{Z}_3 \oplus \mathbf{Z}_{3^{2i}}$. To simplify notation we will write $\sigma(K, \chi)$ as $\sigma_{x,y}$.

3. PROOF OF THEOREM 1.2

Throughout this section we will assume that $4K$ is slice. We will consider all possible metabolizers to the linking form on $(\mathbf{Z}_3 \oplus \mathbf{Z}_{3^{2i}})^4$ and show that each leads to a contradiction to Theorem 2.1.

The following lemma easily follows from known results about linking pairings and the fact that -1 is not a quadratic residue modulo 3. For the sake of completeness we include an elementary proof.

Lemma 3.1. *There is a generating set $\{v, w\}$ for $\mathbf{Z}_3 \oplus \mathbf{Z}_{3^{2i}}$ such that v is of order 3, w is of order 3^{2i} , and the linking form satisfies: $\beta(v, v) = \pm 1/3$, $\beta(w, w) = \pm 1/3^{2i}$, and $\beta(v, w) = 0$.*

Proof. Let a generate the \mathbf{Z}_3 summand and let b generate the $\mathbf{Z}_{3^{2i}}$ summand. Since there is a character to \mathbf{Q}/\mathbf{Z} taking value $1/3^{2i}$ on b , by the nonsingularity of the linking form there is an element x satisfying $\beta(x, b) = 1/3^{2i}$. Write $x = ra + sb$. Since $\beta(a, b)$ is an integer multiple of $1/3$ (a is of order 3), $\beta(b, b)$ must be of the form $t/3^{2i}$ with t not divisible by 3. Let k be the inverse to t in $\mathbf{Z}_{3^{2i}}$. Then $k\beta(b, b) = 1/3^{2i}$. Let $v = a - 3^{2i}\beta(a, b)kb$. It is easily checked that v is of order 3 and $\beta(v, b) = 0$, hence v, b also generate.

By the nonsingularity of the linking form, $\beta(v, v) = \pm 1/3$.

The square of an element is 0 mod 3 if and only if the element itself is such. In $\mathbf{Z}_{3^{2i}}$ there are a total of 3^{2i-1} elements which are 0 mod 3. It follows that there are $3^{2i} - 3^{2i-1}$ elements which are ± 1 mod 3, half of which are additive inverses of the other half, and there are $\frac{3^{2i}-3^{2i-1}}{2}$ distinct squares which are not 0 mod 3. It follows that the above k is $\pm q^2$ for some $q \in \mathbf{Z}_{3^{2i}}$.

Let $w = qb$. □

From now on we will fix the generating set to be as given in the previous lemma.

In order to apply Theorem 2.1 to the knot $4K$, we let $H = H_1(M_{4K})_3 \cong (\mathbf{Z}_3 \oplus \mathbf{Z}_{3^{2i}})^4 \cong (\mathbf{Z}_3)^4 \oplus (\mathbf{Z}_{3^{2i}})^4$. We will let M denote a metabolizer in H . To set up notation, we will represent an element in $(\mathbf{Z}_3)^4 \oplus (\mathbf{Z}_{3^{2i}})^4$ by an ordered 8-tuple and a collection of n elements in $(\mathbf{Z}_3)^4 \oplus (\mathbf{Z}_{3^{2i}})^4$ by an $n \times 8$ matrix, the rows of which represent the individual elements. Each element will be written as

$$u = v \oplus w \in (\mathbf{Z}_3)^4 \oplus (\mathbf{Z}_{3^{2i}})^4.$$

Lemma 3.2. *Let M be a metabolizer for H . Then M cannot be generated by less than four elements.*

Proof. Tensor H and M with \mathbf{Z}_3 . We have $H \otimes \mathbf{Z}_3 \cong (\mathbf{Z}_3)^8$. If M is generated by k' elements, then $M \otimes \mathbf{Z}_3 \cong (\mathbf{Z}_3)^{k'}$, where $k \leq k'$. If $k \leq 3$, then $\text{rk}((H \otimes \mathbf{Z}_3)/(M \otimes \mathbf{Z}_3)) \geq 5$. As $\text{rk}((H/M) \otimes \mathbf{Z}_3) \geq \text{rk}((H \otimes \mathbf{Z}_3)/(M \otimes \mathbf{Z}_3))$, we have a contradiction to the fact that $H/M \cong M$. □

We will call the minimum number of elements required to generate M the rank of M . The proof of Theorem 1.2 is simplest in the case that the rank is greater than 4.

Theorem 3.3. *If $\text{rank}(M) = k$, $k > 4$, then K is not of order 4 in concordance.*

Proof. Consider a minimal generating set $\{(v_i, w_i)\}_{i=1 \dots k}$. These form the rows of a $k \times 8$ matrix which we denote $(V|W)$, where V and W are each $k \times 4$. We will now perform row operations to simplify the generating set. It will be convenient to interchange columns in these matrices as well, but notice that if two columns of W are interchanged, the same columns of V will be interchanged, since these columns correspond to the homology of the cover of a given component of $4K$.

By performing row operations and column interchanges, W can be put in upper triangular form. Hence, the fifth row of W is the trivial vector, $(0, 0, 0, 0) \in (\mathbf{Z}_{3^{2i}})^4$. After further column swaps, the fifth row of V can be put in the form $(\pm 1, \pm 1, \pm 1, 0)$, as these are the only nontrivial elements in $(\mathbf{Z}_3)^4$ with trivial self-linking.

It follows that $3\sigma_{1,0} = 0$, and hence $\sigma_{1,0} = 0$. However, by Theorem 2.6, $\sigma_{1,0} \equiv 1/3 \pmod{\mathbf{Z}}$, giving a contradiction. \square

The rest of this section is devoted to the case that $\text{rank}(M) = 4$.

Lemma 3.4. *Let $\text{rank}(M) = 4$. Then M has a generating set $\{u_j = v_j \oplus w_j \in (\mathbf{Z}_3)^4 \oplus (\mathbf{Z}_{3^{2i}})^4 \mid j = 1, 2, 3, 4\}$ such that the corresponding matrix $(V|W)$ is of the form given below. The $v_{j,k}$ are elements in \mathbf{Z}_3 and the $w_{j,k}$ are elements in \mathbf{Z}_3^{2i} .*

$$\left(\begin{array}{cccc|cccc} v_{1,1} & v_{1,2} & v_{1,3} & v_{1,4} & 1 & 0 & w_{1,3} & w_{1,4} \\ v_{2,1} & v_{2,2} & v_{2,3} & v_{2,4} & 0 & 1 & w_{2,3} & w_{2,4} \\ v_{3,1} & v_{3,2} & v_{3,3} & v_{3,4} & 0 & 0 & 3^{2i-1} & 0 \\ v_{4,1} & v_{4,2} & v_{4,3} & v_{4,4} & 0 & 0 & 0 & 3^{2i-1} \end{array} \right)$$

Proof. Row operations and column swaps (provided the same column swaps are made in V as in W) can be used to make W upper triangular with the diagonal entries nondecreasing powers of 3 such that the remaining entries in the j th row are annihilated by the same power of 3 as is the diagonal entry. Let the diagonal entries be 3^{k_j} with $0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq 2i$.

We first note that $k_4 \neq 2i$:

If $k_4 = 2i$ then the last row is of the form

$$(v_{4,1}, v_{4,2}, v_{4,3}, v_{4,4} \mid 0, 0, 0, 0)$$

with some of the $v_{4,k}$ nonzero. Since the self-linking of this element is 0, exactly 3 of the entries would be nonzero and it would follow that $3\sigma_{1,0} = 0$, implying that $\sigma_{1,0} = 0$, contradicting Theorem 2.6. Hence, we have $0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq 2i - 1$.

It is easily seen that the order of the element u_j represented by row j of this matrix is 3^{2i-k_j} and together the u_j generate a subgroup of order $3^{(8i-\sum k_j)}$. On the other hand, the order of H is 3^{8i+4} and M has the square root order 3^{4i+2} . It follows that $\sum k_j = 4i - 2$.

If $k_4 < 2i - 1$, then the generator u_4 generates a cyclic subgroup of order greater than 3. As $\sum k_j = 4i - 2$, k_4 cannot be zero. It follows that $H/\langle u_4 \rangle$ has rank 8. This implies that $H/\langle u_1, u_2, u_3, u_4 \rangle$ has rank 5 or more. However, by Lemma 2.2, the rank of H/M is 4. Therefore, we have $k_4 = 2i - 1$, $0 \leq k_1 \leq k_2 \leq k_3 \leq 2i - 1$, and $k_1 + k_2 + k_3 = 2i - 1$. As k_3 cannot be 0 either, a similar argument shows that k_3 will have to be $2i - 1$. Therefore we have $k_1 = k_2 = 0$. It is easy to see that the entries above the 1 in the second row and the 3^{2i-1} in the last row can be made 0. \square

Our argument continues to proceed by ruling out possible metabolizers under the assumption that $4K$ is slice.

Lemma 3.5. *Each of the entries $w_{j,k}$ in $(V|W)$ in the form given by Lemma 3.4 may be assumed to be $\pm 1 \pmod 3$. The \mathbf{Z}_3 reductions of the elements $(0, 0, w_{1,3}, w_{1,4})$ and $(0, 0, w_{2,3}, w_{2,4})$ are linearly independent in $(\mathbf{Z}_3)^4$.*

Proof. The self-linking of the first row is computed to be $\frac{\alpha}{3} \pm \frac{(1+w_{1,3}^2+w_{1,4}^2)}{3^{2i}}$ where α is determined by the self-linking of the $v_{1,j}$. If either $w_{1,3}$ or $w_{1,4}$ were $0 \pmod 3$ then it is easily shown that this sum could not be an integer; basically, 0 is not the sum of two nontrivial squares modulo 3. It follows that neither $w_{1,3}$ nor $w_{1,4}$ can be $0 \pmod 3$. A similar argument applies to $w_{2,3}$ and $w_{2,4}$.

If the elements $(0, 0, w_{1,3}, w_{1,4})$ and $(0, 0, w_{2,3}, w_{2,4})$ were dependent over \mathbf{Z}_3 , then by combining the first two rows of $(V|W)$ we would have a metabolizing element of the form $(*, *, *, * \mid \pm 1, \pm 1, 3a, 3b)$. But such an element cannot have self-linking 0. \square

Lemma 3.6. *The metabolizer M contains an element of the type $(1, 1, *, * \mid 0, 0, 3^{2i-1}m, 3^{2i-1}n)$, where m and n are integers.*

Proof. Let $v_{j,k}$, $w_{j,k}$ and $u_j = v_j \oplus w_j$ be as in Lemma 3.4

Suppose that $(v_{3,1}, v_{3,2})$ and $(v_{4,1}, v_{4,2})$ are linearly dependent in $(\mathbf{Z}_3)^2$. Then a nontrivial combination of u_3 and u_4 would yield an element $(0, 0, *, * \mid 0, 0, 3^{2i-1}m, 3^{2i-1}n) \in M$. Note that nontriviality in this case is over \mathbf{Z}_3 . In other words, either m or n is nonzero mod 3. To have self-linking zero the $*$ entries would have to be 0, so that we have $u = (0, 0, 0, 0 \mid 0, 0, 3^{2i-1}m, 3^{2i-1}n) \in M$.

Now, from Lemma 3.5 $(w_{1,3}, w_{1,4})$ and $(w_{2,3}, w_{2,4})$ are linearly independent over \mathbf{Z}_3 , so a linear combination of these yields a vector whose \mathbf{Z}_3 reduction is $(1, 0)$. As the corresponding linear combination of u_1, u_2 is an element in M and therefore links the above u trivially, we have $m \equiv 0 \pmod{3}$. Similarly $n \equiv 0 \pmod{3}$, giving us a contradiction.

It follows that $(v_{3,1}, v_{3,2})$ and $(v_{4,1}, v_{4,2})$ are independent over \mathbf{Z}_3 . Now, by taking an appropriate combination of u_3 and u_4 we can find the desired element of M . \square

Lemma 3.7. *For all $a, b \in \{0, \pm 1\}$, M contains elements of the form $(1, 1, *, * \mid 3^{2i-1}a, 3^{2i-1}b, 3^{2i-1}m, 3^{2i-1}n)$, where $m, n \in \mathbf{Z}$ and exactly one of the $*$ entries is nonzero.*

Proof. Add $3^{2i-1}a$ times the first row and $3^{2i-1}b$ times the second row of the matrix to the element given in the previous lemma. The condition on the two $*$ s comes from the fact that the self-linking of the resulting element must be 0. \square

COMPLETION OF PROOF, THEOREM 1.2. By Theorem 2.6, $\sigma_{1,0}$, $\sigma_{1,3^{2i-1}}$ and $\sigma_{1,2 \cdot 3^{2i-1}}$ are nonzero.

From the previous lemma we have, in the case $a = b = 0$, that either $3\sigma_{1,0} = 0$, $2\sigma_{1,0} + \sigma_{1,3^{2i-1}} = 0$ or $2\sigma_{1,0} + \sigma_{1,2 \cdot 3^{2i-1}} = 0$. The possibility that $3\sigma_{1,0} = 0$ contradicts Theorem 2.6, so either $2\sigma_{1,0} + \sigma_{1,3^{2i-1}} = 0$, or $2\sigma_{1,0} + \sigma_{1,2 \cdot 3^{2i-1}} = 0$.

Similarly, by letting $a = b = 1$ we have either $2\sigma_{1,3^{2i-1}} + \sigma_{1,0} = 0$ or $2\sigma_{1,3^{2i-1}} + \sigma_{1,2 \cdot 3^{2i-1}} = 0$.

Finally, letting $a = b = -1$ we have either $2\sigma_{1,2 \cdot 3^{2i-1}} + \sigma_{1,0} = 0$ or $2\sigma_{1,2 \cdot 3^{2i-1}} + \sigma_{1,3^{2i-1}} = 0$.

Considering the two relations $2\sigma_{1,0} + \sigma_{1,3^{2i-1}} = 0$ and $2\sigma_{1,3^{2i-1}} + \sigma_{1,0} = 0$ together, it follows that $3\sigma_{1,0} = 0$, contradicting Theorem 2.6. Similar considerations with pairs of relations rule out several possibilities.

Only two possibilities remain: the first is that $2\sigma_{1,0} + \sigma_{1,3^{2i-1}} = 0$, $2\sigma_{1,3^{2i-1}} + \sigma_{1,2 \cdot 3^{2i-1}} = 0$, and $2\sigma_{1,2 \cdot 3^{2i-1}} + \sigma_{1,0} = 0$; the second is that $2\sigma_{1,0} + \sigma_{1,2 \cdot 3^{2i-1}} = 0$, $2\sigma_{1,3^{2i-1}} + \sigma_{1,0} = 0$, and $2\sigma_{1,2 \cdot 3^{2i-1}} + \sigma_{1,3^{2i-1}} = 0$. Either case quickly implies that $3^{2i}\sigma_{1,0} = 0$, so $\sigma_{1,0} = 0$, again contradicting 2.6. \square

4. APPLICATIONS

Consider a knot with Alexander polynomial

$$\Delta_K(t) = kt^2 - (2k + 1)t + k, k \geq 0.$$

According to Levine [13] such a knot has finite order in the algebraic concordance group. It will have algebraic concordance order 4 if and only if there is some prime congruent to 3 mod 4 which has odd exponent in $4k + 1$. According to [16], if $4k + 1 = 3m$ with m prime to 3 then K is not of order 4 in concordance. We have the following extension.

Corollary 4.1. *If $\Delta_K(t) = kt^2 - (2k + 1)t + k$ and $4k + 1 = 27m$ with m prime to 3 then K is not of order 4 in concordance.*

Proof. Since the Alexander polynomial is quadratic, $H_1(M_K)$ is of rank at most 2. In the case that the rank is 1, then $H_1(M_K)_3 \cong \mathbf{Z}_{27}$ and hence the main theorem of [17] applies to show that K is not of order 4. In the case that the rank of $H_1(M_K)_3$ is 2, then $H_1(M_K)_3 \cong \mathbf{Z}_3 \oplus \mathbf{Z}_9$ and Theorem 1.2 applies. \square

Doubled Knots. Doubled knots have Alexander polynomials as in Corollary 4.1. According to [1, 2] the k -twisted double of the unknot, D_k , is algebraically slice if and only if $4k + 1 = l^2$ for some integer l . We are thus interested in the case that $4k + 1 = 9m^2$ with m prime to 3. For this to occur, m must be odd: $m = 2n + 1$. Solving gives $k = 9(n^2 + n) + 2$. Furthermore, m will be prime to 3 if $n \not\equiv 1 \pmod{3}$.

A similar calculation shows that D_k satisfies $H_1(D_K) \cong \mathbf{Z}_3 \oplus \mathbf{Z}_m$ with m prime to 3 if $k = 3n + 2$ with $n \not\equiv 0 \pmod{3}$. Hence, we have the corollary:

Corollary 4.2. *For all positive $r \not\equiv 0 \pmod{3}$ and positive $s \not\equiv 1 \pmod{3}$, the knot $D_{3r+2} \# D_{9(s^2+s)+2}$ is of algebraic order 4 but is not of order 4 in concordance.*

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