TOPOLOGY PROCEEDINGS

Volume 37, 2011

Pages 155–176

http://topology.auburn.edu/tp/

Prolongational Concepts for Generalized Flows

by

Ronald A. Knight

Electronically published on May 31, 2010

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS
Volume 37 (2011)
Pages 155-176

E-Published on May 31, 2010

PROLONGATIONAL CONCEPTS FOR GENERALIZED FLOWS

RONALD A. KNIGHT

Abstract. The classical prolongational operators have long yielded significant insight into topological classifications and characterizations of points (sets) in dynamical systems. In this paper we introduce and develop basic properties of f-prolongation and f-prolongational limit operators with the expectation of analyzing the topological properties of points (sets) in generalized flows.

1. Introduction

The author defined generalized dynamical systems or f-flows relative to a continuous function $f : X \longrightarrow X$ and developed some of their basic properties in [4]. The generalization was obtained by replacing the initial flow axiom with $\pi(x, 0) = f(x)$ for each $x \in X$ and, of course, this change necessitated a modification of the group axiom. Also, in [4] and [5] some properties of the orbit, orbit closure, and limit operators were obtained.

Historically, flows were generalized by changing the phase space from the plane of Poincare's work to $\mathbb{R}^n$, to metric spaces, and to Hausdorff Spaces. Moreover, the groups $\mathbb{R}$ and $\mathbb{Z}$ were generalized to topological groups. In each case the generalization kept the axioms intact, however, there was a generalization to phase sets which dropped the continuity axiom. For a classical flow the identity homeomorphism $\pi(x, 0) = x$ was the result of an initial

2010 *Mathematics Subject Classification.* Primary 54H99; Secondary 34C99, 34D99, 37B99, 37C99.

Key words and phrases. Continuous flow, discrete flow, f-flow.

condition for certain differential equations. Motivated by systems of differential equations which have a continuous initial condition $\pi(x, 0) = f(x)$, the author introduced the generalized axioms in [4]. As the author noted there, when $f : X \longrightarrow X$ is a homeomorphism of X, the generalized flow is still just a classical flow. However, when a continuous (nonhomeomorphism of X) $f : X \longrightarrow X$ is used, the class of systems to which the axioms apply is expanded to a super class containing the classical flows as a special case. Hence, the structure of a generalized flow offers nothing new relative to classical flows. However, it gives some structure toward classifying and characterizing generalized nonclassical spaces. That is the direction we are now investigating.

In this paper we shall continue the development of these systems by analyzing properties of f-prolongation and f-prolongational limit operators. Although various notations we shall use are not standardized, the definitions and notations used herein are consistent with those used for generalized flows in [4] and [5], as well as, the writings of many researchers (including the author who used them for classical flows in [1] through [3] and several other papers.)

Before beginning our development of the concepts of this paper, we restate the definition of a generalized flow introduced in [4].

Definition 1.1. Let f be a function defined on a space X. A continuous (discrete) f-flow or simply an f-flow (X, π_f) consists of a topological space X and a function π_f from the product space $X \times \mathbb{R}$ $(X \times \mathbb{Z})$ into the space X satisfying the following axioms:

(1) Initial deformation: $\pi_f(x, 0) = f(x)$ for each $x \in X$.
(2) Group deformation action: $\pi_f\big(\pi_f(x, t), s\big) = \pi_f(f(x), t + s)$ for each $x \in X$ and each $t, s \in \mathbb{R}$ $(t, s \in \mathbb{Z})$.
(3) Continuity: $\pi_f : X \times \mathbb{R} \longrightarrow X$ $(\pi_f : X \times \mathbb{Z} \longrightarrow X)$ is continuous.

For brevity, whenever X is understood, we shall say that π_f is an f-flow when referring to an f-flow (X, π_f). If $\mathbb{R}$ $(\mathbb{Z})$ is replaced by $\mathbb{R}^+ = [0, \infty)$ or $\mathbb{R}^- = (-\infty, 0]$ $(\mathbb{Z}^+$ or $\mathbb{Z}^-)$, then (X, π_f) is called a positive or negative (discrete) f-semiflow, respectively. Each type of f-flow π_f is the corresponding type of classical flow if $f = i_X$, the identity on X, in which case we use the usual notation $\pi = \pi_{i_X}$ and refer to it as a dynamical system or a flow.

A given flow f-flow (X,π_f) on a Hausdorff phase space X will be assumed throughout this paper. Despite the fact that corresponding results hold for discrete f-flows and f-semiflows, we state and prove results throughout this paper primarily for continuous f-flows. Note that whenever f is injective, the mappings f^n are defined on the subspaces $f^{|n|}(X)$ of X for each n in $\mathbb{Z}$.

We conclude this section with the following examples of families of continuous and discrete generalized flows given in [4].

The family of f-flows $\pi_f : (\mathbb{R}\times\mathbb{C})\times\mathbb{R} \longrightarrow \mathbb{R}\times\mathbb{C}$ defined by $\pi_f(x,t) = \big(g(x), p_2(x)\exp(it)\big)$ for $x\in\mathbb{R}\times\mathbb{C}$ and $t\in\mathbb{R}$ are products of generalized flows where $f:\mathbb{R}\times\mathbb{C}\longrightarrow\mathbb{R}\times\mathbb{C}$ is defined by $f(x) = \big(g(x), p_2(x)\big)$ with $g:\mathbb{R}\times\mathbb{C}\longrightarrow\mathbb{R}$ continuous and $p_2:\mathbb{R}\times\mathbb{C}\longrightarrow\mathbb{C}$ the second coordinate projection map. This type of f-flow is not induced by a classical flow and does not induce a classical flow for such maps as $g(x,y) = x^3 - x$. This family of f-flows arises from solutions to systems of partial differential equations. The system $u_t = 0$, $v_t = iv$ with initial conditions $u(x_o,0) = g(x_o)$ and $v(x_o,0) = p_2(x_o)$ for $x_o\in\mathbb{R}\times\mathbb{C}$ has solutions $u(x,t) = g(x)$ and $v(x,t) = p_2(x)\exp(it)$ for each $t\in\mathbb{R}$ and $x\in\mathbb{R}\times\mathbb{C}$ determining these f-flows for any continuous g.

For a continuous function $f : X \longrightarrow X$, let (X,π_f) be defined by $\pi_f(x,n) = f^{n+1}(x)$ on $X\times\mathbb{Z}^+$ or defined on $X\times\mathbb{Z}$ whenever f is an embedding. These discrete f-semiflows and f-flows are just translations of discrete semiflows and flows as long as $f(X) = X$. With $f(X)\subset X$, we have $\pi_f(X\times\mathbb{Z}^+)\subset f(X)$. Gordon Thomas Whyburn considered some dynamical properties of a flow generated by iterates of a continuous f with $f(X)\subset X$ in [6, p. 239].

2. Preliminaries

In order to obtain a useful proposition, we begin with the following construction. Let f be a local homeomorphism at a point x_o of X. Then, we can select a neighborhood V_o of x_o such that $(f|V_o) : V_o \longrightarrow V_1$, where $f(V_o) = V_1$, is a homeomorphism. Moreover, by induction, for each n in $\mathbb{Z}^+$ we can choose points x_i in X and open neighborhoods V_i of x_i such that $(f|V_i) : V_i \longrightarrow V_{i+1}$ is a homeomorphism and $f(x_i) = x_{i+1}$ for each integer i where $-n \le i < n$. Whenever X is locally compact we can select compact sets V_i for each i. Note that $\big(f\circ\pi_f^t\big)(V_i) = \pi_f^t\big(f(V_i)\big) = \pi_f^t(V_{i+1})$ for $-n\le i<n$. This construction leads us to the following definition.

Definition 2.1. If a chain of homeomorphic restrictions as described above exists, we shall refer to it as a local f-homeomorphism chain at x_0 and denote it by $\langle f, x_i, V_i, n\rangle$.

The statements in the following proposition will be particularly useful in our verifications of certain properties of f-prolongation and f-prolongational limit operators in § 3 and § 4, respectively.

Proposition 2.2. *Whenever f is a local homeomorphism at x_0 and $\langle f, x_i, V_i, n\rangle$ is a local f-homeomorphism chain at x_0 for n in $\mathbb{Z}^+$, then for each t and s in $\mathbb{R}$ the following are homeomorphisms:*

$$\pi_f^t|V_i : V_i \to \pi_f^t(V_i)$$

for $-n \le i < n-1$,

$$f|\pi_f^t(V_i) : \pi_f^t(V_i) \to \pi_f^t(V_{i+1})$$

for $-n \le i < n-2$, and

$$\pi_f^t|\pi_f^s(V_i) : \pi_f^s(V_i) \to \pi_f^{t+s}(V_{i+1})$$

for $-n \le i < n-2$. Moreover, for each t and s in $\mathbb{R}$ and for each n in $\mathbb{Z}^+$, we have the following equations:

(1) $(\pi_f^t|V_i)^{-1} = (f|V_i)^{-1} \circ (f|V_{i+1})^{-1} \circ \big(\pi_f^{-t}|\pi_f^t(V_i)\big)$ *for $-n \le i < n-1$,*
(2) $\big(f|\pi_f^t(V_i)\big)^{-1} = (\pi_f^t|V_i) \circ (f|V_i)^{-1} \circ (\pi_f^t|V_{i+1})^{-1}$ *for $-n \le i < n-2$,*
(3) $\big(\pi_f^t|\pi_f^s(V_i)\big)^{-1} = \big(f|\pi_f^s(V_i)\big)^{-1} \circ \big(f|\pi_f^s(V_{i+1})\big)^{-1} \circ \big(\pi_f^{-t}|\pi_f^{t+s}(V_{i+1})\big)$ *for $-n \le i < n-2$,*
(4) $(\pi_f^t|V_{i+k+1}) \circ (f|V_{i+k}) \circ \cdots \circ (f|V_i) = \big(f|\pi_f^t(V_{i+k})\big) \circ \cdots \circ \big(f|\pi_f^t(V_i)\big) \circ (\pi_f^t|V_i)$ *for $-n \le i < i+k+1 \le n$, or simply,*
(5) $\big(\pi_f^t \circ f^m\big)|V_i = f^m \circ \big(\pi_f^t|V_i\big)$ *for $m \ge 0$, for $-n \le i < n$,*
(6) $\big(\pi_f^t|\pi_f^s(V_{i+k+1})\big) \circ \big(f|\pi_f^s(V_{i+k})\big) \circ \cdots \circ \big(f|\pi_f^s(V_i)\big) = \big(f|\pi_f^{t+s}(V_{i+k+1})\big) \circ \cdots \circ \big(f|\pi_f^{t+s}(V_{i+1})\big) \circ \big(\pi_f^t|\pi_f^s(V_i)\big)$ *for $-n \le i < i+k+2 \le n$, or simply,*
(7) $\big(\pi_f^t \circ f^m\big)|\pi_f^s(V_i) = f^m \circ \big(\pi_f^t|\pi_f^s(V_i)\big)$ *for $m \ge 0$, for $-n \le i < n$,*
(8) $(\pi_f^t|V_i) \circ (f|V_i)^{-1} \circ \cdots \circ (f|V_{i+k})^{-1} = \big(f|\pi_f^t(V_i)\big)^{-1} \circ \cdots \circ \big(f|\pi_f^t(V_{i+k})\big)^{-1} \circ (\pi_f^t|V_{i+k+1})$ *for $-n \le i \le i+k < n$,*

(9) $\big(\pi_f^t|\pi_f^s(V_i)\big) \circ \big(f|\pi_f^s(V_i)\big)^{-1} \circ \cdots \circ \big(f|\pi_f^s(V_{i+k})\big)^{-1} = \big(f|\pi_f^{t+s}(V_{i+1})\big)^{-1} \circ \cdots \circ \big(f|\pi_f^{t+s}(V_{i+k+1})\big)^{-1} \circ \big(\pi_f^t|\pi_f^s(V_{i+k+1})\big)$ *for* $-n \leq i \leq i+k < n$,

(10) $\big(\pi_f^t|\pi_f^s(V_i)\big) \circ (\pi_f^s|V_i) = (\pi_f^{t+s}|V_{i+1}) \circ (f|V_i)$ *for* $-n \leq i \leq n-1$, *and*

(11) $\big(\pi_f^t|\pi_f^{s+r}(V_{i+1})\big)\circ\big(\pi_f^s|\pi_f^r(V_i)\big) = \big(\pi_f^{t+s}|\pi_f^r(V_{i+1})\big)\circ\big(f|\pi_f^r(V_i)\big)$ *for* $-n \leq i \leq n-1$.

Proof. Let f be a local homeomorphism at x_0 and let $\langle f, x_i, V_i, n\rangle$ a local f-homeomorphism chain at x_0 where $n > 0$. We shall show first that $\pi_f^t|V_i$ is a homeomorphism for $-n \leq i < n-1$. The mapping $\pi_f^t|V_i : V_i \to \pi_f^t(V_i)$ is injective, and hence, is bijective, since if $(\pi_f^t|V_i)(x) = (\pi_f^t|V_i)(y)$ for some points x and y in V_i and t in $\mathbb{R}$, then we have $\big(\pi_f^{-t}|\pi_f^t(V_i)\big)\big((\pi_f^t|V_i)(x)\big) = \big(\pi_f^{-t}|\pi_f^t(V_i)\big)\big((\pi_f^t|V_i)(y)\big)$, $(\pi_f^{t-t}|V_{i+1})\big((f|V_i)(x)\big)=(\pi_f^{t-t}|V_{i+1})\big((f|V_i)(y)\big)$, $(\pi_f^0|V_{i+1})\big((f|V_i)(x)\big) = (\pi_f^0|V_{i+1})\big((f|V_i)(y)\big)$, $(f|V_{i+1})\big((f|V_i)(x)\big) = (f|V_{i+1})\big((f|V_i)(y)\big)$, and $x = y$. Also, for each $x \in V_i$ and $t \in \mathbb{R}$ we have $\big(\pi_f^{-t}|\pi_f^t(V_i)\big) \circ (\pi_f^t|V_i)(x) = \pi_f(\pi_f(x,t) - t) = \pi_f(f(x), 0) = f^2(x) = (f|V_{i+1}) \circ (f|V_i)(x)$. Therefore, since $\pi_f^{-t}|\pi_f^t(V_i)$, $(f|V_i)^{-1}$, and $(f|V_{i+1})^{-1}$ are continuous, we have $(\pi_f^t|V_i)^{-1}$ continuous because $(\pi_f^t|V_i)^{-1} = (f|V_i)^{-1}\circ(f|V_{i+1})^{-1}\circ\big(\pi_f^{-t}|\pi_f^t(V_i)\big)\big)$. Consequently, $\pi_f^t|V_i$ is a homeomorphism.

Next, we show that $f|\pi_f^t(V_i)$ is a homeomorphism for $-n \leq i < n-2$. In order to verify that $f|\pi_f^t(V_i)$ is an injective mapping, let x and y be points in $\pi_f^t(V_i)$ such that $\big(f|\pi_f^t(V_i)\big)(x) = \big(f|\pi_f^t(V_i)\big)(y)$ where $x = \pi_f^t(p)$ and $y = \pi_f^t(q)$ for $p, q \in V_i$. Then, since $\pi_f^t|V_{i+1}$ and $f|V_i$ are homeomorphisms, we have $f\big(\pi_f(p,t)\big) = f\big(\pi_f(q,t)\big)$, $\pi_f(f(p),t) = \pi_f(f(q),t)$, $f(p) = f(q)$, $p = q$, $\pi_f(p,t) = \pi_f(q,t)$, and $x = y$. Moreover, the mapping $f|\pi_f^t(V_i)$ is a bijection since $f\big(\pi_f^t(V_i)\big) = \pi_f^t\big(f(V_i)\big) = \pi_f^t(V_{i+1})$. The mapping $(\pi_f^t|V_i) \circ (f|V_i)^{-1} \circ (\pi_f^t|V_{i+1})^{-1}$ is the composition of continuous functions, and hence, by employing $(\pi_f^t|V_{i+1})^{-1} = (f|V_{i+1})^{-1} \circ (f|V_{i+2})^{-1} \circ \big(\pi_f^{-t}|\pi_f^t(V_{i+1})\big)$ of (1) in the composition of $(\pi_f^t|V_i) \circ (f|V_i)^{-1} \circ (\pi_f^t|V_{i+1})^{-1}$ and $f|\pi_f^t(V_i)$, we see that (2) holds and $\big(f|\pi_f^t(V_i)\big)^{-1}$ is continuous, and thus, $f|\pi_f^t(V_i)$ is a homeomorphism.

Next, we show that $\pi_f^t|\pi_f^s(V_i)$ is a homeomorphism for $-n \le i < n-2$. In order to establish that $\pi_f^t|\pi_f^s(V_i)$ is an injective mapping, let x and y be points in $\pi_f^s(V_i)$ such that $\pi_f^s(x) = \pi_f^s(y)$. There are points p and q in V_i such that $\pi_f^s(p) = x$ and $\pi_f^s(q) = y$. Hence, since $\pi_f^{t+s}|V_{i+1}$, $f|V_{i+1}$, and $\pi_f^s|V_i$ are homeomorphisms, we have $\pi_f^t\big(\pi_f^s(p)\big) = \pi_f^t\big(\pi_f^s(q)\big)$, $\pi_f^{t+s}\big(f(p)\big) = \pi_f^{t+s}\big(f(q)\big)$, $f(p) = f(q)$, $p = q$, $\pi_f^s(p) = \pi_f^s(q)$, and $x = y$. Moreover, the mapping $\pi_f^t|\pi_f^s(V_i)$ is a bijection since $\big(\pi_f^t|\pi_f^s(V_i)\big)\circ\pi_f^s(V_i) = \pi_f\big(\pi_f(V_i,s),t\big) = \pi_f\big(f(V_i),t+s\big) = \pi_f^{t+s}(V_{i+1})$. The mapping $\big(f|\pi_f^s(V_i)\big)^{-1} \circ \big(f|\pi_f^s(V_{i+1})\big)^{-1} \circ \big(\pi_f^{-t}|\pi_f^{t+s}(V_{i+1})\big)$ is the composition of continuous functions, and hence, by its composition with the mapping $\pi_f^t|\pi_f^s(V_i)$ we see that (3) holds and $\big(f|\pi_f^s(V_i)\big)^{-1}$ is continuous. Wherefore, $\pi_f^t|\pi_f^s(V_i)$ is a homeomorphism.

Verification of (4) through (9) is similar to that of (2.1) of Proposition 2.2 in [4] with the appropriate restrictions, and finally, verification of (10) and (11) is immediate from Definition 1.1 taking into account the restrictions. $\square$

3. f-prolongation Operators

Classically, points (sets) are classified and characterized in terms of the orbit, orbit closure, orbital limit, prolongation, and prolongational limit operators. This includes employing these notions in the examination of various forms of stability and attraction. The first three of the above generalized operators were introduced and studied in [4] and [5]. In this section we shall introduce f-prolongations for generalized flows and obtain some of their properties for the same purpose in generalized flows.

Definition 3.1. For each x in X we define and denote the f-prolongation, positive f-prolongation, and negative f-prolongation operators, respectively, by

(1) $D_f(x) = \{y|\pi_f(x_i,t_i) \to y$ for some nets $x_i \to x$ and $t_i \in \mathbb{R}\}$,

(2) $D_f^+(x) = \{y|\pi_f(x_i,t_i) \to y$ for some nets $x_i \to x$ and $t_i \ge 0\}$, and

(3) $D_f^-(x) = \{y|\pi_f(x_i,t_i) \to y$ for some nets $x_i \to x$ and $t_i \le 0\}$.

For each subset M of X we shall denote $\cup_{x\in M} D_f(x)$, $\cup_{x\in M} D_f^+(x)$, and $\cup_{x\in M} D_f^-(x)$ by $D_f(M)$, $D_f^+(M)$, and $D_f^-(M)$, respectively.

Proposition 3.2. *For each x in X,*

(1) $D_f^+(x) = \cap\{\overline{\pi_f(U,\mathbb{R}^+)}|U \in \eta(x)\}$,
(2) $D_f^-(x) = \cap\{\overline{\pi_f(U,\mathbb{R}^-)}|U \in \eta(x)\}$, *and*
(3) $D_f(x) = \cap\{\overline{\pi_f(U,\mathbb{R})}|U \in \eta(x)\}$.

Proof. Let $y \in D_f^+(x)$. Choose $x_i \to x$ and $t_i \geq 0$ such that $\pi_f(x_i,t_i) \to y$. Then, for U a neighborhood of x, $\pi_f(x_i,t_i) \in \pi_f(U,\mathbb{R}^+)$ for $i \geq i_U$ for some i_U. Thus, $y \in \overline{\pi_f(U,\mathbb{R}^+)}$, and hence, $D_f^+(x) \subset \cap\{\overline{\pi_f(U,\mathbb{R}^+)}|U \in \eta(x)\}$. On the other hand, for $y \in \cap\{\overline{\pi_f(U,\mathbb{R}^+)}|U \in \eta(x)\}$, choose a net $(y_{(V,U)})$ by selecting $y_{(V,U)} \in V \cap \pi_f(U,\mathbb{R}^+)$ for each $V \in \eta(y)$ and $U \in \eta(x)$ where the subscripts have the natural Cartesian product order. There is an $x_{(V,U)} \in U$ and a value $t_{(V,U)} \in \mathbb{R}^+$ such that $y_{(V,U)} = \pi_f(x_{(V,U)}, t_{(V,U)})$ for each $V \in \eta(y)$ and $U \in \eta(x)$. Now, $x_{(V,U)} \to x$ and $y_{(V,U)} \to y$ with $t_{(V,U)} \in \mathbb{R}^+$, and hence, $y \in D_f^+(x)$ and $\cap\{\overline{\pi_f(U,\mathbb{R}^+)}|U \in \eta(x)\} \subset D_f^+(x)$ yielding the first equation. The dual and bilateral equations follow similarly. □

Corollary 3.3. *For each point x in X, $D_f^+(x)$, $D_f^-(x)$, and $D_f(x)$ are closed sets.*

Proposition 3.4. *Statements (1) through (14) of the following and their duals hold for each x in X and t in $\mathbb{R}$. If f is an embedding of X or else X is locally compact and f is a local homeomorphism at x, then equality holds for each set inclusion in (1) through (5) for each n in $\mathbb{Z}$; moreover, (15) through (17) hold for each x in X, and t and s in $\mathbb{R}$.*

(1) $f(D_f(x)) \subset C_f(D_f(x)) \subset D_f(f(x)) \subset D_f(C_f^+(x)) \subset D_f(C_f(x))$
(2) $f(D_f^+(x)) \subset C_f^+(D_f^+(x)) \subset D_f^+(f(x)) \subset D_f^+(C_f^+(x)) \subset D_f^+(C_f(x))$
(3) $f^n(D_f(x)) \subset D_f(f^n(x))$ *for each* $n \in \mathbb{Z}^+$.
(4) $f^n(D_f^+(x)) \subset D_f^+(f^n(x))$ *for each* $n \in \mathbb{Z}^+$.
(5) $\pi_f(D_f(x),t) \subset D_f(\pi_f(x,t))$
(6) $\pi_f(D_f^+(x),t) \subset D_f^+(\pi_f(x,t))$

(7) $D_f(x) = D_f^+(x) \cup D_f^-(x)$
(8) $K_f(x) \subset D_f(x)$
(9) $K_f^+(x) \subset D_f^+(x)$
(10) *$x \in D_f(y)$ implies $f^2(y) \in D_f(x)$ (or $y \in D_f(f^{-2}(x))$ if f is an embedding).*
(11) *$x \in D_f^+(y)$ implies $f^2(y) \in D_f^-(x)$ (or $y \in D_f^-(f^{-2}(x))$ if f is an embedding).*
(12) *$x \in D_f^+(y)$ if and only if $y \in D_f^-(f^{-2}(x))$ when f is an embedding.*
(13) *$D(x) = D_f(f^{-1}(x))$ and $D_f(x) = D(f(x))$ whenever one of π or π_f induces the other.*
(14) *$D^+(x) = D_f^+(f^{-1}(x))$ and $D_f^+(x) = D^+(f(x))$ whenever one of π or π_f induces the other.*
(15) $D_f(\pi_f(x,t)) = D_f(\pi_f(x,s)) = D_f(f(x))$
(16) *$D_f^+(\pi_f(x,t)) \subset D_f^+(\pi_f(x,s))$ for $s \leq t$.*
(17) *$D_f^+(\pi_f(x,t)) \subset D_f^+(f(x))$ for $t \geq 0$.*

Proof. In (1) we have $f(D_f(x)) \subset C_f(D_f(x))$ since $f(M) \subset C_f(M)$ for each $M \subset X$. For $y \in C_f(D_f(x))$ choose $z \in D_f(x)$ and $t \in \mathbb{R}$ such that $y = \pi_f(z,t)$. Choose $x_i \to x$ and $t_i \in \mathbb{R}$ so that $\pi_f(x_i,t_i) \to z$. Then, $\pi_f(f(x_i), t_i + t) = \pi_f(\pi_f(x_i,t_i),t) \to \pi_f(z,t) = y$ where $f(x_i) \to f(x)$, and hence, $y \in D_f(f(x))$ yielding $C_f(D_f(x)) \subset D_f(f(x))$ of (1). Finally in (1), since $f(x) \in C_f^+(x)$, we have $D_f(f(x)) \subset D_f(C_f^+(x)) \subset D_f(C_f(x))$. The inclusions of (2) follow similarly.

To obtain (3), let $n \in \mathbb{Z}^+$ and $y \in f^n(D_f(x))$. There is a $z \in D_f(x)$ such that $y = f^n(z)$. Choose $x_i \to x$ and $t_i \in \mathbb{R}$ so that $\pi_f(x_i,t_i) \to z$. Then, $\pi_f(f^n(x_i),t_i) = f^n(\pi_f(x_i,t_i)) \to f^n(z) = y$ where $f^n(x_i) \to f^n(x)$, and hence, $y \in D_f(f^n(x))$ yielding $f^n(D_f(x)) \subset D_f(f^n(x))$. The inclusions of (4) follow similarly.

To verify (5), let $z \in D_f(x)$ with $x_i \to x$ and $t_i \in \mathbb{R}$ where $\pi_f(x_i,t_i) \to z$. Then, $\pi_f(\pi_f(x_i,t),t_i) = \pi_f(\pi_f(x_i,t_i),t) \to \pi_f(z,t)$ where $\pi_f(x_i,t) \to \pi_f(x,t)$, and hence, $\pi_f(z,t) \in D_f(\pi_f(x,t))$ yielding $\pi_f(D_f(x),t) \subset D_f(\pi_f(x,t))$. Statement (6) follows similarly.

Statements (7), (8), and (9) follow immediately from the definitions.

To establish (10), let $x \in D_f(y)$. Choose nets $y_i \to y$ and (t_i) in $\mathbb{R}$ such that $\pi_f(y_i,t_i) \to x$. Then, $\pi_f(\pi_f(y_i,t_i),-t_i) =$

$\pi_f(f(y_i), t_i - t_i) = f^2(y_i) \to f^2(y)$ with $\pi_f(y_i, t_i) \to x$ and $(-t_i)$ in $\mathbb{R}$. Therefore, $f^2(y) \in D_f(x)$. Whenever f is a homeomorphism of X, $\pi_f(\pi_f(f^{-2}(y_i), t_i), -t_i) = y_i \to y$ with $\pi_f(f^{-2}(y_i), t_i) = f^{-2}(\pi_f(y_i, t_i)) \to f^{-2}(x)$ and $(-t_i)$ is in $\mathbb{R}$. Consequently, $y \in D_f(f^{-1}(x))$ and (10) holds. If (t_i) is in $\mathbb{R}^+$, $(-t_i)$ is in $\mathbb{R}^-$, and so, (11) follows similarly.

Statement (12) follows from (11) of Proposition 3.4 and its dual.

Statements (13) and (14) follow from Definition 3.1, continuity, and the equations $\pi = \pi_f \circ (f^{-1} \times i_{\mathbb{R}})$ and $\pi_f = \pi \circ (f \times i_{\mathbb{R}})$ of Corollaries 2.3 and 2.4 of [4].

If f is an embedding of X, then to verify equality for each set inclusion of (1), let $y \in D_f(C_f(x))$, and in particular, let $y \in D_f(\pi_f(x,t))$. Choose $y_i \to \pi_f(x,t)$ and (t_i) in $\mathbb{R}$ such that $\pi_f(y_i, t_i) \to y$. Note that $\pi_f(f^{-2}(y_i), -t) = f^{-2}(\pi_f(y_i, -t)) \to f^{-2}(\pi_f(\pi_f(x,t), -t)) = f^{-2}(\pi_f(f(x), 0)) = f^{-2}(f^2(x)) = x$ and $\pi_f(\pi_f(f^{-2}(y_i), -t), t + t_i) = \pi_f(f^{-1}(y_i), t_i) = f^{-1}(\pi_f(y_i, t_i)) \to f^{-1}(y)$, so that, $f^{-1}(y) \in D_f(x)$ or $y \in f(D_f(x))$. Hence, $D_f(C_f(x)) \subset f(D_f(x))$, and equality holds for each set inclusion of (1).

Next, let X be locally compact and f be a local homeomorphism at x. As above, let $y \in D_f(C_f(x))$ where $y \in D_f(\pi_f(x,t))$. Select a local f-homeomorphism chain $\langle f, x_i, V_i, n \rangle$ at x where $x_0 = x$, $n > 3$, and V_i is a compact neighborhood of x_i for each i. Choose (y_i) in $\pi_f^t(V_0)$ and (t_i) in $\mathbb{R}$ such that $y_i \to \pi_f^t(x)$ and $\pi_f^{t_i}(y_i) \to y$. Now, by Proposition 2.2, $\left(\pi_f^{-t} | \pi_f^t(V_{-2})\right) \circ \left(f | \pi_f^t(V_{-2})\right)^{-1} \circ \left(f | \pi_f^t(V_{-1})\right)^{-1}(y_i)$ $= \left(f | \pi_f^0(V_{-1})\right)^{-1} \circ \left(f | \pi_f^0(V_0)\right)^{-1} \circ \left(\pi_f^{-t} | \pi_f^t(V_0)\right)(y_i) = (f|V_0)^{-1} \circ (f|V_1)^{-1} \circ \left(\pi_f^{-t} | \pi_f^t(V_0)\right)(y_i) \to (f|V_0)^{-1} \circ (f|V_1)^{-1} \circ \left(\pi_f^{-t} | \pi_f^t(V_0)\right) \circ (\pi_f^t | V_0)(x) = (f|V_0)^{-1} \circ (f|V_1)^{-1} \circ (\pi_f^0 | V_1) \circ (f|V_0)(x) = (f|V_0)^{-1} \circ (f|V_1)^{-1} \circ (f|V_1) \circ (f|V_0)(x) = x$. Also, by Proposition 2.2, $(\pi_f^{t+t_i} | V_0)$ $\circ \left(\pi_f^{-t} | \pi_f^t(V_{-2})\right) \circ \left(f | \pi_f^t(V_{-2})\right)^{-1} \circ \left(f | \pi_f^t(V_{-1})\right)^{-1}(y_i) = \left(\pi_f^{t_i} | \pi_f^t(V_{-1})\right)$ $\circ \left(f | \pi_f^t(V_{-2})\right) \circ \left(f | \pi_f^t(V_{-2})\right)^{-1} \circ \left(f | \pi_f^t(V_{-1})\right)^{-1}(y_i) = \left(\pi_f^{t_i} | \pi_f^t(V_{-1})\right) \circ \left(f | \pi_f^t(V_{-1})\right)^{-1}(y_i) = \left(\pi_f^{t_i} | \pi_f^t(V_{-1})\right)(p_i)$ where we have $p_i = \left(f | \pi_f^t(V_{-1})\right)^{-1}(y_i)$. Because of the compactness of each V_i, we can assume that (y_i) and (t_i) have been selected so that nets (p_i) and $\left(\left(\pi_f^{t_i} | \pi_f^t(V_{-1})\right)(p_i)\right)$ converge to points p and z, respectively. Note that $(\pi_f^{t+t_i}|V_0)\circ\big(\pi_f^{-t}|\pi_f^t(V_{-2})\big)\circ\big(f|\pi_f^t(V_{-2})\big)^{-1}\circ\big(f|\pi_f^t(V_{-1})\big)^{-1}(y_i)$ $\to z$ where $\big(\pi_f^{-t}|\pi_f^t(V_{-2})\big)\circ\big(f|\pi_f^t(V_{-2})\big)^{-1}\circ\big(f|\pi_f^t(V_{-1})\big)^{-1}(y_i)\to x$ so that $z\in D_f(x)$. Since $f\circ\big(\pi_f^{t_i}|\pi_f^t(V_{-1})\big)(p_i)=\big(\pi_f^{t_i}|\pi_f^t(V_{-1})\big)\circ f(p_i)=\big(\pi_f^{t_i}|\pi_f^t(V_{-1})\big)(y_i)\to y$ and $f\circ\big(\pi_f^{t_i}|\pi_f^t(V_{-1})\big)(p_i)\to f(z)$ we have $y=f(z)$. Consequently, $y=f(z)\in f(D_f(x))$, and hence, $y\in f(D_f(x))$ and equality holds for each set inclusion of (1).

Equality holds for each inclusion of (2) similarly whether f is an embedding of X or else X is locally compact and f is a local homeomorphism at x.

Equality for the set inclusions of (3) and (4), whenever f is an embedding of X or else X is locally compact and f is a local homeomorphism at x, follows from equations $f(D_f(x))=D_f(f(x))$ and $f(D_f^+(x))=D_f^+(f(x))$ of (1) and (2) by induction on $n\in\mathbb{Z}$. The equality in (5) will follow from (1) and (15) as soon as (15) is verified.

To obtain equality for the set inclusion of (16) when f is an embedding of X, let $y\in D_f^+(\pi_f(x,t))$ and $s\le t$. There are nets $y_i\to\pi_f(x,t)$ and $t_i\ge 0$ such that $\pi_f(y_i,t_i)\to y$. Now, $\pi_f(f^{-1}(y_i),s-t)\to\pi_f(f^{-1}(\pi_f(x,t)),s-t)=f^{-1}(\pi_f(f(x),s))=\pi_f(x,s)$ and $\pi_f(\pi_f(f^{-1}(y_i),s-t),t_i+t-s)=\pi_f(f(f^{-1}(y_i)),t_i)=\pi_f(y_i,t_i)\to y$. Hence, $y\in D_f^+(\pi_f(x,s))$ and (16) holds.

To verify equality for the set inclusion of (16) whenever X is locally compact and f is a local homeomorphism at x, let $y\in D_f^+(\pi_f(x,t))$ and $s\le t$. Select a local f-homeomorphism chain $\langle f,x_i,V_i,n\rangle$ at x where $x_0=x$, $n>3$, and V_i is a compact neighborhood of x_i for each i. Choose nets (y_i) in $\pi_f^t(V_0)$ and (t_i) in $\mathbb{R}^+$ such that $y_i\to\pi_f^t(x)$ and such that $\pi_f^{t_i}(y_i)\to y$. By Proposition 2.2, $\big(\pi_f^{s-t}|\pi_f^t(V_{-1})\big)\circ\big(f|\pi_f^t(V_{-1})\big)^{-1}(y_i)\to\big(\pi_f^{s-t}|\pi_f^t(V_{-1})\big)\circ\big(f|\pi_f^t(V_{-1})\big)^{-1}\circ(\pi_f^t|V_0)(x)=\big(f|\pi_f^{(s-t)+t}(V_0)\big)^{-1}\circ\big(\pi_f^{s-t}|\pi_f^t(V_0)\big)\circ(\pi_f^t|V_0)(x)=\big(f|\pi_f^s(V_0)\big)^{-1}\circ(\pi_f^{(s-t)+t}|V_1)\circ(f|V_0)(x)=\big(f|\pi_f^s(V_0)\big)^{-1}\circ(\pi_f^s|V_1)\circ(f|V_0)(x)=(\pi_f^s|V_0)\circ(f|V_0)^{-1}\circ(f|V_0)(x)=(\pi_f^s|V_0)(x)$. Also, by Proposition 2.2, we have $\big(\pi_f^{t_i+t-s}|\pi_f^s(V_0)\big)\circ\big(\pi_f^{s-t}|\pi_f^t(V_{-1})\big)\circ\big(f|\pi_f^t(V_{-1})\big)^{-1}(y_i)=\big(\pi_f^{t_i}|\pi_f^t(V_0)\big)\circ\big(f|\pi_f^t(V_{-1})\big)\circ\big(f|\pi_f^t(V_{-1})\big)^{-1}(y_i)=\big(\pi_f^{t_i}|\pi_f^t(V_0)\big)(y_i)\to y$. Thus, for i large enough, the net (t_i+t-s) is in $\mathbb{R}^+$, and hence, $y\in D_f^+(\pi_f(x,s))$ yielding (16).

Similarly, whether f is a homeomorphism of X or else X is locally compact and f is a local homeomorphism at x, $D_f(\pi_f(x,t)) \subset D_f(\pi_f(x,s))$ for any $t, s \in \mathbb{R}$, and hence, (15) must hold.

Finally, whether f is a homeomorphism of X or else X is locally compact and f is a local homeomorphism at x, (17) is a direct result of (16).

Each dual follows in a similar manner completing the proof. □

The bilateral and dual versions of the following corollary and proposition also hold by similar reasoning.

Corollary 3.5. *If f is an embedding of X or else X is locally compact and f is a local homeomorphism at x, then $D_f^+(x)$ is closed and positively f-invariant.*

Proof. The set is closed by Corollary 3.3. Statement (2) of Proposition 3.4 yields positive f-invariance. □

Proposition 3.6. *If (x_i) and (y_i) are nets such that $x_i \to x$, $y_i \to y$, and $y_i \in D_f^+(x_i)$ for each i, then $y \in D_f^+(x)$.*

Proof. Let $U \in \eta(x)$. Then, the net (x_i) is ultimately in U and the net (y_i) is ultimately in $\overline{\pi_f(U, \mathbb{R}^+)}$ by Proposition 3.2. Thus, $y \in \overline{\pi_f(U, \mathbb{R}^+)}$ for each $U \in \eta(x)$, and hence, $y \in \cap_{U \in \eta(x)} \overline{\pi_f(U, \mathbb{R}^+)} = D_f^+(x)$. □

Corollary 3.7. *If M is compact, then $D_f^+(M)$ is closed.*

Proof. Let (y_i) be a net in $D_f^+(M)$ that converges to a point y. There is a net (x_i) in M such that $y_i \in D_f^+(x_i)$. Some subnet (x_j) of (x_i) converges to a point x in the compact set M. Thus, $y \in D_f^+(x) \subset D_f^+(M)$ and $D_f^+(M)$ is closed. □

4. f-prolongational Limit Operators

As we noted in the last section, classically the prolongational limit notion yields classifications and characterizations of certain points (sets) including those which are stable or attractors. In this section we shall introduce for generalized flows the concept of the f-prolongational limit operator and obtain various of its properties in order to examine topologically properties of points (sets).

Definition 4.1. For each x in X we define and denote the f-prolongational limit, positive f-prolongational limit, and negative f-prolon-gational limit operators, respectively, by

(1) $J_f(x) = \{y|\pi_f(x_i, t_i) \to y$ for some nets $x_i \to x$ and $t_i \to \infty\}$,
(2) $J_f^+(x) = \{y|\pi_f(x_i, t_i) \to y$ for some nets $x_i \to x$ and $t_i \to +\infty\}$, and
(3) $J_f^-(x) = \{y|\pi_f(x_i, t_i) \to y$ for some nets $x_i \to x$ and $t_i \to -\infty\}$.

For each subset M of X we shall denote $\cup_{x\in M} J_f(x)$, $\cup_{x\in M} J_f^+(x)$, and $\cup_{x\in M} J_f^-(x)$ by $J_f(M)$, $J_f^+(M)$, and $J_f^-(M)$, respectively.

Proposition 4.2. *For each x in X and each t and s in $\mathbb{R}$ we have (1) through (17) and their duals. If f is an embedding of X or else X is locally compact and f is a local homeomorphism at x, then equality holds for each set inclusion in (1) through (13), and we have (18) and (19).*

(1) $f(J_f(x)) \subset C_f(J_f(x)) \subset J_f(f(x)) \subset J_f(C_f^+(x)) \subset J_f(C_f(x))$
(2) $f(J_f^+(x)) \subset C_f^+(J_f^+(x)) \subset J_f^+(f(x)) \subset J_f^+(C_f^+(x)) \subset J_f^+(C_f(x))$
(3) $\pi_f(J_f(x), t) \subset J_f(\pi_f(x, t))$
(4) $\pi_f(J_f^+(x), t) \subset J_f^+(\pi_f(x, t))$
(5) $J_f(x) = J_f^+(x) \cup J_f^-(x)$
(6) $D_f(x) = C_f(x) \cup J_f(x)$
(7) $D_f^+(x) = C_f^+(x) \cup J_f^+(x)$
(8) *$x \in J_f(y)$ implies $f^2(y) \in J_f(x)$ (or $y \in J_f(f^{-2}(x))$ if f is an embedding).*
(9) *$x \in J_f^+(y)$ implies $f^2(y) \in J_f^-(x)$ (or $y \in J_f^-(f^{-2}(x))$ if f is an embedding).*
(10) *$x \in J_f^+(y)$ if and only if $y \in J_f^-(f^{-2}(x))$ when f is an embedding.*
(11) *$f^n(J_f(x)) \subset J_f(f^n(x))$ for each $n \in \mathbb{Z}^+$ (or $n \in \mathbb{Z}$ if f is an embedding).*
(12) *$f^n(J_f^+(x)) \subset J_f^+(f^n(x))$ for each $n \in \mathbb{Z}^+$ (or $n \in \mathbb{Z}$ if f is an embedding).*

(13) $f(J_f^+(x)) \subset \cap_{t\in M} J_f^+(\pi_f(x,t)) \subset \cap_{t\in M} D_f^+(\pi_f(x,t))$ *where* M *is* $\mathbb{R}^+$ *or* $\mathbb{R}$.
(14) $L_f(x) \subset J_f(x)$
(15) $L_f^+(x) \subset J_f^+(x)$
(16) $J(x) = J_f(f^{-1}(x))$ *and* $J_f(x) = J(f(x))$ *whenever one of* π *or* π_f *induces the other.*
(17) $J^+(x) = J_f^+(f^{-1}(x))$ *and* $J_f^+(x) = J^+(f(x))$ *whenever one of* π *or* π_f *induces the other.*
(18) $J_f(\pi_f(x,t)) = J_f(\pi_f(x,s)) = J_f(f(x))$
(19) $J_f^+(\pi_f(x,t)) = J_f^+(\pi_f(x,s)) = J_f^+(f(x))$

Proof. We have $f(J_f(x)) \subset C_f(J_f(x))$ since $f(M) \subset C_f(M)$ for each $M \subset X$. Let $y \in C_f(J_f(x))$. There exist a $z \in J_f(x)$ and $t \in \mathbb{R}$ such that $y = \pi_f(z,t)$. There are nets $x_i \to x$ and $t_i \to \infty$ such that $\pi_f(x_i,t_i) \to z$. Since $f(x_i) \to f(x)$ and $\pi_f(f(x_i), t_i+t) = \pi_f(\pi_f(x_i,t_i),t) \to \pi_f(z,t) = y$, we have $y \in J_f(C_f(x))$, and hence, $C_f(J_f(x)) \subset J_f(f(x))$. That $J_f(f(x)) \subset J_f(C_f^+(x)) \subset J_f(C_f(x))$ follows from $f(x) \in C_f^+(x) \subset C_f(x)$ completing the proof of (1). The inclusions of (2) follow similarly.

To verify (3) select $y \in \pi_f(J_f(x),t)$ and $z \in J_f(x)$ such that $y = \pi_f(z,t)$. For some nets $x_i \to x$ and $t_i \to \infty$ we have $\pi_f(x_i,t_i) \to z$. Thus, $\pi_f(\pi_f(x_i,t),t_i) = \pi_f(\pi_f(x_i,t_i),t) \to \pi_f(z,t) = y$. Since $\pi_f(x_i,t) \to \pi_f(x,t)$ and $t_i \to \infty$, $y \in J_f(\pi_f(x,t))$, and hence, $\pi_f(J_f(x),t) \subset J_f(\pi_f(x,t))$. The inclusion (4) follows by choosing $t_i \to +\infty$.

Statements (5), (6), (7), (14), and (15) are evident from the definitions. Also, (16) and (17) follow from Definition 4.1, continuity, and the equations $\pi = \pi_f \circ (f^{-1} \times i_{\mathbb{R}})$ and $\pi_f = \pi \circ (f \times i_{\mathbb{R}})$ of Corollaries 2.3 and 2.4 of [4].

Next, to obtain (8) let $x \in J_f(y)$. Choose nets $y_i \to y$ and $t_i \to \infty$ such that $\pi_f(y_i,t_i) \to x$. Then, $\pi_f(\pi_f(y_i,t_i),-t_i) = \pi_f(f(y_i),t_i - t_i) = f^2(y_i) \to f^2(y)$ with $\pi_f(y_i,t_i) \to x$ and $-t_i \to \infty$, and therefore, $f^2(y) \in J_f(x)$. If f is an embedding of X, $\pi_f(\pi_f(f^{-2}(y_i),t_i),-t_i) = y_i \to y$ with $\pi_f(f^{-2}(y_i),t_i) = f^{-2}(\pi_f(y_i,t_i)) \to f^{-2}(x)$ and $-t_i \to \infty$, whence, $y \in J_f(f^{-2}(x))$ and (8) holds. If $t_i \to +\infty$, then $t_i \to -\infty$ so that (9) follows similarly.

Statement (10) is a result (9) of Proposition 4.2 and its dual.

For (11) we use $f(J_f(x)) \subset J_f(f(x))$ of (1) and induction to obtain $f^n(J_f(x)) \subset J_f(f^n(x))$ for $n \in \mathbb{Z}^+$. When f is an embedding of X, let $y \in f^{-1}(J_f(x))$. Then, for $y = f^{-1}(z)$ where $z \in J_f(x)$ we can choose nets $x_i \to x$ and $t_i \to \infty$ such that $\pi_f(x_i, t_i) \to z$. Thus, $\pi_f(f^{-1}(x_i), t_i) = f^{-1}(\pi_f(x_i, t_i)) \to f^{-1}(z) = y$ where $f^{-1}(x_i) \to f^{-1}(x)$ and $t_i \to \infty$ yielding $y \in J_f(f^{-1}(x))$. Hence, $f^{-1}(J_f(x)) \subset J_f(f^{-1}(x))$. By induction on $n \in \mathbb{Z}^+$ we have $f^{-n}(J_f(x)) \subset J_f(f^{-n}(x))$. Statement (12) follows similarly.

To obtain (13) let $z \in f(J_f^+(x))$ and choose $y \in J_f^+(x)$ such that $z = f(y)$. Select nets $x_i \to x$ and $t_i \to +\infty$ such that $\pi_f(x_i, t_i) \to y$. Since $f(\pi_f(x_i, t_i)) \to f(y) = z$, we have, for each $t \in \mathbb{R}$, $f(\pi_f(x_i, t_i)) = \pi_f(f(x_i), t_i) = \pi_f(\pi_f(x_i, t), t_i - t)$ where $\pi_f(x_i, t) \to \pi_f(x, t)$ and $t_i - t \to +\infty$ so that $z \in J_f^+(\pi_f(x, t)) \subset D_f^+(\pi_f(x, t))$. Thus, $f(J_f^+(x)) \subset J_f^+(\pi_f(x, t)) \subset D_f^+(\pi_f(x, t))$ for each $t \in \mathbb{R}$ and (13) follows.

For the remainder of the proof we either let f be an embedding of X or else let X be locally compact and f be a local homeomorphism at x in order to verify that equality holds for each set containment in (1) through (13) and that (18) and (19) hold under those conditions. Separate verifications of (1) and (19) are given relative to the two hypotheses. The remainder of the parts follow by identical reasoning regardless of which assumption is considered.

To obtain equality in (1) when f is an embedding of X, let $y \in J_f(C_f(x))$ and choose $t \in \mathbb{R}$ such that $y \in J_f(\pi_f(x, t))$. There exist nets $x_i \to \pi_f(x, t)$ and $t_i \to \infty$ such that $\pi_f(x_i, t_i) \to y$. Since $\pi_f(f^{-2}(x_i), -t) = f^{-2}(\pi_f(x_i, -t)) \to f^{-2}(\pi_f(\pi_f(x, t), -t)) = f^{-2}(\pi_f(f(x), 0)) = f^{-2}(f^2(x)) = x$, we have $\pi_f(\pi_f(f^{-2}(x_i), -t), t_i + t) = \pi_f(f^{-1}(x_i), t_i) = f^{-1}(\pi_f(x_i, t_i) \to f^{-1}(y)$ so that $f^{-1}(y) \in J_f(x)$ and $y \in f(J_f(x))$. Consequently, $J_f(C_f(x)) \subset f(J_f(x))$ and equality holds in (1).

Next, let X be locally compact and f be a local homeomorphism at x. We verify equality for the set inclusions of (1) by letting $y \in J_f(C_f(x))$ where $y \in J_f(\pi_f(x, t))$, and by selecting a local f-homeomorphism chain $\langle f, x_i, V_i, n \rangle$ at x where $x_0 = x$, $n > 3$, and V_i is a compact neighborhood of x_i for each i. Now choose (y_i) in $\pi_f^t(V_0)$ and (t_i) in $\mathbb{R}$ such that $y_i \to \pi_f^t(x)$, $t_i \to +\infty$, and $\pi_f^{t_i}(y_i) \to y$.

We obtain $\big(\pi_f^{-t}|\pi_f^t(V_{-2})\big) \circ \big(f|\pi_f^t(V_{-2})\big)^{-1} \circ \big(f|\pi_f^t(V_{-1})\big)^{-1}(y_i) = \big(f|\pi_f^0(V_{-1})\big)^{-1} \circ \big(f|\pi_f^0(V_0)\big)^{-1} \circ \big(\pi_f^{-t}|\pi_f^t(V_0)\big)(y_i) = (f|V_0)^{-1} \circ (f|V_1)^{-1} \circ \big(\pi_f^{-t}|\pi_f^t(V_0)\big)(y_i) \to (f|V_0)^{-1} \circ (f|V_1)^{-1} \circ \big(\pi_f^{-t}|\pi_f^t(V_0)\big) \circ (\pi_f^t|V_0)(x) = (f|V_0)^{-1} \circ (f|V_1)^{-1} \circ (\pi_f^0|V_1) \circ (f|V_0)(x) = (f|V_0)^{-1} \circ (f|V_1)^{-1} \circ (f|V_1) \circ (f|V_0)(x) = x$ by Proposition 2.2. Note that $(\pi_f^{t+t_i}|V_0) \circ \big(\pi_f^{-t}|\pi_f^t(V_{-2})\big) \circ \big(f|\pi_f^t(V_{-2})\big)^{-1} \circ \big(f|\pi_f^t(V_{-1})\big)^{-1}(y_i) = \big(\pi_f^{t_i}|\pi_f^t(V_{-1})\big) \circ \big(f|\pi_f^t(V_{-2})\big) \circ \big(f|\pi_f^t(V_{-2})\big)^{-1} \circ \big(f|\pi_f^t(V_{-1})\big)^{-1}(y_i) = \big(\pi_f^{t_i}|\pi_f^t(V_{-1})\big) \circ \big(f|\pi_f^t(V_{-1})\big)^{-1}(y_i) = \big(\pi_f^{t_i}|\pi_f^t(V_{-1})\big)(p_i)$ where we have $p_i = \big(f|\pi_f^t(V_{-1})\big)^{-1}(y_i)$ and $t + t_i \to +\infty$. In view of the compactness of each V_i we can assume that (y_i) and (t_i) have been selected so that nets (p_i) and $\Big(\big(\pi_f^{t_i}|\pi_f^t(V_{-1})\big)(p_i)\Big)$ converge to a pair of points p and z, respectively. Since $\big(\pi_f^{-t}|\pi_f^t(V_{-2})\big) \circ \big(f|\pi_f^t(V_{-2})\big)^{-1} \circ \big(f|\pi_f^t(V_{-1})\big)^{-1}(y_i) \to x$ and $t + t_i \to +\infty$ where $(\pi_f^{t+t_i}|V_0) \circ \big(\pi_f^{-t}|\pi_f^t(V_{-2})\big) \circ \big(f|\pi_f^t(V_{-2})\big)^{-1} \circ \big(f|\pi_f^t(V_{-1})\big)^{-1}(y_i) \to z$, we have $z \in J_f(x)$. Also, we have $f \circ \big(\pi_f^{t_i}|\pi_f^t(V_{-1})\big)(p_i) = \big(\pi_f^{t_i}|\pi_f^t(V_{-1})\big) \circ f(p_i) = \big(\pi_f^{t_i}|\pi_f^t(V_{-1})\big)(y_i) \to y$ and $f \circ \big(\pi_f^{t_i}|\pi_f^t(V_{-1})\big)(p_i) \to f(z)$ yielding $y = f(z) \in f(J_f(x))$, and hence, equality holds for each set inclusion of (1).

Equality holds for each inclusion of (2) similarly whether f is an embedding of X or else X is locally compact and f is a local homeomorphism at x.

The equality for each set inclusion in (3) and (4) whether f is an embedding of X or else X is locally compact and f is a local homeomorphism at x will follow from (1) and (18) and (2) and (19), respectively, once (18) and (19) are verified. Also, equality for the set inclusions of (11) and (12) follows from equations $f(J_f(x)) = J_f(f(x))$ and $f(J_f^+(x)) = J_f^+(f(x))$ of(1) and (2), respectively, by induction on $n \in \mathbb{Z}$.

To show equality in (13) when f is an embedding of X or else X is locally compact and f is a local homeomorphism at x, let $t_o \in M$ and let $y \in \cap_{t\in M} D_f^+(\pi_f(x,t))$ for $M = \mathbb{R}^+$ or $\mathbb{R}$. There are nets x_i in X and (t_i) in $\mathbb{R}^+$ or $\mathbb{R}$ such that $x_i \to x$, $\pi_f(x_i,t_o) = y_i$ for each i, and $\pi_f(f(x_i), t_o + t_i) = \pi_f(\pi_f(x_i,t_o),t_i) = \pi_f(y_i,t_i) \to y$ where $f(x_i) \to f(x)$ and $(t_o + t_i)$ is in $\mathbb{R}^+$ or $\mathbb{R}$ so that $y \in D_f^+(f(x)) = f(D_f^+(x))$. If no subnet (t_j) of (t_i) exists such that $t_j \to +\infty$, then (t_i) is bounded above by some $T > 0$, and hence, we have the impossible condition $y \notin D_f^+(\pi_f(x, T+1))$. Therefore, choose (t_i) such that $t_i \to +\infty$. We now have $t_o + t_i \to +\infty$, so that, $y \in J_f^+(f(x)) = f(J^+(x))$, and hence, equality holds in (13).

To obtain equality for the set inclusion of (19) when f is an embedding of X, let $y \in J_f^+(\pi_f(x,t))$ and $s, t \in \mathbb{R}$. There are nets $y_i \to \pi_f(x,t)$ and $t_i \to +\infty$ such that $\pi_f(y_i, t_i) \to y$. Now, $\pi_f(f^{-1}(y_i), s-t) \to \pi_f(f^{-1}(\pi_f(x,t)), s-t) = f^{-1}(\pi_f(f(x), s)) = \pi_f(x,s)$ and $\pi_f(\pi_f(f^{-1}(y_i), s-t), t_i+t-s) = f^{-1}(\pi_f(\pi_f(y_i,t_i),0)) \to f^{-1}(\pi_f(y,0)) = f^{-1}(f(y)) = y$ where $t_i + t - s \to +\infty$. Hence, $y \in J_f^+(\pi_f(x,s))$ and (19) holds because s and t are arbitrary. For $t = 0$ we also have $J_f^+(\pi_f(x,s)) = J_f^+(f(x))$.

As above, to verify equality for the set inclusion of (18) when X is locally compact and f is a local homeomorphism at x, let $y \in J_f^+(\pi_f(x,t))$ and $s \leq t$. Select a local f-homeomorphism chain $\langle f, x_i, V_i, n\rangle$ at x where $x_0 = x$, $n > 3$, and V_i is a compact neighborhood of x_i for each i. Choose nets (y_i) in $\pi_f^t(V_0)$ and (t_i) in $\mathbb{R}$ such that $y_i \to \pi_f^t(x)$, $t_i \to +\infty$, and $\pi_f^{t_i}(y_i) \to y$. From Proposition 2.2, we have $\big(\pi_f^{s-t}|\pi_f^t(V_{-1})\big)\circ\big(f|\pi_f^t(V_{-1})\big)^{-1}(y_i) \to \big(\pi_f^{s-t}|\pi_f^t(V_{-1})\big)\circ \big(f|\pi_f^t(V_{-1})\big)^{-1} \circ (\pi_f^t|V_0)(x) = \big(f|\pi_f^s(V_0)\big)^{-1} \circ \big(\pi_f^{s-t}|\pi_f^t(V_0)\big) \circ (\pi_f^t|V_0)(x) = \big(f|\pi_f^s(V_0)\big)^{-1} \circ (\pi_f^s|V_1) \circ (f|V_0)(x) = \big(f|\pi_f^s(V_0)\big)^{-1} \circ (\pi_f^s|V_1) \circ (f|V_0)(x) = (\pi_f^s|V_0) \circ (f|V_0)^{-1} \circ (f|V_0)(x) = (\pi_f^s|V_0)(x)$. Since we also have $\big(\pi_f^{t_i+t-s}|\pi_f^s(V_0)\big) \circ \big(\pi_f^{s-t}|\pi_f^t(V_{-1})\big) \circ \big(f|\pi_f^t(V_{-1})\big)^{-1}(y_i) = \big(\pi_f^{t_i}|\pi_f^t(V_0)\big)\circ\big(f|\pi_f^t(V_{-1})\big)\circ\big(f|\pi_f^t(V_{-1})\big)^{-1}(y_i) = \big(\pi_f^{t_i}|\pi_f^t(V_0)\big)(y_i) \to y$ and $t_i + t - s \to +\infty$, we have $y \in J_f^+(\pi_f(x,s))$ yielding (19).

Similarly, whether f is an embedding of X or else X is locally compact and f is a local homeomorphism at x, $J_f(\pi_f(x,t)) \subset J_f(\pi_f(x,s))$ for any $t, s \in \mathbb{R}$, and hence, (18) must hold.

The duals follow in a similar manner completing the proof of the proposition. $\square$

The dual and bilateral versions of the following corollaries, propositions, and theorems through Theorem 4.13 are valid except where noted. Each can be verified by the obvious parallel reasoning.

Corollary 4.3. *If f is an embedding of X or else X is locally compact and f is a local homeomorphism at x, then $J_f^+(x)$ is closed and f-invariant.*

Proof. By (2) and (13) of Proposition 4.2, $C_f^+(J_f^+(x)) = f(J_f^+(x)) = \cap_{t\in\mathbb{R}^+} D_f^+(\pi_f(x,t))$, and thus, $J_f^+(x)$ is positively f-invariant and closed. Since $f(J_f^+(x)) \subset C_f^-(J_f^+(x))$, let $y \in C_f^-(J_f^+(x))$. Choose $z \in J_f^+(x)$ and $t \leq 0$ such that $y = \pi_f(z,t)$. There are nets $x_i \to x$ and $t_i \to +\infty$ such that $\pi_f(x_i,t_i) \to z$. Thus, $\pi_f(f(x_i), t+t_i) = \pi_f(\pi_f(x_i,t_i),t) \to y$ where $f(x_i) \to f(x)$ and $t_i + t \to +\infty$, and hence, $y \in J_f^+(f(x)) = f(J_f^+(x))$, i.e. $C_f^-(J_f^+(x)) = f(J_f^+(x))$. Now $f(J_f^+(x)) = C_f^-(J_f^+(x)) \cup C_f^+(J_f^+(x)) = C_f(J_f^+(x))$ and $J_f^+(x)$ is f-invariant. □

The extended f-flow $(X^*, \pi^*_{f^*})$ on the one point compactification X^* of a locally compact phase space X is examined in [4,pp. 171-2]. We consider the f^*-prolongation and f^*-prolongational limit operators in the following proposition and its corollary.

Proposition 4.4. *Let X be locally compact and let f be an embedding of X. Then, for each x in X,*

$$J_{f^*}^{*+}(x) = \begin{cases} J_f^+(x) & \text{if } J_f^+(x) \text{ is compact} \\ J_f^+(x) \cup \{\infty\} & \text{if } J_f^+(x) \text{ is noncompact and } f^*(\infty) = \infty. \end{cases}$$

Proof. Let $f^*(\infty) = \infty$ and let $J_f^+(x)$ be noncompact. Recall that $J_f^+(x) \subset J_{f^*}^{*+}(x)$. The only point of X^* that a net $(\pi_{f^*}(x_i,t_i))$ with $x_i \to x$ could converge to outside of $J_f^+(x)$ is ∞. The set $D_{f^*}^{*+}(x)$ is compact, and hence, $J_{f^*}^{*+}(x)$ is compact. Thus, $J_{f^*}^{*+}(x) = J_f^+(x) \cup \{\infty\}$.

If $J_f^+(x)$ is compact, then $D_f^+(x)$ is compact and $\infty \notin J_{f^*}^{*+}(x)$, and consequently, $J_{f^*}^{*+}(x) = J_f^+(x)$.

Finally, we claim that if $f^*(\infty) \neq \infty$, then $J_f^+(x)$ is compact and $J_{f^*}^{*+}(x) = J_f^+(x)$. Assume $f^*(\infty) \neq \infty$. Then, $J_f^+(x)$ is a subset of $J_{f^*}^{*+}(x)$ which is a compact in X^*. Since $\infty \notin J_{f^*}^{*+}(x)$, we have $J_{f^*}^{*+}(x) \subset X$; therefore, for any point y in $J_{f^*}^{*+}(x) \setminus J_f^+(x)$ there is a net $(\pi_{f^*}(x_i,t_i))$ which converges to y with $x_i \to x$.

But, since $\pi_{f^*}(x_i, t_i) = \pi_f(x_i, t_i) \in X$ for each i, this also means that y would be in $J_f^+(x)$. Hence, $J_{f^*}^{*+}(x) = J_f^+(x)$ completing the proof. $\square$

Corollary 4.5. *Let X be locally compact and let f be an embedding of X. Then, for each x in X,*

$$D_{f^*}^{*+}(x) = \begin{cases} D_f^+(x) & \text{if } D_f^+(x) \text{ is compact} \\ D_f^+(x) \cup \{\infty\} & \text{if } D_f^+(x) \text{ is noncompact and } f^*(\infty) = \infty. \end{cases}$$

Theorem 4.6. *Let f be an embedding of X, let X be locally compact, and let $x \in X$. Then, $D_f^+(x)$ is compact if and only if $J_f^+(x)$ is nonempty compact.*

Proof. If $D_f^+(x)$ is compact, then $J_f^+(x)$ is a closed subset of $D_f^+(x)$ and is therefore compact. Since $L_f^+(x) \subset J_f^+(x)$, $J_f^+(x)$ is nonempty by Theorem 4.8 of [5] because $K_f^+(x) \subset D_f^+(x)$ means $K_f^+(x)$ is compact, and hence, $L_f^+(x)$ is nonempty compact. Conversely, let $J_f^+(x)$ be nonempty compact. Then, $L_f^+(x) \subset J_f^+(x)$ so that $L_f^+(x)$ is compact. Assume that $L_f^+(x) = \emptyset$. If $C_f(x) \cap J_f^+(x) \neq \emptyset$, then $\pi_f(x,t) \in J_f^+(x)$ for some $t \in \mathbb{R}$. Thus, for $s \in \mathbb{R}$, we have $f(\pi_f(x,s)) = \pi_f(f(x),s) \in \pi_f(J_f^+(x),s) = J_f^+(\pi_f(x,s)) = J_f^+(\pi_f(x,0)) = J_f^+(f(x))$, and $\pi_f(x,s) \in J_f^+(x)$. Therefore, $C_f(x) \subset J_f^+(x)$ and the compactness of $J_f^+(x)$ means that $K_f^+(x)$ is a compact subset of $J_f^+(x)$. By Theorem 4.7 of [5], $L_f^+(x) \neq \emptyset$ contradicting $L_f^+(x) = \emptyset$, and hence, $C_f(x) \cap J_f^+(x) = \emptyset$. Also, $K_f^+(x) = C_f^+(x) \cup L_f^+(x) = C_f^+(x)$, and $C_f^+(x)$ is closed. There is an open set $U \in \eta(J_f^+(x))$ with compact boundary ∂U such that $U \cap C_f^+(x) = \emptyset$. Let $y \in J_f^+(x)$. There exist nets $x_i \to x$ and $t_i \to +\infty$ where $\pi_f(x_i, t_i) \to y$. The set $\pi_f(x_i, [0, t_i])$ is connected for each i, and so, $\pi_f(x_i, [0, t_i]) \cap \partial U \neq \emptyset$ ultimately since the net $(\pi_f(x_i, t_i))$ is ultimately in U. Thus, there is an $s_i \in (0, t_i)$ such that the net $(\pi_f(x_i, s_i))$ is in ∂U. Some subnet $(\pi_f(x_j, s_j))$ of $(\pi_f(x_i, s_i))$ converges to a point $z \in \partial U$. Because $x_j \to x$ and (s_j) is in $\mathbb{R}^+$, $z \in D_f^+(x)$. But this means that $z \in (C_f^+(x) \cup J_f^+(x)) \cap \partial U = \emptyset$ which is absurd. Consequently, $L_f^+(x) \neq \emptyset$, and $J_f^+(x) \neq \emptyset$ follows completing the proof. $\square$

The compactness of $J_f(x)$ in the following theorem need not imply that $J_f(x)$ is connected. Indeed, $J_f^+(x)$ and $J_f^-(x)$ may be nonempty disjoint sets, and hence, $J_f(x) = J_f^+(x) \cup J_f^-(x)$ is not connected. Otherwise, the theorem is valid for both the dual and bilateral versions.

Theorem 4.7. *Let f be an embedding of X, let X be locally compact, and let $x \in X$. If $D_f^+(x)$ or $J_f^+(x)$ is compact, then $D_f^+(x)$ and $J_f^+(x)$ are connected. Conversely, if neither $D_f^+(x)$ nor $J_f^+(x)$ is connected, then they have no compact components.*

Proof. Assume $D_f^+(x)$ is compact but not connected with $D_f^+(x) = G \cup H$ where G and H are nonempty compact disjoint sets. There are disjoint compact neighborhoods $V \in \eta(G)$ and $U \in \eta(H)$ because X is locally compact. Without loss of generality, let $f(x) \in G$. Choose a point y in H. There are nets $x_i \to x$ and (t_i) in $\mathbb{R}^+$ such that $\pi_f(x_i, t_i) \to y$. In fact we can choose (x_i) so that $(f(x_i))$ is in V and $(\pi_f(x_i, t_i))$ is in U. For each i, the compact connected set $\pi_f(x_i, [0, t_i])$ intersects V and U because π_f is continuous. Thus, for each i there is an $s_i \in [0, t_i]$ such that $\pi_f(x_i, s_i) \in \partial V$. Some subnet $(\pi_f(x_j, s_j))$ of $(\pi_f(x_i, s_i))$ converges to a point $z \in \partial V$. But $z \in D_f^+(x)$ and $\partial V \cap D_f^+(x) \neq \emptyset$ which is absurd. Hence, $D_f^+(x)$ is connected.

In order to also see that $J_f^+(x)$ is a connected set, we use (12) and (13) of the Propositions 4.2 and 3.4, respectively, to obtain $J_f^+(x) = f^{-1}(\cap_{t \geq 0} D_f^+(\pi_f(x, t))) = \cap_{t \geq 0} D_f^+(\pi_f(f^{-1}(x), t))$ where $D_f^+(\pi_f(f^{-1}(x), t)) \subset D_f^+(\pi_f(f^{-1}(x), s)) \subset D_f^+(\pi_f(f^{-1}(x), 0)) = D_f^+(x)$ for $0 \leq s \leq t$, i.e. $J_f^+(x)$ is the intersection of nested nonempty compact connected sets.

Conversely, $D_{f^*}^{*+}(x) = D_f^+(x) \cup \{\infty\}$ by Corollary 4.5 since $D_f^+(x)$ is not compact. On the other hand, $D_{f^*}^{*+}(x)$ is compact, and therefore, connected. The set $D_f^+(x)$ is open in the subspace $D_{f^*}^{*+}(x)$ of X^*. It is well known that, for any open set V in a Hausdorff continuum, the set $\overline{V} \setminus V$ contains a limit point of each component of the continuum. The point ∞ is a limit point of each component of $D_f^+(x)$, and hence, no component can be compact. Similarly, the theorem follows for $J_f^+(x)$ completing the proof. $\square$

Proposition 4.8. *Let f be an embedding of X or else X be locally compact and f be a local homeomorphism at x. If (x_i) and (y_i) are nets such that $x_i \to x$, $y_i \to y$, and $y_i \in J_f^+(x)$ for each i, then $y \in J_f^+(x)$.*

Proof. By Proposition 3.6, we have $y \in D_f^+(x)$. Suppose that $y \notin J_f^+(x)$. Then, $y \in C_f^+(x)$ and $y = \pi_f(x, t_y)$ for some $t_y \in \mathbb{R}^+$. If there are nets $x_i' \to x$ and (t_i) in $\mathbb{R}^+$ such that $\pi_f(x_i', t_i) \to y$, then $t_i \not\to +\infty$. No subnet of (t_i) converges to $+\infty$ so that $t_i \leq T$ for some $T \geq 0$ and all i. Some subnet (t_j) of (t_i) must converge to a T_o where $0 \leq T_o \leq T$. Thus, $y \notin \pi_f(x, [T + 1, +\infty))$, i.e. $f(y) \notin \pi_f(f(x), [T+1, +\infty)) = \cup_{t \geq T+1} \pi_f(f(x), t) = \cup_{t-T-1 \geq 0} \pi_f(\pi_f(x, T+1), t-T-1) = C_f^+(\pi_f(x, T+1))$ and $f(y) \notin f(J_f^+(x)) = J_f^+(f(x)) = J_f^+(\pi_f(x, 0)) = J_f^+(\pi_f(x, T+1))$ by (2) and (18) of Proposition 4.2. Hence, $f(y) \notin D_f^+(\pi_f(x, T+1))$. There exists an open $U \in \eta(\pi_f(x, T+1))$ such that $f(y) \notin \overline{\pi_f(U, \mathbb{R}^+)}$. Since $\pi_f(x_i, T+1) \to \pi_f(x, T+1)$, the net $(\pi_f(x_i, T+1))$ is ultimately in U, and hence, ultimately we have $D_f^+(\pi_f(x_i, T+1)) \subset \overline{\pi_f(U, \mathbb{R}^+)}$. Also, $f(y_i) \in f(J_f^+(x_i)) = J_f^+(f(x_i)) = J_f^+(\pi_f(x_i, T+1)) \subset D_f^+(\pi_f(x_i, T+1)) \subset \overline{\pi_f(U, \mathbb{R}^+)}$ holds ultimately, but this contradicts $f(y_i) \to f(y) \notin \overline{\pi_f(U, \mathbb{R}^+)}$. Therefore, $y \in J_f^+(x)$ completing the proof. □

Corollary 4.9. *If M is compact, then $J_f^+(M)$ is closed.*

Proof. The method used to obtain Corollary 3.7 applies here as well. □

The bilateral version of the following theorem is obviously false.

Proposition 4.10. *Let f be an embedding of X and $x \in X$. The maximal f-invariant subset of $D_f^+(x)$ is $J_f^+(x)$.*

Proof. Certainly $J_f^+(x)$ is an f-invariant subset of $D_f^+(x)$. Let M be the maximal f-invariant subset of $D_f^+(x)$. Then, $J_f^+(x) \subset M$. Suppose that there is a point $y \in M \setminus J_f^+(x)$. Then, $y \in C_f^+(x)$ and $y = \pi_f(x, t)$ for some $t \in \mathbb{R}^+$. Now, $C_f(y) = C_f(\pi_f(x, t)) \subset C_f(M) = f(M)$. Since $C_f(\pi_f(x, t)) = \{\pi_f(\pi_f(x, t), s) : s \in \mathbb{R}\} = \{f(\pi_f(x, t+s)) : s \in \mathbb{R}\}$, we have $f(\pi_f(x, t+s)) \subset f(M)$, and hence,

$\pi_f(x, t+s) \subset M$ for each $s \in \mathbb{R}$. Thus, $C_f(x) \subset M$ which means that $D_f^+(x) \subset C_f(x) \cup J_f^+(x) \subset M \subset D_f^+(x)$, and hence, $D_f^+(x) = M$. Either $C_f^-(x) \subset C_f^+(x)$ and $x \in \mathcal{P}_f \cup \mathcal{S}_f$ by Proposition 5.9 of [4], or else $C_f^-(x) \subset J_f^+(x)$, so that for $t \in \mathbb{R}^-$ and for each $s \in \mathbb{R}$, $f(\pi_f(x, t+s)) = \pi_f(f(x), t+s) = \pi_f(\pi_f(x,t), s) \in \pi_f(J_f^+(x), s) = J_f^+(\pi_f(x,s)) = J_f^+(\pi_f(x,0)) = J_f^+(f(x)) = f(J_f^+(x))$ from which $\pi_f(x, t+s) \in J_f^+(x)$ for each $s \in \mathbb{R}$ follows. In either case, $C_f(x) \subset J_f^+(x)$ contradicting $y \in C_f^+(x) \setminus J_f^+(x)$. Hence, $M = J_f^+(x)$, and similarly, $M = J_f^-(x)$ for the dual completing the proof. □

Proposition 4.11. *Let f be an embedding of X and let $x \in X$. If $y \in K_f(x)$, then $J_f^+(f(x)) \subset J_f^+(y)$.*

Proof. We shall begin by verifying that $J_f^+(f(x)) \subset D_f^+(y)$. To do this we show that $z \notin D_f^+(y)$ implies $z \notin J_f^+(f(x))$. For $z \notin D_f^+(y)$ there is an open set $U \in \eta(y)$ such that $z \notin \overline{\pi_f(U, \mathbb{R}^+)}$. Since $y \in K_f(x)$, we have $U \in \eta(\pi_f(x,t))$ for some t in $\mathbb{R}$. Consequently, $D_f^+(\pi_f(x,t)) \subset \overline{\pi_f(U, \mathbb{R}^+)}$. Since $J_f^+(f(x)) = J_f^+(\pi_f(x,0)) = J_f^+(\pi_f(x,t)) \subset D_f^+(\pi_f(x,t))$, we have $z \notin J_f^+(f(x))$. Finally, $J_f^+(f(x))$ is an f-invariant subset of $D_f^+(y)$ which has maximal f-invariant subset $J_f^+(y)$, and thus, $J_f^+(f(x)) \subset J_f^+(y)$. □

Proposition 4.12. *Let f be an embedding of X and let $x \in X$. If $y, z \in L_f^+(x)$, then $f(y) \in J_f^+(z)$ and $J_f^+(z) = D_f^+(z)$.*

Proof. First, $z \in L_f^+(x) \subset K_f(x)$ implies $J_f^+(f(x)) \subset J_f^+(z)$ by Proposition 4.11. Thus, $f(y) \in f(L_f^+(x)) = L_f^+(f(x)) \subset J_f^+(f(x)) \subset J_f^+(z)$. Since y and z are arbitrary, we also have $f(z) \in J_f^+(z)$, and hence, for each $t \in \mathbb{R}$, we have $f(\pi_f(z,t)) = \pi_f(f(z), t) \in \pi_f(J_f^+(z), t) = J_f^+(\pi_f(z,t)) = J_f^+(\pi_f(z,0)) = J_f^+(f(z))$. Thus, $\pi_f(z,t) \in J_f^+(z)$ for each $t \in \mathbb{R}$, which means, $C_f(z) \subset J_f^+(z)$. Therefore, $J_f^+(z) = D_f^+(z)$. □

Theorem 4.13. *Let M be a set with compact boundary. If $f(x) \in M$, $y \in M$, and $y \in D_f^+(x)$, then there is a point $z \in \partial M$ such that $z \in D_f^+(x)$ and $f(y) \in D_f^+(z)$; therefore, if $D_f^+(x) \not\subset M$ for some $x \in X$, then $D_f^+(x) \cap \partial M \neq \emptyset$.*

Proof. If either $f(x)$ or y is in ∂M, the results follow immediately. Let $f(x) \in M^\circ$ and $y \in (X \setminus M)^\circ$. There exist nets $x_i \to x$ and (t_i) in $\mathbb{R}^+$ such that $\pi_f(x_i, t_i) \to y$. Ultimately the net $(f(x_i))$ is in M° and the net $(\pi_f(x_i, t_i))$ is in $(X \setminus M)^\circ$. Since $\pi_f(x_i, [0, t_i])$ is connected for each i, ultimately we have an $s_i \in (0, t_i)$ such that $\pi_f(x_i, s_i) \in \partial M$. Some subnet $(\pi_f(x_j, s_j))$ of $(\pi_f(x_i, s_i))$ converges to a point z in the compact set ∂M. Then, $z \in D^+_f(x)$. Since $\pi_f(\pi_f(x_j, s_j), t_j - s_j) = \pi_f(f(x_j), t_j) = f(\pi_f(x_j, t_j)) \to f(y)$ where $\pi_f(x_j, s_j) \to z$ and $(t_j - s_j)$ is in $\mathbb{R}^+$, we have $f(y) \in D^+_f(z)$.

Finally, the statement $D^+_f(x) \not\subset M$ implies $D^+_f(x) \cap \partial M \neq \emptyset$ is a restatement of the property demonstrated above. □

5. Epilogue

Properties of orbital f-stability and f-attraction will be addressed subsequent sequels to this paper.

References

[1] Ronald A. Knight, *Dynamical systems of characteristic 0*, Pac. Jour. of Math., **41** (1972), 447–457.

[2] ______, *Prolongationally stable transformation groups*, Math. Zeits., **161** (1978), 189–194.

[3] ______, *Iterates of homeomorphisms on locally compact Hausdorff spaces*, Top. and its Appl., **52** (1993), 71–79.

[4] ______, *Initially deformed flows*, Top. Proc., **32** (2008), 167–185.

[5] [1] ______, *Some Topological Properties of Generalized Flows*, Top. Proc., **34** (2009), 283–301.

[6] Gordon Thomas Whyburn, *Analytic Topology*, American Mathematical Society Colloquium Publications, v. 28. New York: American Mathematical Society, 1942.

Mathematics and Computer Science Department, Truman State University, Kirksville, Missouri 63501

E-mail address: rknight@truman.edu

[1]Citations [1], [2], [3], and [4] in *Some Topological Properties of Generalized Flows* correspond to [4], [1], [2], and [3] of its REFERENCES, respectively.