TOPOLOGY PROCEEDINGS

Volume 37, 2011

Pages 177–180

http://topology.auburn.edu/tp/

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Electronically published on July 6, 2010

Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

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http://topology.auburn.edu/tp/
TOPOLOGY PROCEEDINGS
Volume 37 (2011)
Pages 177-180
E-Published on July 6, 2010

HOW DOES UNIVERSALITY OF COPRODUCTS DEPEND ON THE CARDINALITY?

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Abstract. We show that universality of coproducts can break down at any regular cardinal. This is different from coproduct preservation by set-valued functors, which can break down only at measurable cardinals.

1. Some introductory remarks

In this paper we give examples of categories, where coproducts of size α are not universal for some regular cardinal α, but all coproducts of smaller size are universal, as opposed to preservation of coproducts by set-valued functors, which can break down only at *measurable* cardinals (cf. [1]).

Universality of coproducts implies several other important properties, but is not itself implied by them (cf. [2]). Here we ask how it depends on the cardinality of the indexing set. Coproducts of size 1 are always universal; a coproduct of size 0, i.e. an initial object 0 is universal, if and only if every morphism into 0 is an isomorphism. If coproducts of some size ≥ 2 exist and are universal, then the empty coproduct is, too. Then universality of larger coproducts implies universality of smaller ones, if the latter ones exist. If all finite coproducts exist, then they are universal if and only if binary coproducts are.

2010 *Mathematics Subject Classification.* 18A30, 54E15, 06D05.
Key words and phrases. Pullback, universal coproduct, regular cardinal, lattice, distributivity, uniform space, uniform covering.
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It is well-known that in the category of sets or in the category of topological spaces all coproducts are universal. In the category of compact Hausdorff spaces or of uniform spaces, finite coproducts are universal, but infinite ones are not. So we look for cardinals α with the property that in some reasonable category universality of coproducts breaks down at an infinite cardinal α, i.e. coproducts of less than α objects are universal, but coproducts of size α are not. The weakest hypothesis we should assume is the existence of all involved coproducts; universality shall not simply break down by non-existence. So we look for categories $\mathcal{C}$ in which coproducts of size $\leq \alpha$ exist, but are not universal, while coproducts of size $< \alpha$ are.

A necessary condition for the existence of such a $\mathcal{C}$ is the *regularity* of α, i.e. α is not a sum of less than α cardinals $< \alpha$. Indeed, if it is not satisfied, then each coproduct of size α can be represented as a coproduct of less than α coproducts of size $< \alpha$, and if all these smaller coproducts are universal, then the original coproduct is, too.

This present paper is based on the second author's *diplomarbeit*. We thank the referee for some valuable suggestions.

We start with *posets*, considered as categories. Then coproducts are *suprema*, their existence means to work in *complete lattices*, and pullbacks are *binary infima*. Now universality means *distributivity*.

Define $L = \{X \subset Z \mid \#X < \alpha \text{ or } X = Z\}$ for some set of cardinality $\#Z = \alpha$ with the usual set-inclusion. Then arbitrary suprema exist in L; we have $\bigvee_{i\in I} X_i = \bigcup_{i\in I} X_i$ if the cardinality $\#\bigcup_{i\in I} X_i < \alpha$ and $\bigvee_{i\in I} X_i = Z$ otherwise. Then L is a complete lattice. Therefore arbitrary infima exist, in particular finite ones; they are formed by intersection.

Theorem 1.1. *For a regular cardinal α, in L, $\bigvee_{i\in I}(X_i \wedge Y) = \bigvee_{i\in I} X_i \wedge Y$ is true for $\#I < \alpha$, but not for $\#I = \alpha$.*

Proof. Assume $\#I < \alpha$. First, we consider the case $\#X_i < \alpha$ for all $i \in I$. As α is regular, this implies $\#\bigvee_{i\in I} X_i < \alpha$ and hence $\#\bigvee_{i\in I}(X_i \wedge Y) < \alpha$, so all suprema are set unions, and the distributivity law is the usual distributivity law for set union and intersection.

If there is an $i \in I$ with $\#X_i = \alpha$, then $X_i = Z$ must hold. This implies $Y \wedge \bigvee_{i\in I} X_i = Y \wedge Z = Y = \bigvee_{i\in I}(X_i \wedge Y)$.

On the other hand, for $a \in I$ we have $\#X = \alpha$ for $X := Z \setminus \{a\}$, hence $\bigvee_{x\in X}\{x\} = Z$ and $\{a\} \wedge \bigvee_{x\in X}\{x\} = \{a\} \wedge Z = \{a\} \neq \emptyset = \bigvee_{x\in X} \emptyset = \bigvee_{x\in X}(\{a\} \wedge \{x\})$, hence unions (i.e. coproducts) of size α are not distributive (i.e. universal). □

2. A uniform counterexample

This lattice counterexample is not quite satisfactory; it would be desirable to have a counterexample with *disjoint* coproducts, i.e. where the pullback of the two different injections of a coproduct is always an initial object. In a lattice the domain of the pullback of two injections is their meet; hence it is not initial in general. It is well-known that disjointness is equivalent to the statement that every *pre-initial* object is initial (cf. [2]); here an object A is called pre-initial if for every object X there is *at most* one morphism from A to X. Our example is a bicoreflective subcategory of the category of uniform spaces; in particular it is topological over the category of sets and hence complete and cocomplete. Moreover, coproducts are disjoint. For more details on uniform spaces see e.g. [3], [5] or [6].

Let α be an infinite regular cardinal, which will remain fixed throughout the paper. We consider the category $\mathcal{C}$ consisting of all uniform spaces, such that less than α uniform coverings always have a common refinement, which is itself a uniform covering, or equivalently, that the filter of entourages is closed under intersections of less than α members. It yields a bicoreflective subcategory; the coreflector endows the underlying set of a uniform space with the coarsest uniform structure that makes the coarsest uniform refinement of any less than α uniform coverings a uniform covering. If the uniformity is described in terms of entourages, the coreflection is obtained by closing the filter of all entourages under intersections of size smaller than α.

Theorem 2.1. *In $\mathcal{C}$, coproducts of size smaller than α are universal, but coproducts of size α are not.*

Proof. By coreflectivity, coproducts in $\mathcal{C}$ are uniform coproducts; pullbacks in $\mathcal{C}$ can easily be seen to be uniform pullbacks. Now let $f : A \to B := \coprod_{i\in I} B_i$ be a uniformly continuous map and consider a cover $\mathcal{U}$ of B in $\mathcal{C}$ with $\#I < \alpha$ such that for each $i \in I$ the $f_i^{-1}U$, $U \in \mathcal{U}$ form a uniform cover $\mathcal{U}_i$ of $A_i := f^{-1}B_i$ for the restrictions $f_i : A_i \to B_i$ of f. Then for each $i \in I$ the $U \subset A$ with $v_i^{-1}U \in \mathcal{U}_i$ form a uniform open cover of A for the

coproduct injections $v_i : A_i \to A$. Since $A \in \mathcal{C}$ and $\#I < \alpha$, they have a common uniform refinement. This proves that $A \cong \coprod_{i\in I} A_i$, showing that all coproducts of size smaller than α are universal in $\mathcal{C}$.

The discrete uniform space D of size α belongs to $\mathcal{C}$. If coproducts of size α were universal in $\mathcal{C}$, then for each uniform space C in $\mathcal{C}$, $C \times D$ would be a copy of α many copies of C in the canonical way. We claim that this is not the case.

Let C be a set of cardinality α. Call a cover $\mathcal{U}$ of C uniform if it contains a U with $\#C \setminus U < \alpha$. Every intersection of less than α such sets also has this property because α is a regular cardinal. It follows by direct computation that a cover W' of $C \times D$ is uniform if and only if there exists a $U \subset C$ with $C \setminus U < \alpha$ such that $U \times D$ is contained in some member of W'. This proves that C is indeed a uniform space, which even belongs to $\mathcal{C}$

Now assume that C and D both are the set of all ordinals smaller than α; this set has cardinality α. Consider $W := \{(\xi, \eta) | \eta < \eta < \alpha\} \subset C \times D$. Then the cover $\mathcal{W} := \{W\} \cup \bigcup_{\xi\leq\eta<\alpha} \{(\xi, \eta)\}$ is not a uniform cover of $C \times D$. But for each $\eta < \alpha$, the inverse image $V_i := \{\xi | (\xi, \eta) \in W\} = \{\xi | \eta < \xi < \alpha\}$ of W under the map $C \to C \times D$, $\xi \mapsto (\xi, \eta)$ satisfies $\#C \setminus V_i < \alpha$. Thus $\mathcal{W}$ is an open cover in the coproduct structure of $C \times D$ (as a coproduct of all $C \times \{\eta\}$), which therefore does not coincide with the product structure. This proves that coproducts of size α are not universal. □

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